- Let $\alpha \in (0,2)$, $\Omega \subset \mathbb{R}^d$ be a bounded, connected open set and let us consider the following Dirichlet eigenvalue problem

$$\begin{cases} (-\Delta)^{\alpha/2}\varphi_k(x) = \lambda_k \varphi_k(x), & \text{for} \quad x \in \Omega, \\ \varphi_k(x) = 0, & \text{for} \quad x \in \Omega^c. \end{cases}$$

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- **1. Is this true that φ_k has at most k nodal domains?**
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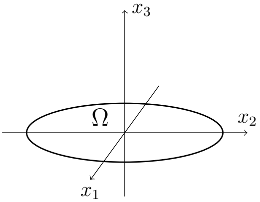
- **Open problems:**
- **1. Is this true that φ_k has at most k nodal domains?** A nodal domain of φ_k - a maximal connected open subset of $\{x \in \Omega : \varphi_k(x) \neq 0\}$.
- 2. **Is this true that for an arbitrary open bounded convex set Ω superlevel sets of φ_1 are convex?** (R. Bañuelos) A superlevel set of φ_1 - $\{x \in \Omega : \varphi_1(x) \geqslant c\}$.

The special case $\alpha = 1$, $d = 2$.

- Let u_k be **the harmonic extension** of φ_k to $\mathbb{R}^3_+$, $u_k(x_1, x_2, 0) = \varphi_k(x_1, x_2)$.



$$K(x_1, x_2, x_3) = \frac{1}{2\pi} \frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}},$$

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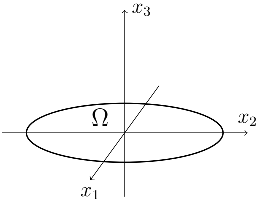

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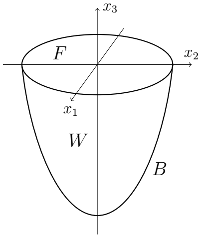
- **The Steklov-Dirichlet problem**

$$\begin{cases} \Delta u_k(x) = 0, \quad \text{for} \quad x \in \mathbb{R}^3_+, \\ u_k(x) = 0, \quad \text{for} \quad x \in \Omega^c \times \{0\}, \\ \dfrac{\partial u_k}{\partial x_3}(x) = -\lambda_k u_k(x), \quad \text{for} \quad x \in \Omega \times \{0\}. \end{cases}$$

- **The Steklov-Neumann problem.** Let $W \subset \mathbb{R}^3$ be a bounded Lipschitz domain.



$$\begin{cases} \Delta u_k(x) = 0, & x \in W, \\ \dfrac{\partial u_k}{\partial x_3}(x) = \nu_k u_k(x), & x \in F, \\ \dfrac{\partial u_k}{\partial \vec{n}}(x) = 0, & x \in B. \end{cases}$$

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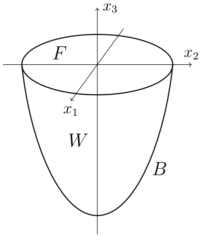

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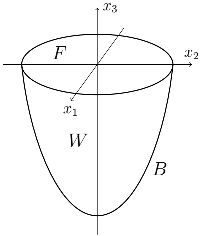

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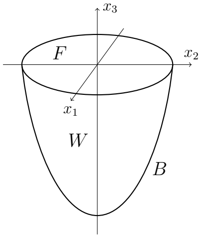

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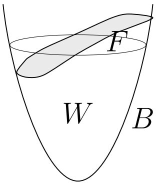
- $A\varphi_k(x) = \nu_k \varphi_k(x)$, $x \in F$,
  A - the Dirichlet to Neumann operator.

- If liquid oscillates freely according to an eigenoscillation φ_k then at every moment the free-surface elevation of liquid is proportional to φ_k.



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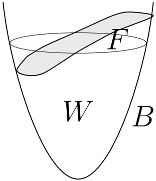
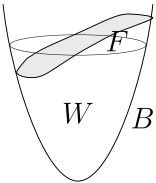


- **Open problem:**

- If liquid oscillates freely according to an eigenoscillation φ_k then at every moment the free-surface elevation of liquid is proportional to φ_k.



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- **3. Is this true that φ_k has at most $k+1$ nodal domains?**

Low eigenvalues of fractional Dirichlet Laplacian ($0 < \alpha < 2$)

Bounded domain $\Omega \subset \mathbb{R}^2$, area A, $\boxed{(-\Delta)^{\alpha/2}\varphi_k = \lambda_k \varphi_k}$

First and second eigenvalue (Rayleigh–Faber–Krahn)

$\lambda_1 A^{\alpha/2}$ minimal for disk (Bañuelos 2001)
$\lambda_2 A^{\alpha/2}$ minimal for two disks infinitely far apart (Brasco–Parini 2016)

Ratio of eigenvalues (PPW) — universal inequality

λ_2/λ_1 maximal for disk?

[conjecture Bañuelos–Kulczycki 2006]

Or, *non-sharp* bound $\lambda_2/\lambda_1 \leq$ const. by simple trial functions?
Other universal inequalities of Hongcang Yang type?

Pólya and Berezin–Li–Yau problems

$\lambda_k > (C_{\mathrm{Weyl}}\, k/A)^{\alpha/2}$?

[False when $d = 1$ by KKMS asymptotic on interval. (Counter-e.g. in $d = 2$??)
Must not contradict Laptev 1997: $\sum_1^n \lambda_k > (C_{\mathrm{Weyl}}/A)^{\alpha/2} n^{1+\alpha/2}/(1 + \alpha/2)$.]

Bounded domain $\Omega \subset \mathbb{R}^2$, diameter D,

$$(-\Delta)^{\alpha/2}\varphi_k = \lambda_k \varphi_k$$

Gap problem (van den Berg–Andrews–Clutterbuck)

$(\lambda_2 - \lambda_1)D^\alpha$ minimal for degenerate rectangle?

[$\alpha = 1$: rough bound by Bañuelos–Kulczycki 2006]

Second eigenvalue

$\lambda_2 D^\alpha$ minimal for disk among general domains?
minimal for square among rectangles? for equilateral among triangles?

[$\alpha = 2$: general conjecture Buçur–Buttazzo–Henrot 2009;
proof for triangles Laugesen–Siudeja 2011]

Eigenvalue sums

$(\lambda_1 + \cdots + \lambda_k)D^\alpha$ minimal for disk? for equilateral among triangles?

[$\alpha = 2$: open for general domains, proof for triangles by Laugesen–Siudeja 2011]

Dirichlet heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$

$$p_t(x,y) \approx t^{-d/\alpha} \wedge \frac{t}{|y-x|^{d+\alpha}} \quad \text{on } \mathbb{R}^d$$

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Regional fractional Laplacian: $\Delta_\Omega^{\alpha/2}$ in bounded Lipschitz Ω

$$\Delta_\Omega^{\alpha/2} u(x) := P.V.\ c \int_\Omega [u(y) - u(x)] |y - x|^{-d-\alpha} dy\,.$$

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(Problem 2)

Prove a power-rate boundary decay of s.

3G vs 4G (and applications to resolving Duhamel's formula)

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Transition density $p(s,x,t,y) = p_{t-s}(y-x)$ of $\Delta^{\alpha/2}$ satisfies 3G:

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(Problem 3/4G)

Prove 4G for other α-stable subordinators, $0 < \alpha < 1$.

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