# Masterclass: Topological Quantum Groups and Hopf Algebras

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## Syllabus

*Algebraic quantum groups and groupoids* [Alfons Van Daele]

In this part, we will mainly give a short introductory treatment of the theory of algebraic quantum groups. They are a special type of locally compact quantum groups and can be studied in a purely algebraic context. Nevertheless, the theory is still very rich and serves as a good and complete model for understanding the algebraic aspects of the more involved and complete analytical theory of locally compact quantum groups.

1. *Duality for finite quantum groups.* We recall the definition of a finite-dimensional Hopf $^*$-algebra $A$ and of its dual $\widehat{A}$. The algebra $A$ acts on itself by left multiplication, while the algebra $\widehat{A}$ acts on $A$ by convolution. The two actions relate by the Heisenberg commutation relation. The duality gives rise to a unitary $W$ in $\widehat{A} \otimes A$. The Heisenberg commutation relations take the form of the pentagon equation for this unitary. We illustrate the formulas with a simple example.

2. *Multiplier Hopf $^*$-algebras.* The duality for finite-dimensional Hopf $^*$-algebras breaks down in the infinite-dimensional case. There is no way to construct the dual of a Hopf algebra if it is infinite-dimensional. However, by allowing algebras without identity and with an adapted notion of a coproduct, more is possible. In this lecture, we will treat multiplier Hopf algebras. They turn out to be the natural objects generalizing Hopf algebras to non-unital algebras. The motivating example comes from a (discrete) group $G$. The algebra $A$ is the algebra of complex functions on $G$ with finite support (and pointwise operations). The coproduct $\Delta$ on $A$ is defined by $\Delta(f)(p,q) = f(pq)$ for $p, q \in G$. It is a homomorphism from $K(G)$ to $C(G \otimes G)$, the algebra of all complex functions on $G \times G$. If $G$ is infinite, we do not have a Hopf algebra but a multiplier Hopf algebra.

3. *Algebraic quantum groups and duality.* In this lecture, we treat multiplier Hopf algebras with integrals (called algebraic quantum groups). Integrals are non-zero invariant linear functionals (like the integral over the Haar measures on a locally

compact group). It turns out that for such objects, there is a nice duality in the sense that the dual of a multiplier Hopf algebra with integrals is again a multiplier Hopf algebra with integrals. The duality of finite-dimensional Hopf algebras is a special case. Indeed, any finite-dimensional Hopf algebra is a multiplier Hopf algebra and integrals automatically exist. The theory of algebraic quantum groups and its dual is good model for the theory of locally compact quantum groups. Many of the algebraic features are present already. Moreover, they include the compact and discrete quantum groups and their duality.

4. *Towards operator algebraic quantum groups.* In this lecture, we treat the passage from algebraic quantum groups to operator algebraic quantum groups. For this we need to start with multiplier Hopf $^*$-algebras with positive integrals. It turns out that the dual in this case is again a multiplier Hopf $^*$-algebra with positive integrals. The Heisenberg commutation relations, as well as the multiplicative unitary (seen as the duality), as we considered it in the first lecture for finite-dimensional Hopf $^*$-algebras, is again considered, but now in this more general setting of $^*$-algebraic quantum groups (with positive integrals). All this allows us to pass to the Hilbert space level and associate an operator algebraic quantum group in this situation. Compact quantum groups and discrete quantum groups are special cases. One of the key results for understanding the operator algebra approach to quantum groups is the so-called Larson Sweedler theorem. Roughly it says that the existence of integrals implies the existence of an antipode. We give a treatment of this result in the case of multiplier Hopf algebras in this lecture also.

5. *Algebraic quantum groupoids.* In this final lecture of the series, we push the theory further into the direction of quantum groupoids. Also here, there is a line starting with finite quantum groupoids (or finite-dimensional weak Hopf algebras) and ending with the theory of measured quantum groupoids. The last one is even more involved than the theory of locally compact quantum groups. Moreover, a decent C$^*$-version is still not available. What we will present here is the intermediate step with weak multiplier Hop algebras with integrals, called here algebraic quantum groupoids. We will approach the theory with a version of the Larson Sweedler theorem, as we did in the previous lecture on operator algebraic quantum groups. The focus will lie on an example where the dual can be constructed explicitly and yields an interesting example of an algebraic quantum groupoid.

*Multiplicative-unitary approach to locally compact quantum groups* [S.L. Woronowicz]

In the last 25 years, multiplicative unitary operators proved to be one of the main tool in the theory of locally compact quantum groups. In this part, we will formulate the theory of manageable multiplicative unitaries and show how they produce quantum groups. We are going to present complete (and in most cases new) proofs. In the original version of this theory an important role was played by Hilbert Schmidt operators. Now we shall use derived pentagon equations that make the proofs much simpler. Next we plan to discuss more recent developments. These are homomorphisms of quantum groups, actions of quantum groups on C* algebras, crossed products and if time permits Landstad-Vaes theory.

1. *Manageable multiplicative unitaries and locally compact quantum groups.* We shall recall the concept of multiplicative unitary operators, manageable unitary operators and adapted unitary operators and formulate the main theorem describing the relation between manageable multiplicative unitaries and locally compact quantum groups. Next we shall discuss the general structure of pentagon equation introducing #-product of unitary operators acting on tensor product of Hilbert spaces. We shall prove that in certain cases #-product is associative.

2. *Derived pentagon equations.* It turns out that any operator adapted to a manageable unitary gives rise to three formulae having form of pentagon equations. We shall derive those formulae and use them to prove the main theorem.

3. *Duality in the theory of locally compact quantum groups.* Locally compact quantum groups appear in dual pairs. Duality is described by a bicharacter playing a fundamental role in the theory. To investigate relation between a group and its dual we use pairs of representations of the algebras of functions on the group and its dual acting on the same Hilbert space. In particular we have Heisenberg and anti-Heisenberg pairs leading to Heisenberg and anti-Heisenberg doubles. These are simplest examples of crossed products.

4. *Actions of quantum groups on C*-algebras. Crossed products.* We shall consider categories of C*-algebras equipped with a right (or left) action of a locally compact quantum group. We shall present Nest-Voigt approach to crossed products. Landstad-Vaes theorem will be presented.

5. *Homomorphisms of quantum groups.* We shall consider a category identified later with the category of locally compact quantum groups. Unitary operators acting on tensor products of Hilbert spaces will be morphisms of the category and #-product will be the composition of morphisms. Objects will be identified as manageable multiplicative unitary operators. We shall introduce universal version of algebras related to quantum groups and identify morphisms with C*-algebra homomorphisms acting between universal algebras.

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