NONCOMMUTATIVE TOPOLOGY

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(NOTES TEX'D BY KONRAD AGUILAR)

Abstract. These notes are for the Lecture Course *Noncommutative Topology for Beginners* that took place at IMPAN during the Fall 2016 Simons Semester: Noncommutative Geometry the Next Generation.

Contents

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1. 19 September 2016

The main focus of today is to begin to answer the questions: why is topology somehow related to C*-algebras and why is the study of C*-algebras sometimes called noncommutative topology?

To start, we consider a motivating example.

Example 1.1. Let X be a compact Hausdorff topological space and let $C(X) = \{f : X \longrightarrow \mathbb{C} : f \text{ is continuous}\}$ be set of complex valued continuous functions with respect to the given compact Hausdorff topology on X and the usual topology on $\mathbb{C}$. To motivate the definition of C*-algebra, we list some properties of the set $C(X)$.

(1) $C(X)$ is an algebra under point-wise operations. For example, $f + g$ for $f, g \in C(X)$ means the continuous function defined by $(f + g)(x) = f(x) + g(x)$.
(2) $C(X)$ has a norm $\|\cdot\|_{C(X)}$ defined by $\|f\|_{C(X)} = \sup_{x\in X} |f(x)|$ for all $f \in C(X)$.
(3) This norms behaves well with the algebra structure:
$\|f+g\|_{C(X)} \leqslant \|f\|_{C(X)} + \|g\|_{C(X)}$, which is already true by definition of norm, and
$\|fg\|_{C(X)} \leqslant \|f\|_{C(X)} \cdot \|g\|_{C(X)}$ for all $f, g \in C(X)$.
(4) The norm $\|\cdot\|_{C(X)}$ is complete.
(5) There is an involution $^* : C(X) \longrightarrow C(X)$ defined pointwise by $f^*(x) = \overline{f(x)}$.
(6) This involution has the following relationship with the norm.
$\|f^*f\|_{C(X)} = \|f\|_{C(X)}$ for all $f \in C(X)$.

We note that in the case that X is a locally compact Hausdorff space, we consider the set $C_0(X) = \{f \in C(X) : \forall\varepsilon > 0, \{x \in X : |f(x)| \geqslant \varepsilon\} \text{ is compact}\}$, which satisfies all of the above properties.

If an algebra $\mathfrak{A}$ (Complex and associative) has a norm that satisfies properties (3) and (4), then we call this a *Banach Algebra*. Now, we define a C*-algebra.

Definition 1.2. Let $\mathfrak{A}$ be a complex normed vector space with norm $\|\cdot\|_{\mathfrak{A}}$. If the norm $\|\cdot\|_{\mathfrak{A}}$ is complete, and

(1) $\mathfrak{A}$ is an algebra for which
$\|ab\|_{\mathfrak{A}} \leqslant \|a\|_{\mathfrak{A}} \cdot \|b\|_{\mathfrak{A}}$ for all $a, b \in \mathfrak{A}$, and
(2) there exists a map $^* : \mathfrak{A} \to \mathfrak{A}$ that is conjugate linear, anti-mutliplicative, and idempotent called the involution such that
$\|a^*a\|_{\mathfrak{A}} = \|a\|^2$ for all $a \in \mathfrak{A}$,

then $\mathfrak{A}$ is a C*-algebra. We say that a C*-algebra is unital if it contains a multiplicative unit.

We note that $C_0(X)$ is an example of a commutative C*-algebra, but also that there are no other examples of commutative C*-algebras. But, what do we mean by "no other examples." This motivates the following definition of isomorphism between C*-algebras.

Definition 1.3. Let $\mathfrak{A}, \mathfrak{B}$ be C*-algebras. $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a *-homomorphism if φ is a homomorphism of the algebra such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathfrak{A}$. (We note that we are not implying that $\mathfrak{A}$ and $\mathfrak{B}$ have the same involution. This is just common notation).

A *-isomorphism is a *-homomorphism that is a bijection.

The fact that there are no other commutative C*-algebras besides ones of the form $C_0(X)$ was due to Gelfand and Naimark.

Theorem 1.4 (Gelfand-Naimark Theorem)**.** *Every commutative C*-algebra is *-isomorphic to $C_0(X)$, for some locally compact Hausdorff space X.*

The next theorem establishes our next step in showing why the study of C*-algebras is sometimes called noncommutative topology.

Theorem 1.5. *Let X_1, X_2 be locally compact Hausdorff spaces.*

*$X_1 \sim X_2$ ($\sim$ means homeomorphic) if and only if $C_0(X_1) \cong C_0(X_2)$ ($\cong$ means *-isomorphic).*

Therefore, this theorem together with the Gelfand-Naimark theorem provide that the objects in the category of C*-algebras (with *-homorphisms) are in one-to-one correspondence with the objects in the category of locally compact Hausdorff spaces (with continuous maps) up to the relations of *-isomorphism and homeomorphism, respectively. Furthermore, one could establish a natural isomorphism between categories (this would require an extra subtle requirement).

Hence, the study of commutative C*-algebras can be seen as the study of topology. Therefore, general C*-algebras can be seen as the study of "noncommutative" topological spaces.

But, this also motivates whether we can generalize certain properties of topological spaces themselves into the noncommuative setting rather than just the entire topological spaces, which we have already done. Let's first look at some topological properties that are easy to generalize to the noncommutatvie setting. We will introduce further definitions as they are needed.

Easy properties:

Example 1.6 (Compactness of a topological space corresponds to unital C*-algebras). . X is a compact Hausdorff space $\iff$ $C_0(X) = C(X)$. But, also $C_0(X) = C(X)$ if and only if $C_0(X)$ has a constant function or if $C_0(X)$ is unital. Thus, a unital C*-algebra generalizes the notion of compactness.

We require the following definition for the next example.

Definition 1.7. Let $\mathfrak{A}$ be C*-algebra. $a \in \mathfrak{A}$ is a projection if $a^2 = a = a^*$.

Example 1.8 (Connectedness and compactness of a topological spaces corresponds to unital projectionless C*-algebras). Let X be a compact Hausdorff topological space. X is connected there exist no disjoint open sest $U_1, U_2 \subset X$ such that $X = U_1 \sqcup U_2$.

Suppose that X is disconnected. So, there exists U_1, U_2 open such that $X = U_1 \sqcup U_2$. Then, the following function

$$f(x) = \begin{cases} 1 & , x \in U_1 \\ 0 & , x \in U_2 \end{cases}$$

is continuous or $f \in C(X)$. We note that f is a projection, $f^2(x) = f(x) = f^*(x)$ for all $x \in X$.

For the other direction. Assume there exists a projection $p \in C(X)$ such that $p(x) \in \{0, 1\}$ and $p(x) = 0$ and $p(y) = 1$ for some $x, y \in X$. If we let $U_1 := p^{-1}(\{0\})$ and $U_2 = p^{-1}(\{1\})$, then $X = U_1 \sqcup U_2$.

Hence, by contraposition X is connected if and only if $C(X)$ is projectionless, which is a term used for the existence of no nontrivial projection.

Example 1.9 (Points in a topological space correspond to irreducible representations of a C*-algebra). Let $x_0 \in C_0(X)$. Define $\mathrm{ev}_{x_0} : f \in C_0(X) \longmapsto f(x_0) \in \mathbb{C}$, which is called the evaluation map at a point x_0. These are one-dimenstional representations of the C*-algebra $C_0(X)$ and are therefore irreducible representations.

Fact: The irreductible representations of $C_0(X)$ are the evaluations ev_x.

This establishes our relationship between the points and irreducible representations.

For the next example, which provides another take on the immediately above example, we need.

Definition 1.10. Let $\mathfrak{A}$ be a C*-algebra. $a \in \mathfrak{A}$ is positive ($a \geqslant 0$) if $a = b^*b$ for some $b \in \mathfrak{A}$.

Definition 1.11. Let $\mathfrak{A}$ be a C*-algebra. Let $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ be a linear functional.

φ is *positive* if $\varphi(a) \geqslant 0$ for all $a \in \mathfrak{A}$ such that $a \geqslant 0$.

φ is a *state* if it is positive and $1 = \|\varphi\| = \sup\{|\varphi(x)| : \|x\|_{\mathfrak{A}} \leqslant 1\}$.

φ is a pure state if is a state and is an extreme poin in the set of states of $\mathfrak{A}$.

Example 1.12 (Points in a topological space correspond to pure states on a C*-algebra). The evaluation maps of the above example ev_x form the set of pure states of $C_0(X)$. Consider the Riesz Representation theorem, which essentially states that there is a one-to-one correspondence between the linear functionals on $C_0(X)$ and Borel measures on X. In particular, for every linear functional φ there exists a unique Borel measure μ_φ on X such that $\varphi(f) = \int_X f d\mu_\varphi$ for all $f \in C_0(X)$.

If φ were a state, then μ_φ would be a positive probability measure.

If φ were a pure state, then μ_φ would be a Dirac point mass or all its mass would be concentrated at a single point x_0.

Next, we move on to a hard example which is a conjecture with a partial answer in the setting of nuclear C*-algebras.

Hard example:

Example 1.13 (A noncommutative Stone-Weierstraß conjecture). For $C([0,1])$, the Weierstraß approximation theorem states that any function can be approximated by polynomials in the norm of Example (1.1). M. H. Stone generalized to the setting of compact Hausdorff spaces in the following way.

Theorem 1.14. *Let X be a compact Hausdorff space. If $A_0 \subset C(X)$ is a subalgebra such that A_0*

- *separates points (if $x \neq y \in X$, then there exists $f \in A_0$ such that $f(x) \neq f(y)$)*
- *contains the constant functions*
- *is self-adjoint ($f \in A_0 \implies f^* \in A_0$)*

then $\overline{A_0}^{\|\cdot\|_{C(X)}} = C(X)$.

In our setting, this is: If $A_0 \subseteq C(X)$ is a unital C*-subalgebra that separates points, then $A_0 = C(X)$.

Now, by the previous example, we saw that points of X correspond to pure states of a C*-algebra. It is then natural to conjecture.

The noncommutative Stone-Stone-Weierstraß conjecture: Suppose $\mathfrak{B}$ is a C*-algebra and $\mathfrak{A} \subseteq \mathfrak{B}$ is a C*-subalgebra that separates the pure states of $\mathfrak{B}$ (if $\varphi \neq \psi$ pure states of $\mathfrak{B}$, then there exists $a \in \mathfrak{A}$ such that $\varphi(a) \neq \psi(a)$), then $\mathfrak{A} = \mathfrak{B}$.

This is still an open problem, which has been proven to be true in the case when $\mathfrak{B}$ is a nuclear C*-algebra.

The next step is to introduce the noncommutative generalization of K-theory for topological spaces. Informally, given a topolocical space X, K-theory for topological spaces associates an Abelian group $K^0(X)$ such that if $X \sim Y$ then the groups $K^0(X)$ and $K^0(Y)$ are isomorphic.

Thus, to generalize this, we will introduce the notion of K-theory for C*-algebras, in which given a C*-algebra $\mathfrak{A}$, we associate an Abelian group $K_0(\mathfrak{A})$. Furthermore, we will show that $K_0(C_0(X)) = K^0(X)$ to provide a suitable generalization.

The first goal of this course will be to move from topological K-theory to K-theory of C*-algebras.

2. 20 September 2016

The plan of the course is to cover 3 major topics. The first topic, which was mentioned at the end of the last lecture is

(1) From topological K-theory to K-theory of C*-algebras.

(2) Brown-Douglas-Fillmore Theory This theory began with the classification of essentially normal operators and turned into a problem about the classification of C*-algebras.

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space. Let $B_1(\mathcal{H})$ denote the closed unit ball of $\mathcal{H}$. Let $\mathfrak{K} = \left\{ T \in B(\mathcal{H}) : \overline{T(B_1(\mathcal{H}))}^{\|\cdot\|_{\mathcal{H}}} \text{ is compact} \right\}$ denote the compact operators.

An operator $N \in B(\mathcal{H})$ is *normal* if $[N, N^*] = 0$.

An operator $T \in B(\mathcal{H})$ is *essentatially normal* if $[T, T^*] \in \mathfrak{K}$.

So, the notion of essentially normal is saying that the commutator of T with T^* is "small."

Definition 2.2. Since $\mathfrak{K}$ is an ideal of $B(\mathcal{H})$, we call $B(\mathcal{H})/\mathfrak{K}$ the *Calkin algebra*, which is a C*-algebra. Let $\pi : B(\mathcal{H}) \longrightarrow B(\mathcal{H})/\mathfrak{K}$ be the quotient map.

