On Fractional Nonlinear Schrödinger Equation in Sobolev Spaces and related problems

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Fractional NLS

We consider the Cauchy problem for the fractional NLS

$$i\partial_t u - (-\Delta)^\sigma u \pm |u|^{p-1}u = 0,\ \ u(0) = u_0 \in H^s(\mathbb{R}^d),$$

where $\sigma \in (0,1)$ with $\sigma \neq \frac{1}{2}$ and $(-\Delta)^\sigma$ is a Fourier multiplier of $|\xi|^{2\sigma}$.

- fNLS is the simplest non-local NLS type equation.
- Ionescu and Pusateri introduced 1d cubic fNLS with $\sigma = \frac{1}{4}$ as a model equation for the water wave equation.
- Models in turbulence (Majda-McLaughlin-Tabak)
- Fractional Quantum mechanics (Laskin)

Dispersive and Strichartz estimates for the linear propagator $e^{it(-\Delta)^\sigma}$

In general, there is loss of regularity in the dispersive estimate

$$\|e^{it(-\Delta)^\sigma} f\|_{L^\infty} \lesssim \frac{1}{|t|^{d/2}} \||\nabla|^{d(1-\sigma)} f\|_{L^1}$$

as well as in Strichartz estimates

$$\|e^{-it(-\Delta)^\sigma} f\|_{L_t^q L_x^r} \lesssim \||\nabla|^{d(1-\sigma)(\frac{1}{2}-\frac{1}{r})} f\|_{L^2},$$

$$\left\| \int_0^t e^{-i(t-s)(-\Delta)^\sigma} F(s) ds \right\|_{L_t^q L_x^r} \lesssim \||\nabla|^{d(1-\sigma)(1-\frac{1}{r}-\frac{1}{\tilde{r}})} F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

where (q, r) and $(\tilde{q}, \tilde{r})$ are admissible pairs, i.e., $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $2 \leq q, r \leq \infty$, $(q, r, d) \neq (\infty, 2, 2)$.

Local well-posedness

Question: Can we prove local well-posedness by a standard contraction mapping argument?

Local well-posedness: Technical difficulties

Define the nonlinear mapping Φ_{u_0} by

$$\Phi_{u_0}(u) = e^{-it(-\Delta)^\sigma} u_0 \pm i \int_0^t e^{-i(t-s)(-\Delta)^\sigma} (|u|^{p-1}u)(s) ds.$$

Applying Strichartz estimates,

$$\begin{aligned} \|\Phi_{u_0}(u)\|_{L_t^q W_x^{s,r}} \lesssim & \, \||\nabla|^{d(1-\sigma)(\frac{1}{2}-\frac{1}{r})} u_0\|_{H^s} \\ & + \||\nabla|^{d(1-\sigma)(1-\frac{1}{r}-\frac{1}{\tilde{r}})} (|u|^{p-1}u)\|_{L_t^{\tilde{q}'} W_x^{s,\tilde{r}'}}. \end{aligned}$$

- Due to loss of regularity, the right hand side has higher order derivatives unless $r = \tilde{r} = 2$.
- After applying Strichartz estimates, fNLS looks like the derivative NLS $i\partial_t u + \Delta u + (|u|^{p-1}u)_x = 0$.

Local well-posedness: Previous results

- Guo-Wang ('10) proved that when $d \geq 2$ and $\sigma > \frac{d}{2d-1}$, if u_0 and F are radially symmetric, there is no loss of regularity in Strichartz estimates. As a consequence, the authors obtained LWP.
- Cho-Hwang-Kwon-Lee ('13) proved that if $\sigma > \frac{1}{2}$ and $s > \frac{1-\sigma}{2}$, then 1d cubic fNLS is locally well-posed in H^s. The authors employed the $X^{s,b}$ analysis. The same proof works on $\mathbb{T}$.

Local well-posedness

Question: Are Strichartz estimates with loss of regularity useless? Can we still use them to prove LWP?

Local well-posedness

Theorem (Hong and S.)

Let $s_c = \frac{d}{2} - \frac{2\sigma}{p-1}$ *and* $s_g = \frac{1-\sigma}{2}$.
(i) (subcritical case) Suppose that

- $s \geq s_g,\ d = 1$ and $2 \leq p \leq 5$;
- $s > s_c,\ d = 1$ and $p > 5$;
- $s > s_c,\ d \geq 2$ and $p \geq 3$.

Then, fNLS is locally well-posed in H^s.
(ii) (critical case) Suppose that

- $s = s_c,\ d = 1$ and $p > 5$;
- $s = s_c,\ d \geq 2$ and $p > 3$.

Then, fNLS is locally well-posed in H^s.

