On the construction of a Lévy-type process associated with a stable-dominated Lévy-type kernel

joint work with K. Bogdan and P. Sztonyk

Bedlewo, June 26–July 2, 2016

Overview

$f \in C^2_\infty(\mathbb{R}^d)$

$$L^z f(x) := p.v. \int_{\mathbb{R}^d\setminus\{0\}} \big(f(x+u) - f(x)\big)\, \nu(z, du),$$

$$\nu(z, A) = \nu(z, -A), \quad \sup_z \int_{\mathbb{R}^d\setminus\{0\}} (1 \wedge |u|^2)\, \nu(z, du) < \infty;$$

$\nu(z, du)$– anisotropic

$f \in C^2_\infty(\mathbb{R}^d)$

$$L^z f(x) := p.v. \int_{\mathbb{R}^d\setminus\{0\}} \big(f(x+u) - f(x)\big)\, \nu(z, du),$$

$$\nu(z, A) = \nu(z, -A), \quad \sup_z \int_{\mathbb{R}^d\setminus\{0\}} (1 \wedge |u|^2)\, \nu(z, du) < \infty;$$

$\nu(z, du)$– anisotropic

$$L = L^z\Big|_{z=x}.$$

$f \in C^2_\infty(\mathbb{R}^d)$

$$L^z f(x) := p.v. \int_{\mathbb{R}^d \setminus \{0\}} \big(f(x+u) - f(x)\big)\, \nu(z, du),$$

$$\nu(z, A) = \nu(z, -A), \quad \sup_z \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |u|^2)\, \nu(z, du) < \infty;$$

$\nu(z, du)$– anisotropic

$$L = L^z \Big|_{z=x}.$$

Question:

1. Does $(L, C^2_\infty(\mathbb{R}^d))$ extend to a gener. of a Feller semigroup

$$T_t f(x) = \mathbb{E}^x f(X_t) \qquad ?$$

2. What info do we have about X? I.e. $\exists$ trans. probab. dens. $p_t(x, y)$, estimates?

Probabilistic interpretation:

$$L^{(0)}f(x) = af'(x) + p.v. \int_{\mathbb{R}^d} \big(f(x+u) - f(x)\big)\nu(du)$$

is the generator of a LP,

$$\mathbb{E}e^{i\xi Z_t} = \exp\{-t(i\xi a + p.v. \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{i\xi u})\nu(du))\}$$

Probabilistic interpretation:

$$L^{(0)}f(x) = af'(x) + p.v. \int_{\mathbb{R}^d} (f(x+u) - f(x))\nu(du)$$

is the generator of a LP,

$$\mathbb{E}e^{i\xi Z_t} = \exp\{-t(i\xi a + p.v. \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{i\xi u})\nu(du))\}$$

Lévy-Ito decomposition:

$$Z_t = at + \int_0^t \int_{|z|\leq 1} z[N(ds,dz) - ds\nu(dz)] + \int_0^t \int_{|z|>1} zN(ds,dz).$$

where $N((0,t],A) := \sharp\{s \in (0,t] : \Delta Z_s \in A\}$,
$\mathbb{E}N(dt,A) = dt\nu(A)$.

Perturb the coefficients (variable jump size!) and get the SDE:

$$dX_t = a(X_t)dt + \int_{|z|\leq 1} \sigma(X_{t-}, z)[N(dt, dz) - dt\nu(dz)] + \int_{|z|>1} \sigma(X_{t-}, z)N(dt, dz).$$

$$\tilde{L}f(x) = a(x)f'(x) + p.v. \int_{\mathbb{R}^d} \big(f(x+\sigma(x,u)) - f(x)\big)\nu(du), \quad f \in C^2_\infty(\mathbb{R}^d)$$

Perturb the coefficients (variable jump size!) and get the SDE:

$$dX_t = a(X_t)dt + \int_{|z|\le 1} \sigma(X_{t-}, z)[N(dt, dz) - dt\nu(dz)]$$
$$+ \int_{|z|>1} \sigma(X_{t-}, z)N(dt, dz).$$

$$\tilde{L}f(x) = a(x)f'(x) + p.v. \int_{\mathbb{R}^d} \big(f(x+\sigma(x,u)) - f(x)\big)\nu(du), \quad f \in C^2_\infty(\mathbb{R}^d)$$

Example:
$\sigma(x, u) = \sigma(x)u$. Then we get

$$dX_t = a(X_t)dt + \sigma(X_{t-})dZ_t,$$

and the “generator"