Let $T \in B(\mathcal{H})$. The *essential spectrum* of T is the spectrum of $\pi(T)$ in the Calkin algebra, $\sigma(\pi(T))$.

We note that the definition of an essentially normal operator T is equivalent to the statement that $\pi(T)$ is normal in $B(\mathcal{H})/\mathfrak{K}$.

Now, Brown-Douglas-Fillmore theory was able to associate an invariant from

$\{$Essential operators with essential spectrum $X\}$ to some abelian group $\mathrm{Ext}(C(X))$.

One can lift this notion to the setting of a C*-algebra $\mathfrak{A}$ as $\mathrm{Ext}(\mathfrak{A})$. To understand this we will learn many important C*-algebra techniques including:

- quasicentral approximate units
- nuclearity and completely positive (c.p.) maps
- Fredholm index
- Voiculescu's Theroem
- Choi-Effors Theorem

(3) Noncommutative theory of retracts Borsuk's theory of retracts provides the notion of absolute retracts and absolute neighborhood retracts.

The noncommutative analogue to the theory of retracts is known as Blackadar's theory.

An important application came in the form of lifting order 0 maps from quotients to the ambient space in the setting of $M_n(\mathbb{C})$, the $n \times n$ complex matrices.

$$
\begin{tikzcd}
 & \mathfrak{B} \arrow[d] \\
M_n(\mathbb{C}) \arrow[ur, dashed, "\text{order } 0"] \arrow[r, "\text{order } 0"'] & \mathfrak{B}/\mathfrak{A}
\end{tikzcd}
$$

follows from the fact that the C*-algebra

$CM_n = C_0((0,1], M_n(\mathbb{C})) = \{f : (0,1] \longrightarrow M_n(\mathbb{C}) : f$ is continuous and vanishes at $0\}$

is a noncommutative absolute retract.

2.1. **Topological K-theory.** We start with Vector Bundles. But, first we give an example.

Example 2.3. [Möbius Strip] Fix $t > 0$. Define $M := ([0,1] \times \mathbb{R})/((0,t) \sim (1,-t))$. The following figure provides a picture of the setting on the left along with a picture of "what happens" when you "glue" $(0,t)$ and $(1,-t)$ on the right. The green line through the green x represents a copy of the vector space $\mathbb{R}$. Also, note that we will view the interval $[0,1]$ as the circle S^1 since we identify 0 with 1.

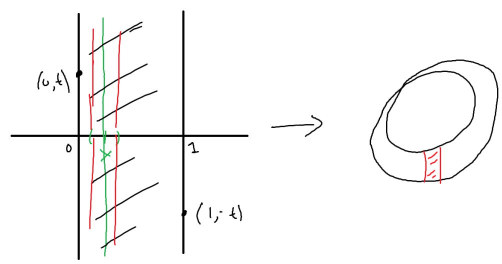

Figure 1. Möbius Strip

With this example in mind, we define.

Definition 2.4. A *vector bundle* (E,p,X) over a topological space X consists of a topological space E, a continuous map $p : E \longrightarrow X$, and a finite-dimensional vector space structure on each $E_x := p^{-1}(\{x\})$ compatible with the topology induced from E.

This means that addition and scalar multiplication of E_x is continuous with resepct to the topology on E. Furthermore, $E = \sqcup_{x\in X} E_x$. Also, we call E_x a fiber.

We usually write $(E,p,X) = E$. And, we assume that our vector spaces are over $\mathbb{R}$ or $\mathbb{C}$ and are finite dimensional.

Now, we list some examples.

Example 2.5.
(1) (trivial bundle) $X \times V$ for a vector space V. The map $p : X \times V \longrightarrow X$ is defined by $p(x,v) := x$.
(2) (Möbius Strip) For Example (2.3) the map $p : M \longrightarrow S^1$ is defined by $p(x,y) := x$.
(3) (Tangent bundle to sphere S^n) Note that $S^n \subset \mathbb{R}^{n+1}$. Consider the following figure. We let $TS^n := \{(x,\xi) : \langle x,\xi\rangle = 0\}$ and $p(x,\xi) = x$.

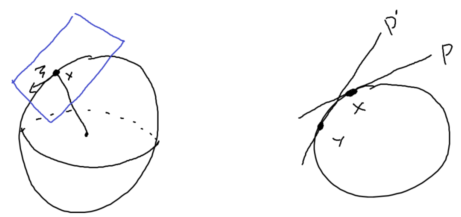

FIGURE 2. Tangent Bundle

To gather some properties of these examples, we define.

Definition 2.6. Let E, F be vector bundles over a space X. A map $\varphi : E \longrightarrow F$ is a *morphism* if

- φ is continuous
- $\varphi(E_x) \subseteq F_x$ for all $x \in X$.
- $\varphi|_{E_x}$ is linear, $\forall x \in X$.

We denote $\varphi_x := \varphi|_{E_x}$ for all $x \in X$.

φ is an *isomorphism* if there exists a morphism $\psi : F \longrightarrow E$ such that $\psi \circ \varphi = id_E$ and $\varphi \circ \psi = id_F$.

Definition 2.7. A vector bundle E is *trivial* if $E \cong X \times V$ from (1) of Example (2.5).

A vector bundle E is *locally trivial* if each point $x \in X$ has a neighborbood U such that $E|_U := p^{-1}(U)$ is trivial.

<u>(1) of Example (2.5)</u> is trivial by definition.

<u>(2) of Example (2.5)</u> is locally trivial since if we take a neigborhood (the green parentheses around x) around x as in the left picture of Figure (1), then this translates to the red band in the right picture of Figure (1). And, we can see that it is locally trivial.

But, this example is not trivial. This is due to the fact that the set $(S^1 \times \mathbb{R}) \setminus S^1$ is disconnected. But, $M \setminus S^1$ is still connected by the "gluing" of $(0, t)$ with $(1, t)$ we can still "wrap around". So, there is no homeomorphism between M and $S^1 \times \mathbb{R}$ that sends S^1 to S^1, which implies that $M \not\cong S^1 \times \mathbb{R}$.

In fact, we can prove something stronger. We can prove that no homeomorphism exits between M and $S^1 \times \mathbb{R}$. Consider the fact that there exists a compact $K \subset S^1 \times \mathbb{R}$ such that for all compact $K' \supseteq K$, we have that $(S^1 \times \mathbb{R}) \setminus K'$ is disconntected. Namely as K one can take the "equator" of the cylinder $S^1 \times \mathbb{R}$. But, for M there exists no compact $K \subset M$ such that for all compact $K' \supseteq K$, $M \setminus K'$ is disconnected. Indeed, assume

there exists such a K. Then, $K \subset [0,1] \times [N,-N]/\sim$. Take $K' := [0,1] \times [N,-N]/\sim$ and $M \setminus K'$ is connected.
(3) of Example (2.5) This example is locally trivial. Consider the right picture of Figure (2), which is just a 2-dimensional representation of the left picture setting. Consider the neighborhood U of x defined by $U := \{y : \langle y, x\rangle > 0\}$. Then, we have the map

$$(y,\xi) \longmapsto (y, \text{Projection of } \xi \text{ onto } P)$$

is an isomorphism from $TS^n|_U$ to $U \times P$ since we are choosing y not orthogonal to x with positive inner product.

3. 21 September 2016

Last time for (3) of Example (2.5), we showed that TS^n is locally trivial. Now, TS^2 is not trivial, but this is not easy and requires the following theorem. A person's name is not associated to this theorem, but a certain phrase is that describes the theorem

Theorem 3.1. *["One cannot comb the hair on a hedgehog"] There is no non-vanishing continuous tangent vector field on S^2.*

So, imagine we are given a hedgehog, which is the sphere. If we were to try to comb it's needles flat, then the idea is that we are forming tangent vector fields.

With this in mind, we can show that TS^2 is not trivial. Assume to the contrary that TS^2 is trivial. Then, there exists an isomorphism

$$S^2 \times \mathbb{R}^2 \xrightarrow[\cong]{\gamma} TS^2.$$

Assume that $z_0 \in \mathbb{R}^2$ such that $z_0 \neq 0$. Then, $\gamma(x, z_0)$ would determine a continuous tangent vector field on S^2, which is a contradiction to Theorem (3.1).
Fact: (Hard) TS^n is trivial only for $n \in \{1,3,7\}$.
!!!! From now on, by vector bundle we mean a **locally trivial** vector bundle!!!!

Remark 3.2 (An aside on connectedness). By the locally trivial assumption, if we let $x \in U$ (a neighborhood of x), then $E|_U \cong U \times V$. So, for all $x, y \in U$, we have that $\dim(E_x) = \dim(E_y)$, or that $\dim(E_x)$ is locally constant. So, if X were connected, then all fibers would be isomorphic to the same vector space. If X were not connected, then we would simply reduce the problem to connected components, where fibers would be isomorphic, locally. It is therefore safe to assume that X is connected.

We will consider another set of vector bundles, but first we need a topological definition.

Definition 3.3. A topological space X is *paracompact* if for each open cover of X there exists a refinement of the open cover such that $\forall x \in X$, there exists a neighborhood of x , which intersects only finitely many sets in the refinement.

An easy example of a paracompact space that is not compact is the interval $(0,1) \subset \mathbb{R}$ with its topology induce from the usal topology on $\mathbb{R}$.
(4) Suppose that X is contractible (homotopic to a point) and paracompact. Then, one can show that each vector bundle over X is trivial.

3.1. **Transition functions.** Transitions functions are a useful tool to show it two vector bundles are isomorphic.

Let E be a vector bundle. There exists a cover $\{U_\alpha\}$ such that

$$U_\alpha \times V \xrightarrow[\cong]{\varphi_\alpha} E|_{U_\alpha},$$

is a trivialization. If $U_\alpha \cap U_\beta \neq \emptyset$, then we have

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times V \arrow[dr, "\cong", "\varphi_\alpha"'] & & (U_\alpha \times U_\beta) \times V \arrow[dl, "\cong"', "\varphi_\beta"] \\
& E|_{U_\alpha \cap U_\beta} &
\end{tikzcd}$$

Thus, we may form the following isomorphism

$$\varphi_{\beta\alpha} := \varphi_\beta^{-1} \circ \varphi_\alpha : (U_\alpha \cap U_\beta) \times V \longrightarrow (U_\alpha \cap U_\beta) \times V,$$

which defines a *transition function.*

The following theorem shows us how transtitions functions may be used to provide an isomorphism.

Theorem 3.4. *Let E, E' be vector bundles over X and let $\{U_\alpha\}$ be a cover of X such that $E|_{U_\alpha}$ and $E'|_{U_\alpha}$ are trivial. Let $\{\varphi_{\beta\alpha}\}$ and $\{\varphi'_{\beta\alpha}\}$ be the corresponding transition functions, then $E \cong E'$ if and only if there exist isomorphisms $h_\alpha : U_\alpha \times V \longrightarrow U_\alpha \times V$ such that $\varphi_{\beta\alpha} = h_\beta^{-1} \circ \varphi'_{\beta\alpha} \circ h_\alpha$.*

Proof. "only if:" Suppose $E \xrightarrow[\cong]{\psi}$. We would like to find h_α such that the following diagram commutes.

$$\begin{tikzcd}
E'|_{U_\alpha} & E|_{U_\alpha} \arrow[l, "\psi|_{U_\alpha}"'] \\
U_\alpha \times V \arrow[u, "\varphi'_\alpha"] & U_\alpha \times V \arrow[l, dashed, "h_\alpha"'] \arrow[u, "\varphi_\alpha"']
\end{tikzcd}$$

Thus, let $h_\alpha := \varphi'^{-1}_\alpha \circ \psi|_{U_\alpha} \circ \varphi_\alpha$. Now, the function $h_\beta^{-1} \circ \varphi'_{\beta\alpha} \circ h_\alpha$ is defined on $U_\alpha \cap U_\beta$. We then have

$$\begin{aligned}
h_\beta^{-1} \circ \varphi'_{\beta\alpha} \circ h_\alpha &= \varphi_\beta^{-1} \circ \psi^{-1}|_{U_\beta} \circ \varphi'_\beta \circ \left(\varphi'_{\beta\alpha}\right) \circ \varphi'^{-1}_\alpha \circ \psi|_{U_\alpha} \circ \varphi_\alpha \\
&= \varphi_\beta^{-1} \circ \psi^{-1}|_{U_\beta} \circ \varphi'_\beta \circ \left(\varphi'^{-1}_\beta \circ \varphi'_\alpha\right) \circ \varphi'^{-1}_\alpha \circ \psi|_{U_\alpha} \circ \varphi_\alpha \\
&= \varphi_\beta^{-1} \circ \psi^{-1}|_{U_\beta} \circ \psi|_{U_\alpha} \circ \varphi_\alpha \\
&= \varphi_\beta^{-1} \circ \varphi_\alpha \quad \text{by defined on } U_\alpha \cap U_\beta \\
&= \varphi_{\beta\alpha},
\end{aligned}$$

which completes this direction.