Local well-posedness

For numerical simplicity, let's fix $d = 3$. Then, $s_c = \frac{3}{2} - \frac{2\sigma}{p-1}$.

Theorem (Hong and S.)

If $p \geq 3$, then fNLS is locally well-posed in H^s for $s > s_c$.

Theorem (Hong and S.)

If $p > 3$, then the fNLS is locally well-posed in H^{s_c}.

LWP: Idea of the Proof (subcritical case)

For further simplification, let's fix $p = 3$. Then $s_c = \frac{3}{2} - \sigma$.

1. For LWP, due to loss of regularity, we are forced to apply Strichartz estimates with $r = \tilde{r} = 2$:

$$\|\Phi_{u_0}(u)\|_{L^\infty_{t\in[0,T)} H^s_x} \lesssim \|u_0\|_{H^s} + \||u|^2 u\|_{L^1_{t\in[0,T)} H^s_x}.$$

Then, by the fractional Leibniz rule,

$$\||u|^{p-1} u\|_{L^1_{t\in[0,T)} H^s_x} \lesssim \|u\|^2_{L^2_{t\in[0,T)} L^\infty_x} \|u\|_{L^\infty_{t\in[0,T)} H^s_x}.$$

Since the contraction mapping argument is perturbative, if we can bound $\|e^{-it(-\Delta)^\sigma} u_0\|_{L^2_{t\in[0,T)} L^\infty_x}$ by $\|u_0\|_{H^s}$, then we will be able to close the argument.

Question: What is the best way?

LWP: Idea of the Proof (subcritical case)

2. Although there is loss of regularity, dispersive and Strichartz estimates are smoothing estimates.

- By Sobolev inequality (formally ignoring failures of endpoint cases),

$$\|e^{-it(-\Delta)^\sigma} f\|_{L^\infty} \lesssim \||\nabla|^{\frac{3}{2}} e^{-it(-\Delta)^\sigma} f\|_{L^2} = \||\nabla|^{\frac{3}{2}} f\|_{L^2} \lesssim \||\nabla|^3 f\|_{L^1}.$$

- By the dispersive estimate,

$$\|e^{-it(-\Delta)^\sigma} f\|_{L^\infty} \lesssim \frac{1}{|t|^{3/2}} \||\nabla|^{3(1-\sigma)} f\|_{L^1}.$$

Thus, in principle, we should take full advantage of Strichartz estimates and avoid using Sobolev inequalities if possible.

To this end, we will use the endpoint Strichartz estimate

$$\|e^{-it(-\Delta)^\sigma} u_0\|_{L_t^2 W_x^{s,6}} \lesssim \||\nabla|^{1-\sigma} u_0\|_{H^s}$$

LWP: Idea of the Proof (subcritical case)

By Sobolev inequality and the endpoint Strichartz estimate, if $s > s_c = \frac{3}{2} - \sigma$, then

$$\|e^{-it(-\Delta)^\sigma} u_0\|_{L_t^2 L_x^\infty} \lesssim \|e^{-it(-\Delta)^\sigma} u_0\|_{L_t^2 W_x^{\frac{1}{2}+,6}} \lesssim \||\nabla|^{1-\sigma} u_0\|_{H^{\frac{1}{2}+}} \leq \|u_0\|_{H^s}.$$

One can easily prove that Φ_{u_0} is contraction in a ball in $L^\infty_{t\in[0,T)} H^s_x \cap L^2_{t\in[0,T)} W_x^{\frac{1}{2}+,6}$.

LWP: Idea of the Proof (critical case)

In the scaling-critical case $s = s_c$, the previous approach doesn't work directly due to the failure of the endpoint Sobolev inequality $\dot{W}^{\frac{1}{2},6} \not\subset L^\infty$.

To overcome this endpoint issue, we make a use of:
- a "slightly" improved Strichartz estimates

$$\Big(\sum_{N \in 2^{\mathbb{Z}}} \|e^{-it(-\Delta)^\sigma} P_N f\|_{L_t^q L_x^r}^2 \Big)^{1/2} \lesssim \| |\nabla|^{3(1-\sigma)(\frac{1}{2}-\frac{1}{r})} f\|_{L^2},$$

$$\Big(\sum_{N \in 2^{\mathbb{Z}}} \Big\| \int_0^t e^{-i(t-s)(-\Delta)^\sigma} P_N F(s) ds \Big\|_{L_t^q L_x^r}^2 \Big)^{1/2} \lesssim \| |\nabla|^{3(1-\sigma)(\frac{1}{2}-\frac{1}{r})} F\|_{L_t^1 L_x^2},$$

where P_N is the Littlewood-Paley projection.
- an interpolation inequality: For $p > 3$, we have

$$\|u\|_{L_{t\in[0,T)}^{p-1} L_x^\infty}^{p-1} \lesssim \Big(\sum_{N \in 2^{\mathbb{Z}}} \|P_N u\|_{L_{t\in[0,T)}^2 \dot{W}_x^{\frac{1}{2},6}}^2 \Big) \cdot \|u\|_{L_{t\in[0,T)}^\infty \dot{H}_x^{\frac{3}{2}-\frac{2\sigma}{p-1}}}^{p-3},$$

which generalizes Lemma 3.1 in [CKSTT, Annals].