Perturb the coefficients (variable jump size!) and get the SDE:

$$dX_t = a(X_t)dt + \int_{|z|\le 1} \sigma(X_{t-}, z)[N(dt, dz) - dt\nu(dz)] + \int_{|z|>1} \sigma(X_{t-}, z)N(dt, dz).$$

$$\tilde{L}f(x) = a(x)f'(x) + p.v. \int_{\mathbb{R}^d} \big(f(x+\sigma(x,u)) - f(x)\big)\nu(du), \quad f \in C^2_\infty(\mathbb{R}^d)$$

Example:
$\sigma(x,u) = \sigma(x)u$. Then we get

$$dX_t = a(X_t)dt + \sigma(X_{t-})dZ_t,$$

and the “generator"

$$\tilde{L}f(x) = a(x)f'(x) + p.v. \int_{\mathbb{R}^d} \big(f(x+\sigma(x)u) - f(x)\big)\nu(du), \quad f \in C^2_\infty(\mathbb{R}^d)$$

$$Lf(x) := p.v. \int \left(f(x+u) - f(x)\right) \nu(x, du)$$

$Lf(x) := p.v. \int \big(f(x+u) - f(x)\big)\, \nu(x, du)$

- parametrix method:

take “some"zero-order approximation p^0, and construct the fund. sol. to the Cauchy problem for $\partial_t - L$, i.e.

$$(\partial_t - L_x)p_t(x,y) = 0, \quad p_t(x,y) \to \delta_x(y),\ t \to 0.$$

in the form

$$p = p^0 + r.$$

p– fund. sol. $\Rightarrow$ $r = p^0 \star \Psi$, where

$$\Psi = \sum_{k=1}^{\infty} \Phi^{k\star}, \quad \Phi = (L - \partial_t)p^0$$

$Lf(x) := p.v. \int \big(f(x+u) - f(x)\big)\, \nu(x, du)$

• parametrix method:
take "some"zero-order approximation p^0, and construct the fund. sol. to the Cauchy problem for $\partial_t - L$, i.e.

$$(\partial_t - L_x)p_t(x,y) = 0, \quad p_t(x,y) \to \delta_x(y),\ t \to 0.$$

in the form

$$p = p^0 + r.$$

p– fund. sol. $\Rightarrow r = p^0 \star \Psi$, where

$$\Psi = \sum_{k=1}^{\infty} \Phi^{k\star}, \quad \Phi = (L - \partial_t)p^0$$

• justification procedure: i.e. prove, that p is the fund. sol. for a Cauchy problem for $\partial_t - A$, A–extension of L.

History

- Levi'1907, Dressel'1940, Feller'36, Friedman'64 (monograph)

History

- Levi'1907, Dressel'1940, Feller'36, Friedman'64 (monograph)
- Drin'1977, Eidelman, Drin'1981, Kochubei'1988, Kolokoltsov'2000

History

- Levi'1907, Dressel'1940, Feller'36, Friedman'64 (monograph)
- Drin'1977, Eidelman, Drin'1981, Kochubei'1988, Kolokoltsov'2000
- Debussche, Fournier'2013, Fournier, J. Printems'2010, Fournier'2013.

History

- Levi'1907, Dressel'1940, Feller'36, Friedman'64 (monograph)
- Drin'1977, Eidelman, Drin'1981, Kochubei'1988, Kolokoltsov'2000
- Debussche, Fournier'2013, Fournier, J. Printems'2010, Fournier'2013.
- Iwasaki'74,77, Kumano-go'81, Hoh'98, Jacob'02, Boettcher'05,08

History

- Levi'1907, Dressel'1940, Feller'36, Friedman'64 (monograph)
- Drin'1977, Eidelman, Drin'1981, Kochubei'1988, Kolokoltsov'2000
- Debussche, Fournier'2013, Fournier, J. Printems'2010, Fournier'2013.
- Iwasaki'74,77, Kumano-go'81, Hoh'98, Jacob'02, Boettcher'05,08
- Portenko, Podolynnyi'94,95, Bogdan, Jakubowski'07, Kim, Song'14

History

- Levi'1907, Dressel'1940, Feller'36, Friedman'64 (monograph)
- Drin'1977, Eidelman, Drin'1981, Kochubei'1988, Kolokoltsov'2000
- Debussche, Fournier'2013, Fournier, J. Printems'2010, Fournier'2013.
- Iwasaki'74,77, Kumano-go'81, Hoh'98, Jacob'02, Boettcher'05,08
- Portenko, Podolynnyi'94,95, Bogdan, Jakubowski'07, Kim, Song'14
- K, Kulik'2014, 2015, K, Kulik, Ganychenko'2015, Kulik'15