"If:" We assume that there exist a family $\{h_\alpha\}$ such that $\varphi_{\beta\alpha} = h_\beta^{-1}\varphi'_{\beta\alpha}h_\alpha$. So, we would like to find a function ψ such that the following diagram commutes.

$$\begin{tikzcd} E'|_{U_\alpha} & E|_{U_\alpha} \arrow[l, dashed, "\psi|_{U_\alpha}"'] \\ U_\alpha \times V \arrow[u, "\varphi'_\alpha"] & U_\alpha \times V \arrow[l, "h_\alpha"'] \arrow[u, "\varphi_\alpha"'] \end{tikzcd}$$

So, on U_α, define $\psi := \varphi'_\alpha \circ h_\alpha \circ \varphi_\alpha^{-1}$.

Remains to prove:(easy exercise) On $U_\alpha \cap U_\beta$

$$\varphi'_\alpha \circ h_\alpha \circ \varphi_\alpha^{-1} \stackrel{?}{=} \varphi'_\beta \circ h_\beta \circ \varphi_\beta^{-1}.$$

□

We now have the tools to prove.

Theorem 3.5. *Every complex vector bundle over S^1 is trivial.*

Proof. Let E be a complex vector bundle over S^1. We want to show that $E \cong S^1 \times V$.

We can think of $S^1 = [0,1]/(0 \sim 1)$. Define $\widetilde{U}_\alpha = (0,1)$ and $\widetilde{U}_\beta = \big[0, \frac{1}{2}\big) \cup \big(\frac{1}{2}, 1\big]$. But, we have that $\widetilde{U}_\alpha$ is an open interval, and by our presentation of S^1, the set $\widetilde{U}_\beta$ is also an open interval. Thus, $E|_{\widetilde{U}_\alpha}$ and $E|_{\widetilde{U}_\beta}$ are trivial. Also, note that $\widetilde{U}_\alpha \cap \widetilde{U}_\beta = \big((0, \frac{1}{2}) \cup (\frac{1}{2}, 1)\big)$. Thus, the transition function

$$\widetilde{\varphi}_{\beta\alpha} : \left(\left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)\right) \times V \longrightarrow E|_{\widetilde{U}_\alpha \cap \widetilde{U}_\beta}.$$

Next, define $U_\alpha = (0,1)$ and $U_\beta = \big[0, \frac{1}{3}\big) \cup \big(\frac{2}{3}, 1\big]$. And, so

$$\varphi_{\beta\alpha} = \widetilde{\varphi}_{\beta\alpha}|_{((0,\frac{1}{3}) \cup (\frac{2}{3},1)) \times V}$$

are defined at $\big\{\frac{1}{3}\big\} \times V$ and $\big\{\frac{2}{3}\big\} \times V$.

We want to find h_β, h_α such that $\varphi_{\beta\alpha} = h_\beta^{-1} \circ h_\alpha$, where in between $h_\beta^{-1} \circ h_\alpha$ we have a transition function for the trivial bundle which can be chosen to be the identity function. Now, consider $h_\beta = id$. Then, note that $h_\alpha : (0,1) \times V \longrightarrow (0,1) \times V$ would be a matrix valued function, whose values are invertible matrices. So, on $\big(0, \frac{1}{3}\big)$ and $\big(\frac{2}{3}, 1\big)$, we would want $h_\alpha = \varphi_{\beta\alpha}$. But, at $\big\{\frac{1}{3}\big\} \times V$ and $\big\{\frac{2}{3}\big\} \times V$, we have that the values of $\varphi_{\beta\alpha}$ are invertible matrices. Since we are in the complex setting, we can produce a continuous path of intertible matrices between two invertible matrices. Thus, on $\big(\frac{1}{3}, \frac{2}{3}\big)$, we define h_α to be this continuous path of invertible matrices.

In other words, to build h_α, we extend $\varphi_{\beta\alpha}$ onto $(0,1) \times V$ by connecting $\varphi_{\beta\alpha}|_{\{\frac{1}{3}\} \times V}$ and $\varphi_{\beta\alpha}|_{\{\frac{2}{3}\} \times V}$ by a continuous path of invertible matrices. □

4. 22 September 2016

Our goal for today is organize a family of bundles over a fixed space X into a semigroup.

Definition 4.1. Fix a topological space X.

Let $[E]$- the set of all vector bundles isomorphic to E.

Let $[n]$ - the set of all vector bundles isomorphic to the $X \times \mathbb{R}^n$. We will also let $[n]$ denote all vector bundles isomorphic to $X \times \mathbb{C}^n$.

Let $V(X)$ be the set of all isomorphism classes of vector bundles over X.

$V(X)$ is the set for which we will equip with an operation to form a semigroup. The operation we will use is the

Whitney sum of 2 vector bundles: Let $E = (E, p, X)$ and $F = (F, q, X)$ be two vector bundles over the same topological space X. The *Whitney sum* $E \oplus F$ will be the triple denoted by $E \oplus F = (E \oplus F, p \oplus q, X)$. Now, we define each term in the triple.

$$E \oplus F = \{(e, f) : p(e) = q(f)\}.$$

$p \oplus q : E \oplus F \longrightarrow X$ is given by

$$(p \oplus q)(e, f) := p(e),$$

and we note that by definition of $E \oplus F$, it is equivalent to replace $p(e)$ with $q(f)$.

The Fibers of $E \oplus F$ are

$$\begin{aligned} (E \oplus F)_x &:= (p \oplus q)^{-1}(\{x\}) \\ &= \{(e, f) : p(e) = x = q(f)\} \\ &= \{(e, f) : e \in E_x, f \in F_x\} \\ &= E_x \oplus F_x. \end{aligned}$$

And, thus, the definition of Whitney sum is consistent with the vector spaces structure of the fibers of E and F.

Easy Observations:

(1) $E \oplus F$ is locally trivial.
(2) $E \oplus F \cong F \oplus E$ by the map $(e, f) \longmapsto (f, e)$
(3) If $E \xrightarrow[\varphi_E]{\cong} E'$ and $F \xrightarrow[\varphi_F]{\cong} F'$, then $E \oplus F \cong E' \oplus F'$ by the map $(e, f) \longmapsto (\varphi_E(e), \varphi_F(f))$.
(4) $[n] \oplus [m] = [n + m]$. Hence, $V(X) \supseteq \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Now, we define the sum of isomorphism classes by $[E] \oplus [F] := [E \oplus F]$, which well-defined by observation (3).

So, $V(X)$ is a semigroup with the Whitney sum, and by observation (2), $V(X)$ is an abelian semigroup. Sometimes we will write $V_{\mathbb{R}}(X)$ or $V_{\mathbb{C}}(X)$ if a distinction needs to be made.

Example 4.2. (1) If X is paracompact and contractible, then $V(X) = \mathbb{N}_0$ by (4) from last lecture.
(2) $V_{\mathbb{C}}(S^1) = \mathbb{N}_0$ by Theorem (3.5).

4.1. $V_{\mathbb{R}}(S^1)$**.** By the end of this section, we will prove that $V_{\mathbb{R}}(S^1)$ =???. Since the Möbius strip M is a real vector bundle over S^1, our approach to answering this question will rely on understanding the Whitney sum of M with other vector bundles. These are the following propositions.

Proposition 4.3. $M \oplus M = [2]$

Proposition 4.4. $M \oplus [n]$ *is not trivial.*

Proposition 4.5. *Any real vector bundle over* S^1 *is either* $[n]$ *or* $M \oplus [n]$ *for some* n*.*

Propositions (4.3, 4.5) will be left as exercises. We now prove Proposition (4.4).

Proof of Proposition (4.4). First, we show $M \oplus [1] \neq [2]$ and the proof that $M \oplus [n] \neq [n+1]$ is absolutely similar.

Recall that $M = ([0,1] \times \mathbb{R})/((0,t) \sim (1,-t)) = \{(x,t) : (0,t) \sim (1,-t)\}$, but $[1] = \{(x,t) : (0,t) \sim (1,t)\}$. Therefore, by definition of the Whitney sum

$$\begin{aligned} M \oplus [1] &= \{((x,t),(x',t')) : x = x', (0,t) \sim (1,-t), (0,t') \sim (1,t')\} \\ &= \{(x,t,t') : (0,t,t') \sim (1,-t,t')\}. \end{aligned} \tag{4.1}$$

We approach with transition functions. Let $U_\alpha = (0,1)$ and $U_\beta = \left[0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right]$. Define $\varphi_\alpha : U_\alpha \times \mathbb{R} \longrightarrow M \oplus [1]|_{U_\alpha}$ by $\varphi(x,t,t') = (x,t,t')$ which is an isomorphism since we avoid $\{0,1\}$.

Next, define $\varphi_\beta : U_\beta \times \mathbb{R} \longrightarrow M \oplus [1]|_{U_\beta}$ by

$$\varphi_\beta := \begin{cases} (x,t,t') & ,x \in \left[0, \frac{1}{3}\right) \\ (x,-t,t') & ,x \in \left(\frac{2}{3}, 1\right] \end{cases}$$

We only check that this function is well-defined since isomorphism is clear. For well-defined, since $0 \sim 1$, we only need to check what happens for these x-values. But, by Expression (4.1), we have

$$\varphi_\beta(0,t,t') = (0,t,t') = (1,-t,t') = \varphi_\beta(1,t,t').$$

Recall, that the transition function $\varphi_{\beta\alpha}$ is defined on $U_\alpha \cap U_\beta$. Now,

$$\varphi_{\beta\alpha}(x,t,t') := \varphi_\beta^{-1} \circ \varphi_\alpha(x,t,t') = \begin{cases} (x,t,t') & ,x \in \left(0, \frac{1}{3}\right) \\ (x,-t,t') & ,x \in \left(\frac{2}{3}, 1\right) \end{cases}. \tag{4.2}$$

But, $\varphi_{\beta\alpha}(x) := \varphi_{\beta\alpha}|_{\{x\} \times \mathbb{R}^2}$ is a matrix-valued function of invertible matrices, as the maps determines linear bijections.

Now, the transition function for [2] is the identity map, thus to show that $M \oplus [1] \neq [2]$ it is necessary and sufficient (by Theorem (3.4)) to show that $\varphi_{\beta\alpha} \neq h_\beta^{-1} \circ h_\alpha$ for some $h_\beta : U_\beta \times \mathbb{R}^2 \longrightarrow U_\beta \times \mathbb{R}^2$ and $h_\alpha : U_\alpha \times \mathbb{R}^2 \longrightarrow U_\alpha \times \mathbb{R}^2$.

For such functions, we would have that $h_\alpha(x) := h_\alpha|_{\{x\} \times \mathbb{R}^2}$ and similarly for h_β, which implies that they are also matrix-valued functions of invertible matrices. Therefore, $\det h_\alpha(x) \neq 0$ is a non-vanishing function, so it is either all positive or all negative. Therefore, $\operatorname{sign}(\det h_\alpha(x))$ is a constant function on $(0,1)$. Similarly, $\operatorname{sign}(\det h_\beta(x))$ is a constant function on $\left[0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right]$. But, this also imples that $\mathrm{sign}(\det(h_\beta^{-1} \circ h_\alpha))$ is a constant function on $U_\alpha \cap U_\beta = \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right)$.

However, a closer look at $\varphi_{\beta\alpha}$ in Equation (4.2), reveals that

$$\varphi_{\beta\alpha}(x) := \varphi_{\beta\alpha}|_{\{x\}\times\mathbb{R}^2} = \begin{cases} I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & , x \in \left(0, \frac{1}{3}\right) \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & , x \in \left(\frac{2}{3}, 1\right) \end{cases} \tag{4.3}$$

where I_2 denotes the 2×2- identity matrix. Therefore, $\mathrm{sign}(\det \varphi_{\beta\alpha})$ is not constant on $U_\alpha \cap U_\beta = \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right)$. Hence, $\varphi_{\beta\alpha} \neq h_\beta^{-1} \circ h_\alpha$, and thus $M \oplus [1] \neq [2]$ by Theorem (3.4).