Ill-posedness

The focusing or defocusing 1d cubic NLS is locally ill-posed in H^s for $s < 0$ (Kenig-Ponce-Vega, Christ-Colliander-Tao). Note that it includes ill-posedness in a subcritical regime, since $s_c = -\frac{1}{2}$ is the critical regularity.

The main tool to show ill-posedness of dispersive PDEs is the Galilean transformations, Lorentzian transformations etc... However, due to non-locality, fNLS doesn't have such a symmetry.

Question: Can we still prove ill-posedness, in particular in a subcritical space?

Ill-posedness

Theorem (Hong and S.)

Let $d = 1, 2, 3$ and $\sigma \in (\frac{d}{4}, 1)$. If p is not an odd integer, we further assume that $p \geq k + 1$, where k is an integer larger than $\frac{d}{2}$. Then, fNLS is ill-posed in H^s for $s \in (s_c, 0)$.

Ill-posedness: Idea of the proof

Observation 1.
Motivated by the Galilean transformation, we introduce the pseudo-Galilean transform

$$(\mathcal{G}_v u)(t,x) = e^{-it|v|^{2\sigma}} e^{iv\cdot x} u(x - 2t\sigma|v|^{2\sigma-2}v), \quad v \in \mathbb{R}^d.$$

Then, $\tilde{u} = \mathcal{G}_v u$ solves

$$i\partial_t \tilde{u} - (-\Delta)^\sigma \tilde{u} \pm |\tilde{u}|^2 \tilde{u} = e^{-it|v|^{2\sigma}} e^{iv\cdot x} (\mathcal{E} u)(t, x - 2\sigma t|v|^{2\sigma-2}v),$$

where $\mathcal{E}$ is a Fourier multiplier of

$$E(\xi) = |\xi + v|^{2\sigma} - |\xi|^{2\sigma} - |v|^{2\sigma} - 2\sigma|v|^{2\sigma-2} v\cdot\xi.$$

One can check that $|E(\xi)| \lesssim |\xi|^{2\sigma}$. Thus, the error term on the right hand side is small if $\| |\xi - v|^\sigma \hat{u}(\xi)\|_{L^2_\xi}$ is small.

Ill-posedness: Idea of the proof

Observation 2.
Fortunately, the counterexample to show ill-posedness in Christ-Colliander-Tao (slowly dispersive solutions) are such solutions. We can adapt their construction for fNLS. However, the condition $\sigma > \frac{d}{4}$ is required.

Open Problems

1. LWP is not known for sub-cubic nonlinearities. $X^{s,b}$ analysis? Probabilistic approach?

2. There is a huge gap between ill-posedness and LWP. In particular, we expect that in 1d, fNLS is ill-posed in H^s for $0 < s < \frac{1-\sigma}{2}$. However, the currently known counterexamples don't seem to work.

Some new solitons

$$i\partial_t u - (-\Delta)^\sigma u + |u|^2 u = 0,$$

Static soliton for the cubic NLS $\sigma = 1$: $e^{it\omega^2} Q_\omega(x)$ with $\omega > 0$, where $Q_\omega(x) = \sqrt{2}\omega sech(\omega x)$ is the ground state for the nonlinear elliptic equation

$$-\Delta u + \omega^2 u - |u|^2 u = 0.$$

Then, since the equation is invariant under the Galilean transformation

$$u(t,x) \mapsto e^{-it|k|^2} e^{ik\cdot x} u(t, x - 2tk), \quad k \in \mathbb{R},$$

boosting a static solution, we obtain a traveling soliton

$$e^{-it(|k|^2 - \omega^2)} e^{ik\cdot x} Q_\omega(x - 2tk)$$

with a velocity of $2k$. The fractional NLS is *almost invariant* under the pseudo-Galilean transformation

$$\mathcal{G}_k : u(t,x) \mapsto e^{-it|k|^{2\sigma}} e^{ik\cdot x} u(t, x - 2t\sigma|k|^{2(\sigma-1)} k) \tag{1}$$

for smooth solutions.

