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Approximation of subordinate semigroups via the Chernoff theorem

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1. SEMIGROUPS AND MARKOV PROCESSES

Def: A family $(T_t)_{t\geq 0}$ of bounded linear operators on a Banach space X is called a C_0-semigroup, if $T_0 = \mathrm{Id}$, $T_s \circ T_t = T_{s+t}$ for all $t, s \geq 0$ and $\lim_{t\to 0} \|T_t\varphi - \varphi\|_X = 0$ for all $\varphi \in X$.

Def: The generator $(L, \mathrm{Dom}(L))$ of a C_0-semigroup $(T_t)_{t\geq 0}$ is defined via

$$L\varphi := \lim_{t\to 0} \frac{T_t\varphi - \varphi}{t}, \quad \mathrm{Dom}(L) := \{\varphi \in X : \text{ this limit exists } \}.$$

Thm: $(T_t)_{t\geq 0}$ is a C_0-semigroup on a BS X with generator $(L, \mathrm{Dom}(L))$ $\Longleftrightarrow$ the Cauchy problem

$$\begin{cases} \frac{df}{dt} = Lf, \\ f(0) = f_0 \end{cases}$$

is correctly posed in X for all $f_0 \in \mathrm{Dom}(L)$. And $f(t) := T_t f_0$ is the unique solution.

Notation: $T_t \equiv e^{tL}$

Rem: Let $(X_t)_{t\geq 0}$ be a Markov process with transition kernel $P(t,x,dy)$. Then $(T_t)_{t\geq 0}$, given by

$$T_t\varphi(x) := \int \varphi(y)P(t,x,dy) \equiv \mathbb{E}^x\left[\varphi(X_t)\right],$$

is a semigroup, can be C_0 on some BS X of functions φ. Therefore,

To construct the C_0-semigroup $T_t \equiv e^{tL}$ on X with a given generator L

$$\Updownarrow$$

To solve the Cauchy problem for the evolution equation $\frac{df}{dt} = Lf$ in X

$$\Updownarrow$$

To find the transition probability $P_t(x,dy)$ of the corresponding Markov process $(X_t)_{t\geq 0}$

2. CHERNOFF APPROXIMATION OF EVOLUTION SEMIGROUPS

The Chernoff theorem [1968]: Let $F:[0,\infty)\to\mathcal{L}(X)$ be such that

- $F(0)=\mathrm{Id}$,
- $\|F(t)\|\leqslant e^{at}$ for some $a\in\mathbb{R}$ and all $t\geqslant 0$,
- the limit $L\varphi := \lim\limits_{t\to 0}\frac{F(t)\varphi-\varphi}{t}$ exists for all $\varphi\in D$, where $D\subset X$: the closure $(L,\mathrm{Dom}(L))$ of (L,D) generates a C_0-semigroup $(T_t\equiv e^{tL})_{t\geq 0}$.

Then

$$e^{tL}\varphi = \lim_{n\to\infty}\left[F(t/n)\right]^n\varphi, \qquad \forall\,\varphi\in X,$$

locally uniformly w.r.t. $t\geq 0$.

Notation: $F(t)\sim T_t$.

Ex1: Let L be a bounded operator on X. Then $F(t) := \mathrm{Id} + tL \sim e^{tL}$. Hence

$$e^{tL} = \lim_{n\to\infty} \left[\mathrm{Id} + \frac{t}{n} L\right]^n.$$

Ex2: $F_k(t) \sim e^{tL_k}, k = 1, ..., m \quad \Longrightarrow \quad F_1(t) \circ ... \circ F_m(t) \sim e^{t(L_1 + ... + L_m)}$.

Rem: Take $k = 2$ and $F_k(t) := e^{tL_k}$ to get the Daletskii–Lie–Trotter formula:

$$e^{t(L_1+L_2)} = \lim_{n\to\infty} \left[e^{tL_1/n} \circ e^{tL_2/n}\right]^n.$$

Rem: If $F(t)$ are integral operators, one has

$$e^{tL}\varphi = \lim_{n\to\infty} \left[F(t/n)\right]^n \varphi =$$

$$= \lim_{n\to\infty} \int \cdots \int \ldots \varphi(x_n)\, dx_1 \cdots dx_n \quad \leftarrow \text{ Feynman formula} =$$

$=$ path integral (Feynman–Kac formula / Feynman path integral)

Some recent results: $F(t)$ are constructed: $F(t) \sim e^{tL}$ on X, where

- the operator L generates a Feller process (B., Schilling, Smolyanov 2012)

- $L\varphi(x) := \operatorname{tr}(A(x)\mathrm{Hess}\varphi(x)) + b(x) \cdot \nabla\varphi(x) + c(x)\varphi(x)$

  in $C_\infty(\mathbb{R}^d)$ and in $L^p(\mathbb{R}^d)$, $p \in [1, \infty]$
  (B., Grothaus, Smolyanov 2010, 2016; Plyashechnik 2013)

- the same $L+$ Dirichlet boundary conditions
  (B., Grothaus, Smolyanov 2010)