We note that for the argument for $M \oplus [n] \neq [n+1]$, the argument would be the same and we would simply replace the top matrix in Equation (4.3) with I_{n+1} and the bottom matrix with -1 in the top left entry and 1's on the remaining diagonal entries with 0's elsewhere. And, it is easy to see that we would reach the same conclusion. □

Remark 4.6. We note that $M \oplus [n]$ is the sum of a not trivial vector bundle with a trivial vector bundle that is not trivial. But, this is not always the case. For instance, $TS^2 \oplus [1] = [3] = [2] \oplus [1]$, which also provides that in general $V(X)$ might not have the cancellation property.

To finish our discussion of real vector bundles over S^1, we present.

Theorem 4.7. $V_{\mathbb{R}}(S^1) \cong (\mathbb{Z}_2 \times \mathbb{N}) \cup (0,0)$.

Proof. Define $f : (\mathbb{Z}_2 \times \mathbb{N}) \cup (0,0) \longrightarrow V_{\mathbb{R}}(S^1)$ by

$$f(0,n) := [n] \text{ and } f(1,n) := M \oplus [n-1]$$

Note that $(1,0) \notin (\mathbb{Z}_2 \times \mathbb{N}) \cup (0,0)$ and f is well-defined by Proposition (4.3). To show that f is a homomorphism is easy. Surjectivity is provided by Proposition (4.5). And, injecvitivity is Proposition (4.4). □

5. 23 September 2016

The first objectvie of today is to show that the map $X \longmapsto V(X)$ is a contravariant functor between the category of topolocial spaces (with continuous maps) and the category of abelian semigroups (with homomorphisms). We will cover the defininition of contravariant functor as well.

Now, the map $X \longmapsto V(X)$ already sends objects to objects, but how do we send a morphism (continuous maps) $X \xrightarrow{\varphi} Y$ to a morphism (homomorphism) $V(X) \xleftarrow{?} V(Y)$. We note that the fact the arrow is in the opposite direction is why we will have a contravariant functor and not a covariant functor. Now, let $E = (E, p, Y)$ be a vector bundle over Y. Let X be a topological space and let $X \xrightarrow{\varphi} Y$ be a continuous function. Consider the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

This diagram suggests the following definition.

Definition 5.1. Let $E = (E, p, Y)$ be a vector bundle over Y. Let X be a topological space and let $X \xrightarrow{\varphi} Y$ be a continuous function.

Define $\varphi^* E = (\varphi^* E, \varphi^* p, X)$ by

$$\varphi^* E = \{(x, e) : \varphi(x) = p(e)\},$$

which is motivated by the above diagram.

Let $\varphi^* p : \varphi^* E \longrightarrow X$ be given by $\varphi^* p(x, e) := x$.

Next, we establish that this defines a vector bundle.

Proposition 5.2. *$\varphi^* E$ is a vector bundle.*

Proof. What remains is to check that we produce fibers that are vector spaces and that we have local triviality (recall that all our vector bundles are assumed to be locally trivial).

<u>Fibers:</u> Fix $x_0 \in X$. Then,

$$(\varphi^* E)_{x_0} = \varphi^* p^{-1}(\{x_0\}) = \{(x_0, e) : \varphi(x_0) = p(e)\} = E_{\varphi(x_0)}, \tag{5.1}$$

which is therefore a vector space that satisfies the definition of a fiber of a vector bundle.

<u>Local triviality:</u> Let $x_0 \in X$, then $\varphi(x_0) \in Y$. Since E is locally trival, we have that there exists a neighborhood U of $\varphi(x_0)$ such that $U \times V \xrightarrow[\gamma]{\cong} E|_U$, where V is a vector space. By continuity, $\varphi^{-1}(U)$ is a neigborhood of x_0. Thus, we will show that $\varphi^* E|_{\varphi^{-1}(U)}$ is trivial.

Define $\widetilde{\gamma} : \varphi^{-1}(U) \times V \longrightarrow \varphi^* E|_{\varphi^{-1}(U)}$ by

$$\widetilde{\gamma}(x, v) := (x, \gamma(\varphi(x), v)).$$

We show that this map is well-defined. So, we show that $(x, \gamma(\varphi(x), v)) \in \varphi^* E$. But, by triviality of E_U, we have that $\gamma(\varphi(x), v) \in E_{\varphi(x)}$. But, then we have that $p(\gamma(\varphi(x), v)) = \varphi(x)$. Therefore, $(x, \gamma(\varphi(x), v)) \in \varphi^* E$, where $x \in \varphi^{-1}(U)$. $\square$

Consider the following 2 propositions. Recall the Whitney sum from the previous lecture.

Proposition 5.3. *$\varphi^*(E \oplus F) = \varphi^* E \oplus \varphi^* F$.*

Proposition 5.4. *If $E \cong E'$, then $\varphi^* E \cong \varphi^* E'$.*

We note the following consequence of these results.

First, by Proposition (5.4), the map $\varphi^* : V(Y) \longrightarrow V(X)$ defined by $\varphi^*[E] := [\varphi^* E]$ is well-defined.

Second, by Proposition (5.3), the map φ^* is a semigroup homomorphism.

Now, we are in a position to discuss our contravariant functor, but first let's recall the definition.

Definition 5.5. Let $\mathcal{C}, \mathcal{D}$. Denote the class of objects of $\mathcal{C}$ by $\mathsf{obj}(\mathcal{C})$, and similarly for $\mathcal{D}$, and the class of morphisms $\mathcal{C}$ by $\mathsf{hom}(\mathcal{C})$ and similarly for $\mathcal{D}$. A *contraviant functor* F from $\mathcal{C}$ to $\mathcal{D}$ is a function that acts on both objects and morphisms by:

(1) $X \in \mathsf{obj}(\mathcal{C}) \longmapsto F(X) \in \mathsf{obj}(\mathcal{D}$.
(2) If $X, Y \in \mathsf{obj}(\mathcal{C})$ and $\varphi \in \mathsf{hom}(\mathcal{C})$ such that $X \xrightarrow{\varphi} Y$, then $F(\varphi) \in \mathsf{hom}(\mathcal{D})$ such that $F(Y) \xrightarrow{F(\varphi)} F(X)$.
(3) If $X \in \mathsf{obj}(\mathcal{C})$, then $F(id_X) = id_{F(X)}$.
(4) If $\psi, \varphi \in \mathsf{hom}(\mathcal{C})$, then $F(\psi \circ \varphi) = F(\varphi) \circ F(\psi)$.

Therefore, let V denote our contravariant functor from the category of topological spaces with continuous maps as the morphisms to the category of abelian semigroups with homomorphisms as the morphism. On objects, $V : X \longmapsto V(X)$ and on morphisms $V : \varphi \longmapsto \varphi^*$. We have already establised properites (1) and (2) for V to be a contravariant functor. We leave the remaining 2 properties as exercises, which we list a propositions.

Proposition 5.6. $(id_X)^* = id_{V(X)}$.

Proposition 5.7. $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

5.1. **Grothendieck Group.** To move towards K-theory, we need to build an abelian group from an abelian semigroup. For example, this construction will provide the group $\mathbb{Z}$ with addition from the semigroup $\mathbb{N}$ with addition.

Grothendieck Group construction

Let H be an abelian semigroup.

Let $x - y$ denote formal differences of $x, y \in H$. We introduce the following relation.

$$x - y \sim x' - y' \iff x + y' + z = x' + y + z \text{ for some } z \in H. \tag{5.2}$$

The proof of the next proposition will be left as an exercise, in which the proof of transitivity will reveal why we have the z in the above relation.

Proposition 5.8. *Show that Relation (5.2) is an equivalence relation.*

Therefore, let $[x - y]$ denote the equivalence class of $x - y$ with respect to the equivalence relation $\sim$.

Define $[x - y] + [x_1 - y_1] := [(x + x_1) - (y + y_1)]$.

The neutral element is $[x-x]$, which is independent of x. Indeed, $x-x \sim y-y$ since $x+y = y+x$ by abelian.

Finally, we define the inverse $[x-y]^{-1} := [y-x]$.

Definition 5.9. Let H be an abelian semigroup, then the Grothendieck group of H is

$$\sigma(H) = \{[x-y] : x, y \in H\}$$

with operations defined above.

Example 5.10.

(1) If $H = (\mathbb{N}, +)$, then $\sigma(H) = (\mathbb{Z}, +)$ by the map $[n_1 - n_2] \longmapsto n_1 - n_2$.

(2) If $H = (\mathbb{N}, \cdot)$, then $\sigma(H) = (\mathbb{Q}_+, \cdot)$ by the map $[n_1 - n_2] \longmapsto \frac{n_1}{n_2}$.

(3) Let $H = (\mathbb{N} \cup \{\infty\}, +)$. Since $(\infty + n = \infty)$, for all n_1, n_2, m_1, m_2 we have that $n_1 + m_2 + \infty = \infty = m_2 + n_2 + \infty$ implies that $n_1 - n_2 \sim m_1 - m_2$. Thus, the group $\sigma(H) = 0$, the trivial group. This leads to the following proposition.

   Proposition 5.11. *If H has an ∞ element ($\infty + h = \infty, \forall h \in H$), then $\sigma(H) = 0$.*

(4) If $H = (\mathbb{N}_0, \cdot)$, then $0 \cdot n = 0$ for all $n \in \mathbb{N}_0$. Therefore, 0 is an ∞ element and $\sigma(H) = 0$ by the previous proposition.

Let H be an abelian semigroup. Fix $k \in H$. Consider the following map $i : H \longmapsto \sigma(H)$ defined by $i(x) = [(x+k) - k]$.

We note that this map is independent of the choice of k. Indeed, $x+k-k \sim x+m-m$ since $x+k+m = x+m+k$ by abelian. Also, in the case that H has a neutral element, we can define $i(x) = [x-0]$.

Proposition 5.12. *i is injective if and only if H has cancellation.*

Proof. <u>If:</u>

$$\begin{aligned} i(x) = i(y) &\implies [(x+k)-k] = [(y+k)-k] \\ &\implies (x+k)-k \sim (y+k)-k \\ &\implies x+2k+z = y+2k+z \text{ for some } z \in H \\ &\implies x = y \text{ by cancellation} \end{aligned}$$

<u>only if:</u>

$$\begin{aligned} x+m = y+m &\implies [(x+m)-m] = [(y+m)-m] \\ &\implies i(x) = i(y) \\ &\implies x = y \text{ by injectivity.} \end{aligned}$$

□

<u>Universal property of $\sigma(H)$</u>

Theorem 5.13. *Let H be an abelian semigroup and let G be an abelian group. If $\varphi : H \longrightarrow G$ is a homomorphism, then there exists a unique homorphism $\psi : \sigma(H) \longrightarrow G$ such that the following diagram commutes.*

$$\begin{tikzcd} H \arrow[r, "\varphi"] \arrow[d, "i"'] & G \\ \sigma(H) \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

In other words, any homomorphism $\varphi : H \longrightarrow G$ can be extended in a unique way to a homomorphism $\psi : \sigma(H) \longrightarrow G$ via $i : H \longrightarrow \sigma(H)$.

Proof. Define $\psi([x-y]) := \varphi(x) - \varphi(y)$. Well-defined is clear. Now, we check that the diagram commutes.

$$\begin{aligned} \psi \circ i(x) &= \psi(i(x)) \\ &= \psi([(x+k)-k]) \\ &= \varphi(x+k) - \varphi(k) \\ &= \varphi(x) + \varphi(k) - \varphi(k) \\ &= \varphi(x). \end{aligned}$$

For uniqueness, let $\widetilde{\psi}$ be another extension.

First, we show that $[x-y] = i(x) - i(y)$.

$$\begin{aligned} i(x) - i(y) &= [(x+k)-k] - [(y+k)-k] \\ &= [(x+k)-k] + [k-(y+k)] \\ &= \left[\underbrace{(x+2k)-(y+2k)}_{\sim x-y} \right] \\ &= [x-y]. \end{aligned}$$

Therefore,

$$\begin{aligned} \widetilde{\psi}([x-y]) &= \widetilde{\psi}(i(x)-i(y)) \\ &= \widetilde{\psi}(i(x)) - \widetilde{\psi}(i(y)) \\ &= \varphi(x) - \varphi(y) \\ &= \psi([x-y]). \end{aligned}$$

□

6. 26 September 2016

Today, given a topological space X (compact or locally compact), we will construct the group $K^0(X)$ from the Grothendieck groups of the previous day. First, we need some more properties of extending maps. The next serves as a Corollary to Theorem (5.13).