History

- Levi'1907, Dressel'1940, Feller'36, Friedman'64 (monograph)
- Drin'1977, Eidelman, Drin'1981, Kochubei'1988, Kolokoltsov'2000
- Debussche, Fournier'2013, Fournier, J. Printems'2010, Fournier'2013.
- Iwasaki'74,77, Kumano-go'81, Hoh'98, Jacob'02, Boettcher'05,08
- Portenko, Podolynnyi'94,95, Bogdan, Jakubowski'07, Kim, Song'14
- K, Kulik'2014, 2015, K, Kulik, Ganychenko'2015, Kulik'15
- Chen, Zhang'13, Kim, Song, Vondracek'16

History

- Levi'1907, Dressel'1940, Feller'36, Friedman'64 (monograph)
- Drin'1977, Eidelman, Drin'1981, Kochubei'1988, Kolokoltsov'2000
- Debussche, Fournier'2013, Fournier, J. Printems'2010, Fournier'2013.
- Iwasaki'74,77, Kumano-go'81, Hoh'98, Jacob'02, Boettcher'05,08
- Portenko, Podolynnyi'94,95, Bogdan, Jakubowski'07, Kim, Song'14
- K, Kulik'2014, 2015, K, Kulik, Ganychenko'2015, Kulik'15
- Chen, Zhang'13, Kim, Song, Vondracek'16
- Kuhn'16

Conditions:

$$\nu_0(drd\theta) = r^{-1-\alpha} dr \mu_0(d\theta), \quad \theta \in \mathbb{S}$$

$$\nu_0(B(x,r)) \le m_0 r^{\gamma},\ |x| = 1,\ 0 < r < 1/2$$

Conditions:

$$\nu_0(drd\theta) = r^{-1-\alpha} dr \mu_0(d\theta), \quad \theta \in \mathbb{S}$$

$$\nu_0(B(x,r)) \le m_0 r^\gamma, \; |x| = 1, \; 0 < r < 1/2$$

A1. “Ballance condition": α, γ and d satisfy $\alpha + \gamma > d$.

Conditions:

$$\nu_0(drd\theta) = r^{-1-\alpha} dr \mu_0(d\theta), \quad \theta \in \mathbb{S}$$

$$\nu_0(B(x,r)) \leq m_0 r^\gamma, \ |x| = 1, \ 0 < r < 1/2$$

A1. “Ballance condition": α, γ and d satisfy $\alpha + \gamma > d$.

A2. $\exists M_0 > 0$, $\eta \in (0,1]$ s.t.

$$M_0^{-1} \nu_0(A) \leq \nu(x,A) \leq M_0 \nu_0(A), \quad x \in \mathbb{R}^d, \ A \subset \mathbb{R}^d,$$

$$|\nu(x_1,A) - \nu(x_2,A)| \leq M_0 \left(|x_1 - x_2|^\eta \wedge 1\right) \nu_0(A), \quad x_1, x_2 \in \mathbb{R}^d, \ A \subset \mathbb{R}^d.$$

Theorem 1

1. $p_t(x,y)$ is continuous in $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.
2. $\exists C > 0$ s.t. $\forall t > 0,\ x,y \in \mathbb{R}^d$,

$$\left|\partial_t^k p_t(x,y)\right| \le C t^{-k} e^{ct} G_t^{(\alpha+\gamma)}(y-x), \quad k = 0,1. \tag{1}$$

where $G^{(\beta)}(x) = (1 \vee |x|)^{-\beta}$, $G_t^{(\beta)}(x) = t^{-d/\alpha} G^{(\beta)}(xt^{-1/\alpha})$.

3.

$$\begin{aligned}\left|p_t(x_1,y) - p_t(x_2,y)\right| \le C\Big(\Big(\frac{|x_1 - x_2|}{t^{1/\alpha}}\Big)^\theta \wedge 1\Big) e^{ct} \\ \cdot \Big(G_t^{(\alpha+\gamma)}(y - x_1) + G_t^{(\alpha+\gamma)}(y - x_2)\Big)\end{aligned} \tag{2}$$

$\forall x_1, x_2, y \in \mathbb{R}^d,\ t > 0$, and $0 < \theta < \eta \wedge \alpha \wedge (\alpha + \gamma - d)$.