- $L\varphi(x) := a(x)\Delta_K\varphi(x) + b(x) \cdot \nabla_K\varphi(x) + c(x)\varphi(x)$

  in $C_b(K)$, K is a compact Riemannian manifold
  (Butko 2016, 2008 via Smolyanov, Weizsäcker, Wittich 2007)

  in $C_\infty(K)$, K is a star graph
  (Butko 2015 via Kostrykin, Potthoff, Schrader 2012)

3. SUBORDINATE SEMIGROUPS

To construct a subordinate semigroup/process, one needs:

(1): Original (parent) C_0 contraction semigroup $(T_t \equiv e^{tL})_{t\geq 0}$ on a BS X / Markov process $(X_t)_{t\geq 0}$.

(2): Subordinator, i.e. $(\xi_t)_{t\geq 0} \Longleftrightarrow (\eta_t)_{t\geq 0} \Longleftrightarrow f \Longleftrightarrow (\sigma, \lambda, \mu)$, where

- $(\xi_t)_{t\geq 0}$ is a Lévy-process with a.s. non-decreasing paths;
- $(\eta_t)_{t\geq 0}$ is such that $\mathbb{P}(\xi_t \in A) = \eta_t(A)$; it is a convolution semigroup supported by $[0,\infty)$, i.e. η_t are Borel measures on $[0,\infty)$, $\eta_t([0,\infty)) \leq 1$, $\eta_t * \eta_s = \eta_{t+s}$ and $\eta_t \rightharpoonup \delta_0$ as $t \to 0$;
- f is a Bernstein function, such that $\mathcal{L}[\eta_t] = e^{-tf}$;
- (σ, λ, μ) are Lévy characteristics of f, i.e. $\sigma, \lambda \geq 0$, μ is a Radon measure on $(0,\infty)$ with $\int_{0+}^{\infty} \frac{s}{1+s}\mu(ds)$ such that

$$f(z) = \sigma + \lambda z + \int_{0+}^{\infty} (1 - e^{-sz})\mu(ds), \quad \forall z : \mathrm{Re}\, z \geq 0.$$

Def: The family $(T^f_t)_{t\geq 0}$ defined on the Banach space X by

$$T^f_t \varphi := \int_0^\infty T_s\varphi\, \eta_t(ds), \quad \forall\, \varphi \in X,$$

is called subordinate to $(T_t)_{t\geq 0}$ w.r.t. $(\eta_t)_{t\geq 0}$.

• $(T^f_t)_{t\geq 0}$ is a C_0 contraction semigroup corresponding to the subordinate process $(X_{\xi_t})_{t\geq 0}$.

• Let $(L^f, \mathrm{Dom}(L^f))$ be the generator of $(T^f_t)_{t\geq 0}$, then $\mathrm{Dom}(L)$ is a core for L^f and

$$L^f\varphi = -\sigma\varphi + \lambda L\varphi + \int_{0+}^\infty (T_s\varphi - \varphi)\mu(ds) \equiv -f(-L)\varphi, \quad \forall\, \varphi \in \mathrm{Dom}(L),$$

where $(L, \mathrm{Dom}(L))$ generates the parent semigroup $(T_t)_{t\geq 0}$.

4. CHERNOFF APPROXIMATION OF SUBORDINATE SEMIGROUPS

Let the parent semigroup $(T_t)_{t\geq 0}$ be unknown explicitly. Then:

$$T_t^f\varphi := \int_0^\infty T_s\varphi\,\eta_t(ds) \quad\Longrightarrow\quad \text{unknown}$$

and

$$L^f\varphi = -\sigma\varphi + \lambda L\varphi + \int_{0+}^\infty (T_s\varphi - \varphi)\mu(ds) \quad\Longrightarrow\quad \text{unknown}$$

Task: To find $\mathcal{F}(t) \sim T_t^f$ if $F(t) \sim T_t$ is given.

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Therefore,

$$T_t^f \equiv e^{tL^f} = e^{t(L_2+L_1+L_0)}$$

with

$L_2 : L_2\varphi = -\sigma\varphi \Longrightarrow e^{tL_2}\varphi = e^{-t\sigma}\varphi,$

$L_1 : L_1\varphi = \lambda L\varphi \Longrightarrow e^{tL_1}\varphi = e^{t(\lambda L)}\varphi = e^{(t\lambda)L}\varphi \equiv T_{t\lambda}\varphi \Longrightarrow F(t\lambda) \sim e^{tL_1},$

$L_0 : L_0\varphi = \int_{0+}^{\infty}(T_s\varphi - \varphi)\mu(ds) = L^{f_0}$, where $f_0 \Longleftrightarrow (0, 0, \mu) \Longleftrightarrow (\eta_t^0)_{t\geq 0}$.

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Assume $\mathcal{F}_0(t) \sim e^{tL_0}$ is constructed. Then

$$\mathcal{F}(t) := e^{-t\sigma} \circ F(t\lambda) \circ \mathcal{F}_0(t) \sim T_t^f.$$

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But how to construct $\mathcal{F}_0(t)$?