Corollary 6.1. *Let H_1, H_2 be two abelian semigroups. Recall the map i from Proposition (5.12). If $\varphi : H_1 \longrightarrow H_2$ is a homomorphism, then there exists a unique homomorphism*

$\psi : \sigma(H_1) \longrightarrow \sigma(H_2)$ *such that the following diagram commutes.*

$$\begin{tikzcd} H_1 \arrow[r, "\varphi"] \arrow[d, "i"'] & H_2 \arrow[d, "i"] \\ \sigma(H_1) \arrow[r, dashed, "\psi"] & \sigma(H_2) \end{tikzcd}$$

Proof. Apply Theorem (5.13) to the map $i \circ \varphi$. $\square$

Definition 6.2 (K-theory)**.** Let X be a compact Hausdorff space. Define $K^0(X)$: $\sigma(V(X))$.

Proposition 6.3. *K^0 is a contravariant functor from the category of compact Hausdorff spaces with continuous maps to the category of groups with homomorphisms.*

Proof. Let $\varphi : X \longrightarrow Y$ be a continuous map, then by Corollary (6.1)), we have that there exists a unique map ψ such that the following diagram commutes.

$$\begin{tikzcd} V(Y) \arrow[r, "\varphi^*"] \arrow[d, "i"'] & V(X) \arrow[d, "i"] \\ K^0(Y) = \sigma(V(Y)) \arrow[r, dashed, "\psi"] & \sigma(V(X)) \end{tikzcd}$$

Define $K^0(\varphi) := \psi$.

It remains to check that $K^0(\varphi \circ \psi) = K^0(\psi) \circ K^0(\varphi)$. But, $K^0(\varphi \circ \psi)$ is the unique extension of $(\varphi \circ \psi)^*$. However, $K^0(\varphi), K^0(\psi)$ are unique extension of φ^*, ψ^*, respectively. Proposition (5.7) showed that $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$, which implies that $K^0(\varphi \circ \psi) = K^0(\psi) \circ K^0(\varphi)$ by uniqueness of extenstions by Corollary (6.1). $\square$

We note that by K^0, we mean the complex case whereas $K^0_{\mathbb{R}}$ is reserved for the real case.

Example 6.4. (1) $K^0(\{x_0\}) = \sigma(\mathbb{N}_0) = \mathbb{Z}$.
(2) $K^0(S^1) = \mathbb{Z}$.
(3) $K^0_{\mathbb{R}}(S^1) = \sigma(\mathbb{Z}_2 \times \mathbb{N} \cup \{(0,0)\}) = \mathbb{Z}_2 \times \mathbb{Z}$.

The next proposition will lead us to the lcoally compact case.

Proposition 6.5. *If X is a compact space, then $K^0(X)$ contains $\mathbb{Z}$ as a direct summand. And, we denote $K^0(X) = \mathbb{Z} \oplus \widetilde{K^0}(X)$.*

(Caution: $V(X) \supseteq \mathbb{N}_0 \not\Longrightarrow K^0(X) \supseteq \mathbb{Z}$. For instance, $\mathbb{N}_0 \cup \{\infty\} \supset \mathbb{N}_0$ but $\sigma(\mathbb{N}_0 \cup \{\infty\}) = 0$ and $\sigma(\mathbb{N}_0) = \mathbb{Z}$).

Proof. Fix $x_0 \in X$. Consider

$$\begin{tikzcd} \{x_0\} \arrow[r, hook, "i"] & X \arrow[r, two heads, "p"] & \{x_0\} \end{tikzcd}$$

and we note that i is an injection and p is a surjection. By Theorem (6.3), we have

$$K^0(\{x_0\}) \xleftarrow{K^0(i)} K^0(X) \xleftarrow{K^0(p)} K^0(\{x_0\})$$

However, $p \circ i = id_{\{x_0\}} \implies K^0(i) \circ K^0(p) = id_{K^0(\{x_0\})} \implies K^0(i)$ is a surjection and $K^0(p)$ is an injection. Hence, since $\mathbb{Z} = K^0(\{x_0\})$,

$$\mathbb{Z} \xleftarrow{K^0(i)} K^0(X) \xleftarrow{K^0(p)} \mathbb{Z}.$$

Thus, if we show that there is an isomorphism $\gamma : K^0(X) \longrightarrow K^0(i)(K^0(X)) \oplus \ker K^0(i)$, then we would be done since $K^0(i)(K^0(X)) = \mathbb{Z}$. Define γ by

$$\gamma(g) = \left(K^0(i)(g), g - K^0(p)\left(K^0(i)(g)\right)\right).$$

To check that this is well-defined, note that

$$\begin{aligned} K^0(i)\left(g - K^0(p)\left(K^0(i)(g)\right)\right) &= K^0(i)(g) - \left(K^0(i) \circ K^0(p)\right) \circ K^0(i)(g) \\ &= K^0(i)(g) - id_{K^0(\{x_0\})} \circ K^0(i)(g) = 0. \end{aligned}$$

Therefore, $g - K^0(p)\left(K^0(i)(g)\right) \in \ker K^0(i)$.

It is left as an exercise to prove that γ is an isomorphism.

We note that $\widetilde{K^0}(X) := \ker K^0(i)$ and that this is independent of the choice of x_0 since if $G = \mathbb{Z} \oplus G_1 = \mathbb{Z} \oplus G_2$, then $G_1 \cong G_2$. □

Now, assume that X is a non-compact locally compact Hausdorff space. Let X^+ denote its one-point compactification. By Proposition (6.5), we have that

$$K^0(X^+) = \mathbb{Z} \oplus \widetilde{K^0}(X^+).$$

Thus, we define.

Definition 6.6. Let X be a non-compact locally compact space and denote its one-point compactification by X^+. Define

$$K^0(X) := \widetilde{K^0}(X^+).$$

We note that in order for this K^0 to be a contravariant functor, we need to consider instead the category of locally compact Huasdorff spaces with *proper* continuous maps because a proper continuous map on a locally compact Hausdorff space extendts to a continuous map on the one-point compactification, which is not the case if our map is only assumed to be continuous.

<u>Homotopy invariance</u> If $f, g : X \longrightarrow Y$ are homotopic, then $K^0(f) = K^0(g)$. (We will prove this later in the the more general context of K-theory for C*-algebras.)

<u>Fact:</u> Let X be a locally compact space. If $Y \subseteq X$ is closed, then the sequence

$$K^0(X \setminus Y) \longrightarrow K^0(X) \longrightarrow K^0(Y)$$

is exact.

Definition 6.7. $K^{-1}(X) := K^0(X \times \mathbb{R})$

$K^{-n}(X) := K^0(X \times \mathbb{R}^n)$.

Note that if $Y \subseteq X$ is closed, then $Y \times \mathbb{R}^n$ is closed in $X \times \mathbb{R}^n$. Furthermore, there exist maps δ such that the following is a long exact sequence.

$$\begin{tikzcd}
 & \cdots \arrow[dl, "\delta"] & \\
K^{-2}(X \setminus Y) \arrow[r] & K^{-2}(X) \arrow[r] & K^{-2}(Y) \arrow[dll, "\delta"'] \\
K^{-1}(X \setminus Y) \arrow[r] & K^{-1}(X) \arrow[r] & K^{-1}(Y) \arrow[dll, "\delta"'] \\
K^{0}(X \setminus Y) \arrow[r] & K^{0}(X) \arrow[r] & K^{0}(Y)
\end{tikzcd} \tag{6.1}$$

One of the main benefits to working with the groups K^0 instead of semigroups is that we can use Bott periodicity, which proves very useful for calclating K-theory.
<u>Bott periodicity:</u> There is a natural isomorphism between $K^0(X)$ and $K^{-2}(X)$ and hence between $K^{-n}(X)$ and $K^{-n-2}(X)$. By natural isomorphism, we mean a natural transformation, which satisfues the following. Given a map $X \longrightarrow Y$ there exist maps η_X and η_Y such that the following diagram commutes.

$$\begin{tikzcd}
K^0(X) \arrow[d, "\eta_X"'] & K^0(Y) \arrow[l] \arrow[d, "\eta_Y"'] \\
K^{-2}(X) & K^{-2}(Y) \arrow[l]
\end{tikzcd}$$

With this the Sequence (6.1) becomes the 6-term exact sequence.

$$\begin{tikzcd}
K^{-1}(X \setminus Y) \arrow[r] & K^{-1}(X) \arrow[r] & K^{-1}(Y) \arrow[d, "\delta"'] \\
K^0(Y) \arrow[u, "\delta"'] & K^0(X) \arrow[l] & K^0(X \setminus Y) \arrow[l]
\end{tikzcd}$$

By Bott periodicity we may calculate the following K-groups.

Example 6.8. (1) $K^0(\mathbb{R}^2) = K^0(\{x_0\}) = \mathbb{Z}$.
(2) Since $S^2 = \mathbb{R}^{2^+}$, we have $K^0(S^2) = \mathbb{Z} \oplus K^0(\mathbb{R}^2) = \mathbb{Z}^2$.

Next, to move toward K-theory of C*-algebras, we must describe vector bundles in algebrac terms. We begin with

Theorem 6.9 (Swan's theorem)**.** *Let X be a compact Hausdorff space. If E is a vector bundle over X, then there exists a vector bundle E' over X such that $E \oplus E' \cong$ trivial.*

We already observed this phenomenon in the case of the non-trivial vector bundle M. Indeed, $M \oplus M = [2]$.

7. 27 September 2016

We continue our journey to reformulate K-theory in a more algebraic way to translate to C*-algebras. This begins with Swan's Theorem (6.9). In order to prove Swan's theorem we will spend the day proving results that will lead to Swan's theorem. The first of which is:

Theorem 7.1. *Let E be a vector bundle over X. Let $p : E \longrightarrow E$ be an idempotent morphism (that is, p is a morphism such that $p^2 = p$, where $p^2 = p \circ p$.) Recall that for a morphism, $p_x := p|_{E_x}$ for all $x \in X$. Then,*

(i) Ran $p = \sqcup_{x\in X} p_x$ *and*

(ii) $\ker p = \sqcup_{x\in X} p_x$

are vector bundles over X.

Remark 7.2. In general, range (and kernel) over morphism needn't be a vector bundle. For example, consider $X = [0, 1]$ and the mprhism $\varphi : X \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ defined by $\varphi(x, t) = (x, xt)$. Now, if

$x \neq 0$, then $\mathrm{Ran}(\varphi_x) = \mathbb{R}$ and if

$x = 0$, then $\mathrm{Ran}(\varphi_x) = 0$, which breaks localy triviality. The philosophy for why this does not work is because the map from linear transformations $T \longmapsto \dim(\mathrm{Ran}T)$ is not continuous (although, it is semicontinuous).

Proof. We need to establish local triviality. First note that $p_x^2 = p_x$ for all $x \in X$.

Claim 7.3. *If $x, x_0 \in X$ are close, then there exists an invertible linear map f_x such that $p_x = f_x^{-1} p_{x_0} f_x$.*

Proof of claim. Define $f_x := 1 - p_{x_0} - p_x + 2p_{x_0}p_x$. Now, if x is close to x_0, then p_x is close to p_{x_0} and hence $-p_{x_0} - p_x$ is close to $-2p_x = -2p_x^2$ by idempotent. But, this is close to $-2p_{x_0}p_x$. Therefore, f_x is close to $1 - 2p_{x_0}p_x + 2p_{x_0}p_x = 1$, the identity. This implies that f_x is invertible. But, by idempotent,

$$\begin{aligned}
p_{x_0} f_x &= p_{x_0}(1 - p_{x_0} - p_x + 2p_{x_0}p_x) \\
&= p_{x_0} - p_{x_0}^2 - p_{x_0}p_x + 2p_{x_0}^2 p_x \\
&= p_{x_0} - p_{x_0} - p_{x_0}p_x + 2p_{x_0}p_x \\
&= -p_{x_0}p_x + 2p_{x_0}p_x \\
&= (1 - p_{x_0} - p_x + 2p_{x_0}p_x)p_x \\
&= f_x p_x,
\end{aligned}$$

and thus, $p_{x_0}p_x = f_x p_x$ implies that $p_x = f_x^{-1} p_{x_0} f_x$. □

Hence, let U be a neighborhood of x_0 such that for $x \in U$, f_x is invertible and $p_x = f_x^{-1} p_{x_0} f_x$. Now, define

$$\mathrm{Ran}\ p|_U = \sqcup_{x\in U} \mathrm{Ran}\ p_x \xrightarrow{\gamma} U \times \mathrm{Ran}\ p_{x_0}$$

by $\gamma((x,v)) := (x, f_x v)$. For well-defined, let $v \in \text{Ran } p_x = \text{Ran } f_x^{-1} p_{x_0} f_x$. Then, $v = f_x^{-1} p_{x_0} f_x w \implies f_x v = p_{x_0} f_x w \in \text{Ran } p_{x_0}$.