4. $p_t(x,y)$ is the tr. probab. den. of a strong Markov process X_t on $\mathbb{R}^d$, and the the resp. generator is closure of $(L, C_\infty^2(\mathbb{R}^d))$.

Choice of p^0: $p_t^z(y-x)$ corresponds to the Lévy kernel $\nu(z,du)$,

$$p_t^0(x,y) := p_t^z(y-x)\big|_{z=y}.$$

$$G^{(\beta)}(x) = (|x| \vee 1)^{-\beta}, \quad G_t^{(\beta)}(x) = t^{-d/\alpha} G^{(\beta)}\left(xt^{-1/\alpha}\right)$$

Lemma 1

$\forall \beta \in \mathbb{N}_0^d$, $\exists c > 0$, depending only on ν_0, β, and M_0, s.t.

$$|\partial_x^\beta p_t^z(x)| \le ct^{-|\beta|/\alpha} G_t^{(\alpha+\gamma)}(x), \quad t > 0,\ x, z \in \mathbb{R}^d.$$

$\Phi = (\partial_t - L)p^0$, η: from Hölder cont. of $\nu(x,du)$

Choice of p^0: $p_t^z(y-x)$ corresponds to the Lévy kernel $\nu(z,du)$,

$$p_t^0(x,y) := p_t^z(y-x)\big|_{z=y}.$$

$$G^{(\beta)}(x) = (|x| \vee 1)^{-\beta}, \quad G_t^{(\beta)}(x) = t^{-d/\alpha} G^{(\beta)}\left(xt^{-1/\alpha}\right)$$

Lemma 1

$\forall \beta \in \mathbb{N}_0^d$, $\exists c > 0$, depending only on ν_0, β, and M_0, s.t.

$$|\partial_x^\beta p_t^z(x)| \le ct^{-|\beta|/\alpha} G_t^{(\alpha+\gamma)}(x), \quad t > 0,\ x, z \in \mathbb{R}^d.$$

$\Phi = (\partial_t - L)p^0$, η: from Hölder cont. of $\nu(x,du)$

Lemma 2

$$\left|\Phi_t(x,y)\right| \le C_\Phi t^{-1}(1 \wedge |y-x|^\eta) G_t^{(\gamma+\alpha)}(y-x), \quad x, y \in \mathbb{R}^d,\ t > 0.$$

$\theta > 0$, $\kappa \in (d-\alpha, d]$

$$H_t^{(\kappa,\theta)}(x) = \left(t^{-\theta/\alpha} \wedge \left(\frac{|x|}{t^{1/\alpha}} \vee 1 \right)^{\theta} \right) G_t^{(\kappa+\alpha)}(x). \tag{3}$$

$\theta > 0$, $\kappa \in (d-\alpha, d]$

$$H_t^{(\kappa,\theta)}(x) = \left(t^{-\theta/\alpha} \wedge \left(\frac{|x|}{t^{1/\alpha}} \vee 1 \right)^{\theta} \right) G_t^{(\kappa+\alpha)}(x). \qquad (3)$$

Proposition

Assume that $\alpha + \kappa - d > \theta$. Then $H_t = H_t^{(\kappa,\theta)}(x)$ satisfy the sub-convolution property

$$(H_{t-s} * H_s)(x) \leq C_H H_t(x)$$

with some constant C_H, and

$$c \leq \int_{\mathbb{R}^d} H_t^{(\kappa,\theta)}(x)dx \leq C, \quad t > 0.$$

$$\big|\Phi\big|_t(x,y) \le Ct^{-1+\theta/\alpha} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0,$$

This gives

$$\big|\Phi\big|_t^{\star k}(x,y) \le \frac{C_1 C_2^k}{\Gamma(k\theta/\alpha)} t^{-1+k\theta/\alpha} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0,$$

and thus

$$\left|\Phi\right|_t(x,y) \le Ct^{-1+\theta/\alpha} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0,$$

This gives

$$\left|\Phi\right|_t^{\star k}(x,y) \le \frac{C_1 C_2^k}{\Gamma(k\theta/\alpha)} t^{-1+k\theta/\alpha} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0,$$

and thus

$$\left|\Psi\right|_t(x,y) \le Ct^{-1+\theta/\alpha} e^{ct} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0,$$

$$\left(p^0 \star \Psi\right)_t(x,y) \le Ct^{\theta/\alpha} e^{ct} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0.$$

and

$$\big|\Phi\big|_t(x,y) \le Ct^{-1+\theta/\alpha} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0,$$