Therefore, it is still natural to consider an ansatz of the form

$$u_{\omega,k}(t,x) = e^{-it(|k|^{2\sigma}-\omega^{2\sigma})} e^{ik\cdot x} Q_{\omega,k}(x - 2t\sigma|k|^{2\sigma-2}k), \tag{2}$$

which will lead to a natural family of moving solitons with frequency ω and speed k. The profile $Q_{\omega,k}$ then solves the pseudo-differential equation

$$\mathcal{P}_k Q_{\omega,k} + \omega^{2\sigma} Q_{\omega,k} - |Q_{\omega,k}|^2 Q_{\omega,k} = 0, \tag{3}$$

where

$$\mathcal{P}_k = e^{-ik\,x}(-\Delta)^{\sigma} e^{ik\,x} - |k|^{2\sigma} + 2i\sigma|k|^{2\sigma-2} k \cdot \nabla_x.$$

Theorem (Hong and S.)

Let $\sigma \in (\frac{1}{2}, 1)$. For any $k \in \mathbb{R}$, there exists $Q_{\omega,k} \in H^1(\mathbb{R})$ solving (3) for some $\omega > 0$. Furthermore, we have $Q_{\omega,k} \in C^\infty(\mathbb{R})$.

The key observation to prove the theorem is that the pseudo-differential operator $\mathcal{P}_k$ is an elliptic operator. Indeed, $\mathcal{P}_k$ is a Fourier multiplier, $\widehat{\mathcal{P}_k f}(\xi) = p_k(\xi)\hat{f}(\xi)$, with the symbol

$$p_k(\xi) = |\xi + k|^{2\sigma} - |k|^{2\sigma} - 2\sigma|k|^{2\sigma-2}k \cdot \xi.$$

So, we have $p_k(0) = 0$. Differentiating $p_k(\xi)$, we get or $k \neq 0$

$$\begin{aligned} p_k'(\xi) &= 2\sigma|\xi + k|^{2\sigma-2}(\xi + k) - 2\sigma|k|^{2\sigma-2}k && \Rightarrow \quad p_k'(0) = 0, \\ p_k''(\xi) &= 2\sigma(2\sigma - 1)|\xi + k|^{2\sigma-2} && \Rightarrow \quad p_k''(\xi) \geq 0. \end{aligned} \tag{4}$$

Thus, $p_k(\xi)$ is non-negative for all ξ for $\sigma > 1/2$.

$$\begin{aligned} p_k(\xi) &= |\xi|^{2\sigma} + O(|\xi|^{2\sigma-1}) && \text{as} \quad \xi \to \infty, \\ p_k(\xi) &= \sigma(2\sigma - 1)|k|^{2\sigma-2}|\xi|^2 + O(|\xi|^3) && \text{as} \quad \xi \to 0. \end{aligned} \tag{5}$$

Therefore, the operator $\mathcal{P}_k$ behaves like $(-\Delta)^\sigma$ in high frequencies, and it behaves like $\sigma(2\sigma - 1)|k|^{2\sigma-2}(-\Delta)$ in low frequencies.

Theorem (Gagliardo-Nirenberg inequality for $\mathcal{P}_1$)

For $\theta \in (0,1)$, we have

$$\|u\|_{L^4}^4 \lesssim \|u\|_{L^2}^{\frac{4\sigma-1}{\sigma}} \|\mathcal{P}_1^{1/2} u\|_{L^2}^{\frac{1}{\sigma}} + \alpha \|u\|_{L^2}^3 \|\mathcal{P}_1^{1/2} u\|_{L^2} - \left\{ \|u\|_{L^2}^{\frac{4\sigma-1}{\sigma}} \|\mathcal{P}_1^{1/2} u\|_{L^2}^{\frac{1}{\sigma}} \right\}^{1-\theta} \left\{ \alpha \|u\|_{L^2}^3 \|\mathcal{P}_1^{1/2} u\|_{L^2} \right\}^{\theta},$$

where $\alpha = \alpha(\sigma) = 1/\sqrt{\sigma(2\sigma-1)}$.

Weinstein Functional

$$\mathcal{W}(u) := W_1(u) + W_2(u) - W_3(u),$$

where

$$W_1(u) := \frac{\|u\|_{L^2}^{\frac{4\sigma-1}{\sigma}} \|\mathcal{P}_1^{1/2} u\|_{L^2}^{\frac{1}{\sigma}}}{\|u\|_{L^4}^4}, \quad W_2(u) := \alpha \frac{\|u\|_{L^2}^3 \|\mathcal{P}_1^{1/2} u\|_{L^2}}{\|u\|_{L^4}^4},$$

and

$$W_3(u) = W_1(u)^{1-\theta} W_2(u)^{\theta}.$$

Minimization problem:

$$\frac{1}{c_{GN}} = \inf_{u \in H^1} \mathcal{W}(u).$$

Tools:

- Profile decomposition
- Argument by contradiction (use of W_3)
- Pohozaev identities
- Use of the minimality properties of well-known ground states

Thank you