Next, define

$$\ker p|_U = \sqcup_{x \in U} \ker p_x \xrightarrow{\theta} U \times \ker \ p_{x_0}$$

by $\theta((x,v)) = (x, f_x^{-1} v)$. The proof of well-defined follows similarly. □

We can think of Swan's theorem as a way to find a complement of E denoted E' such that $E \oplus E'$ is trivial. The next theorem establishes a notion of complement for vector bundles, which transfers the difficulty of finding a complement to finding a suitable morphism, which is part (3) of the following theorem.

Theorem 7.4. *Let E, F be vector bundles over X. The following are equivalent.*

(1) *There exists a vector bundle E' over X such that $E \oplus E' \cong F$.*
(2) *There exist morphisms $\alpha : F \longrightarrow E$ and $\beta : E \longrightarrow F$ such that $\alpha \circ \beta = id_E$.*
(3) *There exists a morphism $\alpha : F \longrightarrow E$ such that α_x is surjective for all $x \in X$.*

Before we prove this theorem, let's begin with a remark of the case of vector spaces instead of vector bundles to motiviate the proof of (3) $\implies$ (2).

Remark 7.5. Let M, N be vector spaces and let $\alpha : M \longrightarrow N$ be a surjective linear map. We want to find $\beta : N \longrightarrow M$ linear such that $\alpha \circ \beta = id_N$. As we are in the case of vectro spaces $M/\ker\alpha \cong N$ and $M \cong N \oplus \ker\alpha$. Since we are only considering finite dimensional case, we can view α as a rectangular matrix

$$\alpha = \left[\begin{array}{c|c} \gamma & 0 \end{array}\right],$$

where $\gamma : N \longrightarrow N$ denotes an isomorphism and 0 denotes $\ker\alpha$. Now, define β as

$$\beta := \left[\begin{array}{c} \gamma^{-1} \\ \hline 0 \end{array}\right],$$

where 0 represents $\ker\alpha$. Therefore,

$$\alpha \circ \beta = \left[\begin{array}{c|c} \gamma & 0 \end{array}\right] \left[\begin{array}{c} \gamma^{-1} \\ \hline 0 \end{array}\right] = \mathbb{1}_N.$$

Proof. We start with the hardest implication (3) $\implies$ (2). Now, condition (3) implies that $\alpha(E_x) = F_x$ for all $x \in X$. As E_x and F_x are vector spaces, we will apply the techniques of the above remark in a consistent way between different x's.

By locally trivial, there exists a cover $\{U_i\}$ of X such that $F|_{U_i} = U_i \times M$ and $E|_{U_i} = U_i \times N$. Let $x \in U_i$. First, we construct β on U_i. Now, $M \cong N \oplus \ker\alpha_x$. Thus, following the remark, there is an isomorphism $\gamma_x : N \longrightarrow N$ such that

$$\alpha_x = \left[\begin{array}{c|c} \gamma_x & 0 \end{array}\right],$$

where 0 denotes $\ker\alpha_x$ and for all $y \in U_i$,

$$\alpha_y = \left[\begin{array}{c|c} \gamma_y & \theta_y \end{array}\right].$$

γ_y is close to γ_x, and hence γ_y is an isomorphism. We note that it might be the case that we need a finer cover than $\{U_i\}$ for this, but this can be done without consequence. Thus, we may define

$$\beta_y := \left[\begin{array}{c}\gamma_y^{-1} \\ \hline 0\end{array}\right].$$

Therefore, $\alpha_y \circ \beta_y = \left[\begin{array}{c|c}\gamma_y & \theta_y\end{array}\right] \left[\begin{array}{c}\gamma_y^{-1} \\ \hline 0\end{array}\right] = \mathbb{1}_{\{y\}\times N}$.

We define $\beta_i : E|_{U_i} \longrightarrow F|_{U_i}$ by $(\beta_i)_x := \beta_y$. We thus have

$$\alpha|_{U_i} \circ \beta_i = id_{E|_{U_i}}. \tag{7.1}$$

Next, we need to glue the β_i's together to obtain $\beta : E \longrightarrow F$.

To do this, let $\{\eta_i\}$ be a partition of unity corresponding to $\{U_i\}$. Define $\beta : E \longrightarrow F$ by if $e \in E_x$, then

$$\beta(e) := \sum \eta_i(x)\beta_i(e),$$

(assuming that $\eta_i(x)\beta_i(e) = 0$ when $\eta_i(x) = 0$). Therefore, if $e \in E_x$, then

$$\begin{aligned}\alpha \circ \beta(e) &= \alpha\left(\sum \eta_i(x)\beta_i(e)\right) \\ &= \sum \eta_i(x)\alpha|_{U_i}(\beta_i(e)) \\ &= \sum \eta_i(x)e \text{ by Equation (7.1)} \\ &= 1e = e \text{ by partition of unity.}\end{aligned}$$

Thus, $\alpha \circ \beta = id_E$.

(2) $\implies$ (1). For this, we will prove that $F \cong E \oplus \ker(\beta \circ \alpha)$. By assumption,

$$(\beta \circ \alpha)^2 = \beta \circ \underbrace{\alpha \circ \beta}_{=id_E} \circ \alpha = \beta \circ \alpha$$

is idempotent. Thus, by the previous theorem, $\ker(\beta \circ \alpha)$ is a vector bundle. Therefore, $E \oplus \ker(\beta \circ \alpha)$ is a vector bundle. Hence, we can show that the following map is an isomorphism.

Let $\gamma : F \longrightarrow E \oplus \ker(\beta \circ \alpha)$ be defined by $\gamma(f) = (\alpha(f), f - \beta \circ \alpha(f))$. Easy exercise.

(1) $\implies$ (3). Define $\alpha : F \longrightarrow E$ by $\alpha((e, e')) = e$. □

8. 28 September 2016

Today, we will prove Swan's Theorem (6.9). But first, we need one more notion.

Definition 8.1. A *section* of a vector bundle $E = (E, p, X)$ is a continuous map $s : X \longrightarrow E$ such that $s(x) \in E_x$ for all $x \in X$. That is, $p \circ s(x) = x$ for all $x \in X$.

We let $\Gamma(E)$ be the set of all sections. And, $\Gamma(E)$ is an abelian group with addition defined point-wise $(s + t)(x) = s(x) + t(x)$ for $s, t \in \Gamma(E), x \in X$, which is well-defined since E_x is a vector space.

For example, if we consider the trivial bundle $S^1 \times \mathbb{R}$, then a section is a continuous map s that assigns to each $x \in S^1$ some value $s(x) \in E_x = \mathbb{R}$ such that s is continuous. Consider the following figure:

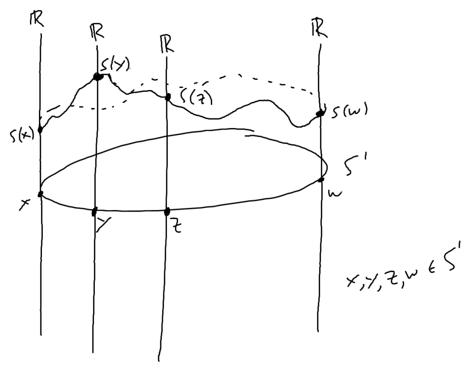

FIGURE 3. Section of $S^1 \times \mathbb{R}$

Definition 8.2. Sections $s_1, \ldots, s_N \in \Gamma(E)$ are *linearly independent* if for $x \in X$, the vectors $s_1(x), s_2(x), \ldots, s_N(x) \in E_x$ are linearly independent.

Proposition 8.3. *The vector bundle* $[n]$ *has* n *linearly independent sections.*

Proof. First consider the trival bundle $X \times V$, where $\dim V = n$. Fix some basis of V denoted $v_1, \ldots, v_n$. For $i = 1, \ldots, n$, define constant sections s_i by

$$s_i(x) := (x, v_i)$$

which are linearly independent since $v_1, \ldots, v_n$ are linearly independent.

Next, assume that E is a vector bundle such that $X \times V \xrightarrow[\gamma]{\cong} E$.

For $i = 1, \ldots, n$, define $\widetilde{s_i} \in \Gamma(E)$ by $\widetilde{s_i}(x) = \gamma \circ s_i$. □

We list two observation, which will be useful in the proof of Swan's theorem.

Observation 8.4. Let E be a vector bundle over X. If $f \in C(X)$ and $s \in \Gamma(E)$, then $fs \in \Gamma(E)$ since $f(x) \in \mathbb{C}$ and $s(x) \in E_x$, which is a vector space, and $(fs)(x) = f(x)s(x) \in E_x$.

Observation 8.5. Let E be a vector bundle over X. If $s_1, \ldots, s_N \in \Gamma(E)$, then we can define $\alpha : X \times \mathbb{R}^N \longrightarrow E$ by

$$\alpha((x, \mu_1, \ldots, \mu_N)) := \sum_{i=1}^{N} \mu_i \underbrace{s_i(x)}_{\in E_x} \in E_x.$$

We rewrite Swan's theorem and we note that Swan's theorem is valid for either the $\mathbb{C}$ or $\mathbb{R}$ case.

Theorem 8.6 (Swan's Theorem)**.** *Let* X *be a compact Hausdorff space. If* E *is a vector bundle over* X*, then there exists a vector bundle* E' *over* X *such that* $E \oplus E'$ *is trivial.*

Proof. By Theorem (7.4) from yesterday, it is enough to construct $\alpha : X \times \mathbb{R}^N \longrightarrow E$ such that α_x is a surjection for all $x \in X$ for some N.

Let $\{U_i\}_{i=1}^r$ be a cover of X such that $E|_{U_i} = U_i \times \mathbb{R}^{n_i}$. (We note that if X is connected, then $n_i = n_j$ for $i, j = 1, \ldots, r$). By Proposition (8.3), $E|_{U_i}$ has n_i linearly independent sections $s_1^{(i)}, \ldots, s_{n_i}^{(i)}$. Therefore, for $x \in U_i$, the vectors $s_1^{(i)}(x), \ldots, s_{n_i}^{(i)}(x) \in \mathbb{R}^{n_i}$ are n_i linearly independent vectors, which thus form a basis for $\mathbb{R}^{n_i}$. Hence, we note for all $x \in U_i, \xi \in E_x, \exists \lambda_1^{(i)}, \ldots, \lambda_n^{(i)} \in \mathbb{R}$ such that

$$\xi = \sum_{j=1}^{n_i} \lambda_j^{(i)} s_j^{(i)}(x). \tag{8.1}$$

From this, we build sections on all of E. Let $\{\eta_i\}$ be a partition of unity corresponding to $\{U_i\}$, then for $i = 1, \ldots, r, j = 1, \ldots, n_i$ define

$$\sigma_j^{(i)} := \eta_i s_j^i \in \Gamma(E)$$

by Observation (8.5). Therefore, we have $(n_1 + \cdots + n_r)$ sections. This will be our N in the definition of α. Indeed, define

$$\alpha : X \times \mathbb{R}^{n_1 + \cdots + n_r} \longrightarrow E$$

by

$$\alpha\left(x, \mu_1^{(1)}, \ldots, \mu_{n_1}^{(1)}, \ldots, \mu_1^{(r)}, \ldots, \mu_{n_r}^{(r)}\right) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \mu_j^{(i)} \sigma_j^{(i)}(x).$$

By Theorem (7.4), all that remains to check is that for $x \in X$, the map α_x is surjective. Now, there exists η_i such that $\eta_i(x) \neq 0 \implies x \in U_i$. By Equation (8.1), for $\xi \in E_x$, we have since $\eta_i(x) \neq 0$

$$\xi = \sum_{j=1}^{n_i} \lambda_j^{(i)} s_j^{(i)}(x) = \sum_{j=1}^{n_i} \frac{\lambda_j^{(i)} \sigma_j^{(i)}(x)}{\eta_i(x)}$$

and by definition of α,

$$\alpha\left(x, 0 \ldots, 0, \overbrace{\frac{\lambda_1^{(i)}}{\eta_i(x)}, \ldots, \frac{\lambda_{n_i}^{(i)}}{\eta_i(x)}}^{i^{th}\text{-block}}, 0 \ldots, 0 \right) = \xi.$$

$\square$

Remark 8.7. In general, we don't know what N is for the trivial bundle $X \times \mathbb{R}^N$ in the proof of Swan's theorem. But, we can find calculate an N (not necessarily the smallest N) in the case that the covering dimension of X is finite ($\dim_{\text{cov}} X < \infty$). This is an application of <u>Ostrand's theorem</u>, which states:

Let X be compact. Then, $\dim_{\text{cov}} X = n < \infty$ if and only if for every cover $\{\widetilde{W_i}\}$ there exists a refinement $\{W_i\}$ such that this refinement can be split into $n+1$ families, $\mathcal{W}_1, \ldots, \mathcal{W}_{n+1}$ such that for fixed $j = 1, \ldots, n+1$, we have that $U \cap V = \emptyset$ if $U, V \in \mathcal{W}_j$.