This gives

$$\big|\Phi\big|_t^{\star k}(x,y) \le \frac{C_1 C_2^k}{\Gamma(k\theta/\alpha)} t^{-1+k\theta/\alpha} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0,$$

and thus

$$\big|\Psi\big|_t(x,y) \le Ct^{-1+\theta/\alpha} e^{ct} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0,$$

$$\big(p^0 \star \Psi\big)_t(x,y) \le Ct^{\theta/\alpha} e^{ct} H_t^{(\gamma,\theta)}(y-x), \quad x,y \in \mathbb{R}^d, \quad t>0.$$

and

$$p_t(x,y) \le Ce^{ct} G_t^{(\gamma+\alpha)}(x,y), \quad x,y \in \mathbb{R}^d,\ t>0.$$

Verification: Hölder continuity of $\Psi = \sum_{k=1}^{\infty} \Phi^{\star k}$

$$\big|\Psi_t(x_1, y) - \Psi_t(x_2, y)\big| \le C\big(|x_1 - x_2|^{\theta-\epsilon} \wedge 1\big)\, t^{-1+\epsilon/\alpha}\Big(H_t^{(\gamma,\theta)}(y - x_1) + H_t^{(\gamma,\theta)}(y - x_2)\Big).$$

Verification: Hölder continuity of $\Psi = \sum_{k=1}^{\infty} \Phi^{\star k}$

$$\left|\Psi_t(x_1, y) - \Psi_t(x_2, y)\right| \le C\left(|x_1 - x_2|^{\theta-\epsilon} \wedge 1\right) t^{-1+\epsilon/\alpha}\Big(H_t^{(\gamma,\theta)}(y - x_1) + H_t^{(\gamma,\theta)}(y - x_2)\Big).$$

"a la Kochubei", define

$$\mathcal{D} := \{f : \lim_{\epsilon\to 0} L^{(\epsilon)} f < \infty\}, \quad L^{(\epsilon)} f := \int_{|u|>\epsilon} (\dots).$$

Verification: Hölder continuity of $\Psi = \sum_{k=1}^{\infty} \Phi^{\star k}$

$$\big|\Psi_t(x_1,y)-\Psi_t(x_2,y)\big| \le C\big(|x_1-x_2|^{\theta-\epsilon}\wedge 1\big)t^{-1+\epsilon/\alpha}\Big(H_t^{(\gamma,\theta)}(y-x_1) + H_t^{(\gamma,\theta)}(y-x_2)\Big).$$

"a la Kochubei", define

$$\mathcal{D} := \{f : \lim_{\epsilon\to 0} L^{(\epsilon)} f < \infty\}, \quad L^{(\epsilon)} f := \int_{|u|>\epsilon} (\dots).$$

Then $p_t(\cdot,y)\in\mathcal{D}$, and

$$Lp_t(x,y) = Lp_t^0(x,y) + \int_0^t \int_{\mathbb{R}^d} Lp_{t-s}^0(x,z)\Psi_s(z,y)dzds,$$

$$\partial_t p_t(x,y) = \partial_t p_t^0(x,y) + \Psi_t(x,y) + \int_0^t \int_{\mathbb{R}^d} \partial_t p_{t-s}^0(x,z)\Psi_s(z,y)dzds.$$

Since $\Phi = (L-\partial_t)p^0$, we get $\Psi = \Phi + \Psi\star\Phi$.

On the martingale problem:

$$Lf(x) = -\mathcal{F}^{-1}\Big((q(x,\xi)\mathcal{F}f(\xi)\Big)_{\xi\to x}$$

$$q(x,\xi) = p.v.\int_{\mathbb{R}^d}(1-e^{i\xi u})\nu(x,du)$$

On the martingale problem:

$$Lf(x) = -\mathcal{F}^{-1}\Big((q(x,\xi)\mathcal{F}f(\xi)\Big)_{\xi\to x}$$

$$q(x,\xi) = p.v. \int_{\mathbb{R}^d} (1 - e^{i\xi u})\nu(x, du)$$

Hoh'94: $q(x,\xi)$– continuous, $q(x,\xi) \le c(1+|\xi|^2)$
$\Rightarrow$ exists a solution to the MP

On the martingale problem:

$$Lf(x) = -\mathcal{F}^{-1}\Big((q(x,\xi)\mathcal{F}f(\xi)\Big)_{\xi\to x}$$

$$q(x,\xi) = p.v. \int_{\mathbb{R}^d} (1 - e^{i\xi u})\nu(x, du)$$

Hoh'94: $q(x,\xi)$– continuous, $q(x,\xi) \leq c(1+|\xi|^2)$
$\Rightarrow$ exists a solution to the MP

We say that $\mathbb{P}$ is a solution of the *martingale problem* for $(\mathcal{L}, C_0^\infty(\mathbb{R}^d))$, if for all $u \in C_0^\infty(\mathbb{R}^d)$

$$u(X_t) - u(X_0) - \int_0^t (\mathcal{L}u)(X_s)ds, \quad t \geq 0,$$

is a martingale.