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Note, $\widetilde{\mathcal{F}}_0(t) := \int_0^{\infty} F(s)\eta_t^0(ds) \nsim e^{tL_0} = \int_0^{\infty} T_s\eta_t^0(ds)$ since $\widetilde{\mathcal{F}}_0'(0) \neq L_0$.

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Thm: Let $m:(0,\infty)\to\mathbb{N}_0$ be monotone and $m(t)\to\infty$ as $t\to 0$, e.g., $m(t):=[1/t]$. Then $\mathcal{F}_0(t)\sim e^{tL_0}$, where

$$\mathcal{F}_0(t)\varphi := \int_0^\infty \left[F(s/m(t))\right]^{m(t)}\varphi\,\eta_t^0(ds), \quad \varphi\in X.$$

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Then $L_0:\ L_0\varphi=\int_{0+}^\infty (T_s\varphi-\varphi)\mu(ds)$ is bounded $\Longrightarrow$ $\mathrm{Id}+tL_0\sim e^{tL_0}$.

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Then $L_0:\ L_0\varphi=\int_{0+}^\infty (T_s\varphi-\varphi)\mu(ds)$ is bounded $\Longrightarrow \mathrm{Id}+tL_0\sim e^{tL_0}$.

Thm: Let $m:(0,\infty)\to\mathbb{N}_0$ be monotone and $m(t)\to\infty$ as $t\to 0$, e.g., $m(t):=[1/t]$. Then $\mathcal{F}_0^\mu(t)\sim e^{tL_0}$, where

$$\mathcal{F}_0^\mu(t)\varphi := \varphi + t\int_{0+}^\infty \left([F(s/m(t))]^{m(t)}\varphi-\varphi\right)\mu(ds), \quad \varphi\in X.$$

Example: $L: L\varphi(x) = \frac{1}{2}\mathrm{tr}(A(x)\mathrm{Hess}\varphi(x)) + B(x)\cdot\nabla\varphi(x) - C(x)\varphi(x)$.
In case (a):

$$\mathcal{F}(t)\varphi(q) := e^{-t\sigma}\int\limits_{0+}^{\infty}\int_{\mathbb{R}^{d(m(t)+1)}} e^{-t\lambda C(q_{m(t)+2}) - \frac{s}{m(t)}\sum\limits_{k=1}^{m(t)} C(q_{k+1})}\times$$

$$\times\, e^{-\sum\limits_{k=1}^{m(t)+1} A^{-1}(q_{k+1})B(q_{k+1})\cdot(q_{k+1}-q_k)} e^{-\frac{s}{m(t)}\sum\limits_{k=1}^{m(t)} A^{-1}(q_{k+1})B(q_{k+1})\cdot B(q_{k+1})}\times$$

$$\times\, e^{-\frac{t\lambda}{2}A^{-1}(q_{m(t)+2})B(q_{m(t)+2})\cdot B(q_{m(t)+2})}\varphi(q_1)\times$$

$$\times\left[p_A(t\lambda, q_{m(t)+1}, q_{m(t)+2})\prod_{k=1}^{m(t)} p_A(s/m(t), q_k, q_{k+1})\right]\prod_{k=1}^{m(t)+1} dq_k\eta_t^0(ds)$$

$q_{m(t)+2} := q$ and $p_A(t,x,y) := \frac{1}{\sqrt{\det A(x)(2\pi t)^d}}\exp\left(-\frac{A^{-1}(x)(x-y)\cdot(x-y)}{2t}\right)$;

In case (b):

$$\mathcal{F}^{\mu}(t)\varphi(q) :=$$

$$= \frac{\exp(-t(\sigma+\lambda C(q)))}{\sqrt{\det A(q)(2\pi t\lambda)^d}} \int\limits_{\mathbb{R}^d} e^{\frac{-A^{-1}(q)(q-q_{m(t)+1}+t\lambda B(q))\cdot(q-q_{m(t)+1}+t\lambda B(q))}{2t\lambda}} \times$$

$$\times \left(\varphi(q_{m(t)+1}) + t \int\limits_{0+}^{\infty} \left[\int\limits_{\mathbb{R}^{dm(t)}} e^{-\frac{s}{m(t)}\sum_{k=1}^{m(t)} C(q_{k+1})} \prod_{k=1}^{m(t)} \left(\det A(q_{k+1})(2\pi t\lambda)^d \right)^{-1/2} \right.\right.$$

$$\times e^{-\sum_{k=1}^{m(t)} \frac{A^{-1}(q_{k+1})\left(q_{k+1}-q_k+\frac{s}{m(t)}B(q_{k+1})\right)\cdot\left(q_{k+1}-q_k+\frac{s}{m(t)}B(q_{k+1})\right)}{2s/m(t)}}$$

$$\left.\left. \times \varphi(q_1)dq_1\ldots dq_{m(t)} - \varphi(q_{m(t)+1}) \right] \mu(ds) \right) dq_{m(t)+1}$$

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