Obstacle problems for integro-differential operators

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(joint work with L. Caffarelli and J. Serra)

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- For a general symmetric Lévy process,

$$Lu(x) = a_{ij}\partial_{ij}u + \int_{\mathbb{R}^n}\left(\frac{u(x+y)+u(x-y)}{2} - u(x)\right)d\nu(y),$$

with (a_{ij}) nonnegative definite and $\int_{\mathbb{R}^n}\min(1,|y|^2)d\nu(y) < \infty$.

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- Elliptic integro-differential operator:

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- More general class of operators:

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Theorem (Caffarelli-Salsa-Silvestre; Invent. Math. 08)

Solutions are $u \in C^{1+s}$*. Moreover, for each* $x_0 \in \Gamma(u)$

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- *Almgren frequency function* $\rightsquigarrow$ Blow-ups are homogeneous

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Open question: Regularity for obstacle problem for more general operators

$$Lu(x) = \int_{\mathbb{R}^n} \left(\frac{u(x+y) + u(x-y)}{2} - u(x) \right) K(y)dy$$

of order $2s$?

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New results: We establish new regularity results for

$$0 < \frac{\lambda}{|y|^{n+2s}} \leq K(y) \leq \frac{\Lambda}{|y|^{n+2s}}, \qquad K(y) \text{ homogeneous}$$

Integro-differential operators: new results

Theorem (Caffarelli-R-Serra; preprint arXiv '16)

Assume L is as above. Then, for each free boundary point x_0,

(a) *either*

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for all $\alpha \in (0,s)$ such that $1+s+\alpha < 2$.

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for all $\alpha \in (0, s)$ *such that* $1 + s + \alpha < 2$.

Moreover, the set of points satisfying (a) is open and $C^{1,\gamma}$ *for all* $\gamma < s$.

Furthermore, if x_0 *is regular then*

$$u(x) - \varphi(x) = c_0\, d^{1+s}(x) + o(|x - x_0|^{1+s+\gamma}),$$

with $c_0 > 0$.

The proof

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5. Show that the regular set is open, hence the free boundary is C^1
6. Prove that the free boundary is $C^{1,\gamma}$, and then the expansion

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5. By a boundary Harnack in C^1 domains, prove that $v_i \equiv \lambda v_j$.
6. This yields that u is 1D, and Ω is a half-space.

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- Obstacle problems with non-homogeneous kernels

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The end

Thank you!