Kulik'15:

Check Ethier, Kurtz'86, Lem. 4.3.4: $\forall g(t,x)$, $g(\cdot,x) \in C^1((0,T))$, $g(t,\cdot) \in D(\mathcal{L})$ the process

$$g(t, X_t) - \int_0^t (\partial_s + \mathcal{L}_x) g(s, X_s) ds$$

is a martingale w.r.t. P.

Let

$$P_t f(x) = \int p_t(x,y) f(y) dy.$$

Applying to $g^T(t,x) = P_{T-t} f(x)$, we get

$$\mathbb{E}^P P_0 f(X_T) = \mathbb{E}^P P_T f(X_0),$$

then use EK'86, Th.4.4.2: solution is unique, and is a str. M. pr.

Anisotropy and adding the drift: what to do

$$Lf(x) = b(x)\nabla f(x) + p.v. \int_{\mathbb{R}^d} \big(f(x+\sigma(x)u) - f(x)\big)\nu(du)$$

$$dX_t = b(X_t)dt + \sigma(X_{t-})dZ_t^{(\alpha)}$$

Anisotropy of $Z^{(\alpha)}$ + matrix-valued σ— BIG problem!

Anisotropy and adding the drift: what to do

$$Lf(x) = b(x)\nabla f(x) + p.v. \int_{\mathbb{R}^d} \big(f(x+\sigma(x)u) - f(x)\big)\nu(du)$$

$$dX_t = b(X_t)dt + \sigma(X_{t-})dZ_t^{(\alpha)}$$

Anisotropy of $Z^{(\alpha)}$ + matrix-valued σ— BIG problem!

1. Big problem, when σ rotates the support of ν.
2. In general, single-kernel estimate is not enough, need "compound kernel structure"

$$p_t^z(x) \leq (g_t * P_t)(x), \quad x \in \mathbb{R}^d, t \in (0, T]$$

SDE framework: Adding a drift

Assume now $Z^{(\alpha)}$ –rot. sym., $L = b(x)\nabla + \sigma^{\alpha}(x)L^{(0)}$, $\sigma : \mathbb{R}^d \to \mathbb{R}$.

SDE framework: Adding a drift

Assume now $Z^{(\alpha)}$ –rot. sym., $L = b(x)\nabla + \sigma^\alpha(x)L^{(0)}$, $\sigma : \mathbb{R}^d \to \mathbb{R}$.

When the drift dominates $Z^{(\alpha)}$, i.e. $\alpha \in (0,1]$: Chose

$$p_t^0(x,y) := p_t^z(\theta_t(y) - x)\big|_{z=y}.$$

where θ_t is the solution to

$$d\theta_t = -b(\theta_t)dt, \quad \theta_0 = x.$$

(Problem: smoothness of b, see K.Kulik '14, Kulik'15)

SDE framework: Adding a drift

Assume now $Z^{(\alpha)}$ –rot. sym., $L = b(x)\nabla + \sigma^\alpha(x)L^{(0)}$, $\sigma : \mathbb{R}^d \to \mathbb{R}$.

When the drift dominates $Z^{(\alpha)}$, i.e. $\alpha \in (0,1]$: Chose

$$p_t^0(x,y) := p_t^z(\theta_t(y) - x)\Big|_{z=y}.$$

where θ_t is the solution to

$$d\theta_t = -b(\theta_t)dt, \quad \theta_0 = x.$$

(Problem: smoothness of b, see K.Kulik '14, Kulik'15)

$$p_t^0(x,y) \le CG_t^{(\alpha+d)}(\theta_t(y) - x).$$

The kernel

$$Q_t(x,y) := H_t^{(\theta,d)}(\theta_t(y) - x) \asymp H_t^{(\theta,d)}(y - \theta_t^{-1}(x))$$

also satisfies the sub-conv. property!

Thank you!