Now, if $W_i \cap W_j = \emptyset$ and $E|_{W_i}$ trivial and $E|_{W_j}$ trivial, then $E|_{W_i \sqcup W_j}$ trivial. With this observation in mind, in the proof of Swan's theorem, consider the cover $\{U_i\}$ such that U_1 is the union of all subsets in in the first family $\mathcal{W}_1$ and so on. In the connected case, this produces the following formula.

$$(\dim_{\text{cov}} X + 1) \cdot \dim(\text{fiber}).$$

If not connected, then by compactness there are only finitely many connected components and by locally triviality, we can replace dim(fiber) with the maximum dimension of all the fibers.

For example, if $E = M$, since the covering dimension of S^1 is 1 and the dimension of the fiber is 1, then $(1 + 1) \cdot 2$ implies that $M \oplus E' = [2]$.

In the case when $E = TS^2$. The covering dimension of S^2 is 2 and the dimension of the fiber is 2 since the fibers are planes from the second lecture. Thus, $(2+1) \cdot 2 = 6$ and $TS^2 \oplus E' = [6]$. But, of course, we were already able to do better. Indeed, $TS^2 \oplus [1] = [3]$.

Next, we move to the realm of modules.

An abelian group M is a (left)*module* over an algebra A (with product $\bullet$) if there is a map $(a, m) \in A \times M \longmapsto am \in M$ that satisifes, $(a_1 \bullet a_2)m = a_1(a_2 m)$ and $(a_1 + a_2)m = a_1 m + a_2 m$.

We already have an example of such a structure.

Proposition 8.8. *If E is a vector bundle over X, then $\Gamma(E)$ is a module over $C(X)$.*

Proof. Note that we already stated in the definition of sections that $\Gamma(E)$ is an abelian group. For the rest, use Observation (8.5). □

Let's cover some more examples of modules.

Example 8.9. (1) $\bigoplus_{j=1}^n C(X) =: C(X)^n$ is a module over $C(X)$ (free-module of rank n) by the operation $f(f_1, \ldots, f_n) = (ff_1, \ldots, ff_n)$

(2) Free-module Let A be an algebra and I a set. Then

$$A^I := \{(a_\alpha)_{\alpha \in I} : \text{only finitely many } a_\alpha \text{ are} \neq 0\}$$

is a module over A be the operation $a(a_\alpha)_{\alpha \in I} = (aa_\alpha)_{\alpha \in I}$.

(3) Let $X = [0, 1]$, then $C_0((0, 1])$ is a module over $C(X)$ since if $f(0) = 0$, then $fg(0) = 0$ for any $f \in C(X)$.

9. 29 September 2016

The goal of today is to state the Serre-Swan theorem and prove a part of it. This theorem provides an equivalence between the category of vector bundles over a fixed compact Hausdorff space and the category of finitely generated projective modules over $C(X)$, which is another important step to our move to K-theory of C*-algebras.

The majority of today will be taken up by proving a proposition that holds many of the important properties used to prove Serre-Swan. But, first some definitions.

Definition 9.1. Let M, N be modules over A. $\gamma : M \longrightarrow N$ is a *module homomorphism* if $\gamma(am) = a\gamma(m)$ for all $a \in A, m \in M$.

Definition 9.2. A module P over A is *projective* if it is a direct summand of a free module. That is, $\exists$ a module N over A such that $P \oplus N$ is free.

This definition is not the same a saying that P is a submodule of a free module. Indeed, $C_0((0,1])$ is a submodule of the free module $C([0,1])$ but it is not a direct summand of $C([0,1])$. This is because $C([0,1])$ has codimension 1 in $C([0,1])$ and thus any vector space to complement of $C_0((0,1])$ in $C([0,1])$ has the form $C_0((0,1]) \oplus \mathbb{C}f_0 = C([0,1])$ for some $f_0 \in C([0,1])$. But, $\mathbb{C}f_0$ is not a submodule of $C([0,1])$. This is because there exists $f \in C([0,1])$ such that $f_0 f \notin \mathbb{C}f_0$.

Next, we give an equivalent definition to projective module, in which the proof of equivalence is left as an exercise.

Definition 9.3. A module P over A is projective if for all modules M over A and any surjective module homomorphism $\gamma : M \longrightarrow P$ there exists a module homomorphism $h : P \longrightarrow M$ such that $\gamma \circ h = id_P$.

We note that we always assume that the algebra A is unital.

Definition 9.4. A module M over A is *finitely generated* if there exists finitely many $m_1, \ldots, m_n \in M$ such that for each $m \in M$ there exist $a_1, \ldots, a_n \in A$ such that

$$m = \sum_{j=1}^{n} a_j m_j.$$

Example 9.5.

(1) The free module of rank n, A^n is n-generated. Consider the elements of the form $(0, \ldots, 0, \underbrace{1}_{j}, 0, \ldots, 0)$ for $j = 1, \ldots, n$.

(2) (non-example) $C_0((0,1])$ is not a finitely generated module over $C([0,1])$. The proof of this is left as an exercise.

Now, we are ready to state the large proposition.

Proposition 9.6. *Let E, F be vector bundles over a compact Hausdorff space X.*

(1) $\Gamma(E)$ *is a module over* $C(X)$.
(2) *A (iso)morphism* $\varphi : E \longrightarrow F$ *induces a module (iso)homomorphism* $\varphi_* : \Gamma(E) \longrightarrow \Gamma(F)$.
(3) $\Gamma([n]) \cong C(X)^n$.
(4) $\Gamma(E \oplus F) = \Gamma(E) \oplus \Gamma(F)$.
(5) $\Gamma(E)$ *is a finitely generated projective module.*

Proof of (1). was done yesterday as Proposition (8.8). ☐

Proof of (2). Consider the following diagram where $\varphi : E \longrightarrow F$ is a morhpism and $s \in \Gamma(E)$.

$$\begin{tikzcd} E \arrow[r, "\varphi"] & F \\ X \arrow[u, "s"] & \end{tikzcd}$$

Therefore, $\varphi \circ s \in \Gamma(F)$ since $\varphi \circ s(x) = \varphi(\underbrace{s(x)}_{\in E_x}) \in F_x$ by definition of morphism. Thus, define

$$\varphi_* : s \in \Gamma(E) \longmapsto \varphi \circ s \in \Gamma(F),$$

which we already showed to be well-defined. For the module homomoprhism property. Let $f \in C(X), s \in \Gamma(E)$. Recall that φ is linear on fibers.Therefore,

$$\begin{aligned} \varphi_*(fs)(x) &= (\varphi \circ (fs))(x) \\ &= \varphi(\underbrace{f(x)s(x)}_{\in E_x}) \\ &= \varphi_x(f(x)s(x)) \\ &= f(x)\varphi_x(s(x)) \\ &\implies \varphi_*(fs) = f\varphi_*(s). \end{aligned}$$

If φ were an isomorphism, then $\varphi_*^{-1} = \left(\varphi^{-1}\right)_*$. □

Proof of (3). Let $[n] = X \times \mathbb{C}^n$. Fix $s \in \Gamma([n])$. Then, s is of the form $s(x) = (x, f_s(x))$ for each $x \in X$, where $f_s(x) \in \mathbb{C}^n$. But then , $f_s \in C(X, \mathbb{C}^n) = \oplus_{j=1}^n C(X) = C(X)^n$. Thus, there exist $f_1, \dots, f_n \in C(X)$ such that $f_s(x) = (f_1(x), \dots, f_n(x))$. Therefore, it follows that

$$s \in \Gamma([n]) \longmapsto f_s \in C(X)^n$$

is a bijection, and the fact that this map is a module homomorphism follows easily. □

Proof of (4). Let $s \in \Gamma(E \oplus F)$. Then, recalling the properties of the (Whitney) sum of vector bundles, we have that $s(x) \in (E \oplus F)_x = E_x \oplus F_x$. Hence, define $s_1 : X \longrightarrow E, s_2 : X \longrightarrow F$ by $s(x) = (s_1(x), s_2(x)) \in E_x \oplus F_x$ for all $x \in X$. Then, $s_1 \in \Gamma(E), s_2 \in \Gamma(F)$. □

Proof of (5). First, we establish the following claim.

Claim 9.7. *A direct summand of a finitely generated module is finitely generated.*

Proof of claim. Assume that M, N are modules over A and R is a finitely generated module over A such that $M \oplus N = R$. Let $r_1, \dots, r_n \in R$ be a finte set of generators. Then, for each $j = 1, \dots, n$ we have $r_j = (m_j, n_j)$ and $m_1, \dots, m_n$ is a finite gerating set for M. □

By Swan's theorem, there exists a vector bundle E' over X such that $E \oplus E' = [n]$ for some n. Therefore,

$$C(X)^n \overset{(3)}{\cong} \Gamma([n]) = \Gamma(E \oplus E') \overset{(4)}{=} \Gamma(E) \oplus \Gamma(E').$$

Hence, by the claim, since $C(X)^n$ is finitely generated, we have that $\Gamma(E)$ is finitely generated projective. □

As a corollary to this proposition, we have.

Corollary 9.8. *Let X be a compact Hausdorff space. Then, $\Gamma : E \longmapsto \Gamma(E)$ is a covariant functor from the category of vector bundles over X with morphisms to the category of finitely generated projective modules over $C(X)$.*

Proof. It remains to check composition. $(\varphi \circ \psi)_*(s) = \varphi \circ \psi \circ s = \varphi_*(\psi_*(s))$. □

Remark 9.9. If X were not compact (locally compact), then we could not use Swan's theorem, which in the compact case shows that $\Gamma(E)$ is a finitely generated projective module. Also, Swan's theorem is used to provide injectivity in the following result.

Now, we state the Serre-Swan theorem, and proof some of it today.

Theorem 9.10 (Serre-Swan Theorem)**.** *With the same setting of the previous Corollary, the map $\Gamma : E \longmapsto \Gamma(E)$ is a bijection.*

Proof. We start with injectivity. So, we will show that if $\Gamma(E) \cong \Gamma(F)$, then $E \cong F$. We will begin by showing something stronger. We will show that any module homomorphism $\gamma : \Gamma(E) \longrightarrow \Gamma(F)$ is induced by some morphism $\varphi : E \longrightarrow F$ such that $\gamma = \varphi_*$. We will then establish the same for isomorphisms. First, we show take care of the case of trivial bundles.

Lemma 9.11. *If $\gamma : \Gamma([n]) \longrightarrow \Gamma([m])$ is a module homomorphism, then there exists a morphism $\varphi : E \longrightarrow F$ such that $\varphi_* = \gamma$.*

Proof of Lemma. Let $[n] = X \times \mathbb{C}^n$ (or $\mathbb{R}^n$) and $[m] = X \times \mathbb{C}^m$ (or $\mathbb{R}^m$).

Let $v_1, \ldots, v_n \in \mathbb{C}^n$ be a basis and $w_1, \ldots, w_m \in \mathbb{C}^m$ be a basis. We define constant sections $\overline{v_i} \in \Gamma([n])$ by $\overline{v_i}(x) = (x, v_i)$ and similarly define $\overline{w_i} \in \Gamma([m])$. Let $s \in \Gamma([m])$. Fix $x \in X$, then $s(x) = (x, y)$ such that $y \in \mathbb{C}^m$. Hence, there exist $\lambda_1(x), \ldots, \lambda_m(x) \in \mathbb{C}$ such that

$$s(x) = (x, y) = \left(x, \sum_{i=1}^{m} \lambda_m(x) w_i\right).$$

Therefore, $s = \sum_{i=1}^{m} \lambda_i(\cdot)\overline{w_i}$ and we only need to see how γ acts on the constant sections.

$$\gamma(\overline{v_i}) = \sum \lambda_j^i(\cdot)\overline{w_j}.$$

Now, by linearity, we only need to define the morphism $\varphi : [n] \longrightarrow [m]$ on the basis. So, define

$$\varphi_x(v_i) := \sum_{j=1}^{m} \lambda_j^i(x) w_j. \tag{9.1}$$

It is left as an exercise to check that $\varphi_* = \gamma$. □

Now, let $\gamma : \Gamma(E) \longrightarrow \Gamma(F)$. By Swan's theorem $E \oplus E' = [n]$ and $F \oplus F' = [m]$. In particular, by (4) of Proposition (9.6), we have that $\Gamma(E) \oplus \Gamma(E') = \Gamma([n])$ and $\Gamma(F) \oplus \Gamma(F') = \Gamma([m])$. Thus, we may define

$$\begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix} : \Gamma([n]) \longrightarrow \Gamma([m]).$$

By the above lemma, there exists $\varphi : [n] \longrightarrow [m]$ such that $\varphi_* = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}$. But, as $E \oplus E' = [n]$ and $F \oplus F' = [m]$, we may write $\varphi = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_3 & \varphi_4 \end{pmatrix}$, where $\varphi_1 : E \longrightarrow F$. And,

$$\varphi_* = \begin{pmatrix} \varphi_{1*} & \varphi_{2*} \\ \varphi_{3*} & \varphi_{4*} \end{pmatrix} = \begin{pmatrix} \gamma & 0 \\ 0 & 0 \end{pmatrix}.$$

implies that $\varphi_{1*} = \gamma$.

Tomorrow, we will begin by verifying the case of isomorphism. □

10. 30 September 2016

Today we finish the proof of the Serre-Swan Theorem (9.10).

Continuation of proof of Theorem (9.10). Last time we proved that a module homomorphism $\gamma : \Gamma(E) \longrightarrow \Gamma(F)$ is induced by a morphism $\varphi : E \to F$ such that $\gamma = \varphi_*$.

Today, we begin by show that if γ is an isomorphism then so is φ. Assume that γ is an isomorphism, then $\gamma^{-1} : \Gamma(F) \longrightarrow \Gamma(E)$ is an isomoprhism and there exists a morphism $\psi : F \longrightarrow E$ such that $\gamma^{-1} = \psi_*$. But, then

$$(\varphi \circ \psi)_* = \varphi_* \circ \psi_* = id_{\Gamma(F)} \text{ and } (\psi \circ \varphi)_* = id_{\Gamma(E)}.$$

Therefore, the following claim would complete our proof of injectivity of the functor γ.

Claim 10.1. *If $\delta_* = id$, then $\delta = id$.*

Proof of claim. By definition and assumption, $\delta_*(s) = \delta \circ s = s$ for all $s \in \Gamma(E)$. In particular, we have that if $x \in X$, then $s(x) \in E_x$ and $\delta(s(x)) = \delta_x(s(x)) = s(x)$. But, we want $\delta(x) = e$ where $e \in E_x$. Thus, it is enough to find $s \in \Gamma(E)$ such that $s(x) = e$.

Fix $e \in E = \sqcup_{x \in X} E_x$. So, there exists $x \in S$ such that $e \in E_x$. By local triviality, let U_x be a neighborhood of x such that $E|_{U_x}$ is trivial. Let $\widetilde{e}(x)$ denote the constant section on $E|_{U_x}$. Let η be a function with support U_x such that $\eta(x) = 1$. Define $s = \eta\widetilde{e}$. □

Therefore, injectivity is complete.

Surjectivity: Let M be a finitely generated projective module over $C(X)$. Assume that M is n-generated. We want to show that there exists a vector bundle over X such that $\Gamma(E) = M$. To prove this, we use a lemma.

Lemma 10.2. *If M is an n-generated projective module over A, then M is a direct summand of A^n (the free module of rank n.)*

Proof. We will use the second definition of projective module Definition (9.3). Let $m_1, \ldots, m_n$ be the generators of M. Define a surjection $\gamma : A^n \longrightarrow M$ by

$$\gamma(a_1, \ldots, a_n) := \sum_{j=1}^{m} a_j m_j.$$

Thus, since M is projective, there exists $h : M \longrightarrow A^n$ such that $\gamma \circ h = id_M$. Therefore, one can show that

$$x \in A^n \longmapsto (\gamma(x), x - h \circ \gamma(x)) \in M \oplus \ker \gamma$$

is an isomorphism. $\square$

By the lemma, $M \oplus R = C(X)^n$. We define a skew projection

$$p : (m, r) \in C(X)^n \longmapsto (m, 0) \in C(X)^n,$$

and thus $p^2 = p$. But, by Proposition (9.6) (3), $p : \Gamma([n]) \longrightarrow \Gamma([m])$ is a morphism. Therefore, by the part of the proof of Serre-Swan from yesterday there exists $\varphi : [n] \longrightarrow [n]$ such that $p = \varphi_*$. And, $\varphi^2 = \varphi$ is idempotent since p is idempotent. Therefore, by Theorem (7.1), we have that Ran φ is a vector bundle.

We will show that $M \cong \Gamma(\text{Ran } \varphi)$. We can already day that $M = \text{Ran } p$. But,

$$\text{Ran } p = \text{Ran } \varphi_* = \{\varphi \circ s : s \in \Gamma([n])\} = \{x \mapsto \varphi_x(s(x)) : s \in C(X, \mathbb{C}^n)\}.$$

And, $\Gamma(\text{Ran } \varphi) = \{g : X \longrightarrow \mathbb{C}^n : g(x) \in \text{Ran } \varphi_x\}$. Thus, we have that Ran $p \subseteq \Gamma(\text{Ran } \varphi)$. Let $g \in \Gamma(\text{Ran } \varphi)$. Since φ is idempotent and $g(x) \in \text{Ran } \varphi_x$, we have

$$pg(x) = \varphi_x(g(x)) = g(x).$$

Thus, $pg = g$ and $g \in \text{Ran } p$, which completes the proof. $\square$

Corollary 10.3. *Let X be a compact Hausdorff space. Then, $V(X) \cong$ the semigroup of finitely generated projective modules over $C(X)$. Hence,*

$$K_0(X) = \sigma(\textit{the semigroup of finitely generated projective modules over } C(X)).$$

Now, the right-hand side of $V(X)$ of the above corollary is a statement that makes sense for any ring, and this gives algebraic K_0.

Definition 10.4 (Algebraic K_0)**.** Let A be a ring. Define

$$K_0^{\text{alg}}(A) := \sigma(\text{semigroup of finitely generated projective modules over } A).$$

Corollary 10.5. $K^0(X) = K_0^{\text{alg}}(C(X))$.

To move to C*-algebras we will formulate algebraic K_0 in C*-algebraic terms. We begin with a definition.

Definition 10.6. Let A be a C*-algebra. We define $M_\infty(A) := \cup_{n\in\mathbb{N}} M_n(A)$, where for each $n \in \mathbb{N}$, we have $M_n(A) \hookrightarrow M_{n+1}$ by the map $a \longmapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

Also, if $p \in M_n(A), q \in M_m(A)$ are projections ($p^2 = p = p^*$), then $p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \in M_{n+m}(A)$ is a projection.

Theorem 10.7. *Let A be a unital C*-algebra. The following three semigroups are isomorphic.*

(1) $V_1(A) = \{[M] : M$ *is a finitely generated projective module over* $A\}$, *where* $[M] = [M']$ *when* $M \cong M$ *as modules.*
(2) $V_2(A) = \{[q] : q \in M_\infty(A), q^2 = q\}$ *and* $[q] = [q']$ *if there exist* $x, y \in M_\infty(A)$ *such that* $q = xy$ *and* $q' = yx$.
(3) $V_3(A) = \{[p] : p \in M_\infty(A), p^2 = p = p^*\}$, *where* $[p] = [p']$ *if there exists* $v \in M_\infty(A)$ *such that* $p = vv^*$ *and* $p' = v^*v$.

We will prove these isomorphisms and use (3) to define K-theory for C*-algebras.

Proof. (1) $\iff$ (2). Let M be a finitely generated projective module of d-generators. By Lemma (10.2), we have that $M \oplus N = A^d$. Define $p_M : A^d \longrightarrow A^d$ by $p_M((m, n)) := (m, 0)$, which is a module homomorphism.

Lemma 10.8. *Consider A as the rank 1 module A^1. If $f : A \longrightarrow A$ is module homomorphism, then there exists $a_0 \in A$ such that $f(a) = aa_0$ for all $a \in A$.*

Proof. By definition, $f(ab) = af(b)$. If $b = 1$, then for all $a \in A$, we have

$$f(a) = f(a1) = af(1).$$

Let $a_0 = f(1)$. □

We will continue the proof the next day... □

11. 3 October 2016

We continue with the proof of the following theorem.

Theorem 11.1. *Let A be a unital C*-algebra. The following three semigroups are isomorphic.*

(1) $V_1(A) = \{[M] : M$ *is a finitely generated projective module over* $A\}$, *where* $[M] = [M']$ *when* $M \cong M$ *as modules.*
(2) $V_2(A) = \{[q] : q \in M_\infty(A), q^2 = q\}$ *and* $[q] = [q']$ *if there exist* $x, y \in M_\infty(A)$ *such that* $q = xy$ *and* $q' = yx$.

(3) $V_3(A) = \{[p] : p \in M_\infty(A), p^2 = p = p^*\}$, *where* $[p] = [p']$ *if there exists* $v \in M_\infty(A)$ *such that* $p = vv^*$ *and* $p' = v^*v$.

Proof. (1) $\iff$ (2). Let M be a finitely generated projective module of d-generators. By Lemma (10.2), we have that $M \oplus N = A^d$. Define $p_M : A^d \longrightarrow A^d$ by $p_M((m,n)) := (m, 0)$, which is a module homomorphism. Last time we already prooved part (1) of the following lemma.

Lemma 11.2. *Consider A as the rank 1 module, A^1.*

(1) *If $f : A \longrightarrow A$ is module homomorphism, then there exists $a_0 \in A$ such that $f(a) = aa_0$ for all $a \in A$.*
(2) *If $f : A^d \longrightarrow A^d$ is a module homomorphism, then there exists $b = (b_{ij}) \in M_d(A)$ such that*

$$f\begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^d a_j b_{1j} \\ \vdots \\ \sum_{j=1}^d a_j b_{dj} \end{pmatrix}.$$

Proof of part (2) of Lemma. f can be written as $f = \begin{pmatrix} f_{11} & \cdots & f_{1d} \\ \vdots & \ddots & \vdots \\ f_{d1} & \cdots & f_{dd} \end{pmatrix}$ where $f_{ij} : A \longrightarrow A$. It follows easily from f being a module homomoprhism that all f_{ij}'s are module homomorphisms.

Thus, by part (1) of the Lemma, there exist $b_{ij} \in A$ such that $f_{ij}(a) = ab_{ij}$ for all $a \in A$. Therefore,

$$\begin{aligned} f\begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} &= \begin{pmatrix} f_{11} & \cdots & f_{1d} \\ \vdots & \ddots & \vdots \\ f_{d1} & \cdots & f_{dd} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=1}^d f_{1j}(a_j) \\ \vdots \\ \sum_{j=1}^d f_{dj}(a_j) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=1}^d a_j b_{1j} \\ \vdots \\ \sum_{j=1}^d a_j b_{dj} \end{pmatrix}. \end{aligned}$$

□

Hence, by the Lemma, p_M is defined by some matrix $(b_{ij}) \in M_d(A)$. Also, $p_M^2 = p_M \implies (b_{ij})^2 = (b_{ij})$ and is thus an idempotent in $M_\infty(A)$. To construct our map from $V_1(A)$ to $V_2(A)$, we use the following claim.

Claim 11.3. *If M, M' are two finitely generated projective modules over A^d such that $M \cong M'$, then $(b_{ij}) \sim (b'_{ij})$ as idempotents.*

Proof of claim. By construction of the (b_{ij}), is is enough to show that there exist $F, G : A^d \longrightarrow A^d$ such that $p_M = FG$ and $p_{M'} = GF$.

Now, by projective modules, $M \oplus N = A^d$ and $M' \oplus N' = A^d$. By assumption, there exist isomorphisms $F_0 : M' \longrightarrow M$ and $G_0 : M \longrightarrow M'$. On A^d, define

$$F(a) = \begin{cases} F_0(a) & : a \in M' \\ 0 & : a \in N' \end{cases} \text{ and } G(a) = \begin{cases} G_0(a) & : a \in M \\ 0 & : a \in N \end{cases},$$

then $FG(a) = \begin{cases} F_0G_0(a) = a & : a \in M \\ 0 & : a \in N \end{cases} = p_M(a)$. Simliarly, $GF = p_{M'}$. □

By the claim, the map $[M] \in V_1(A) \longmapsto [(b_{ij})] \in V_2(A)$ is well-defined.

Injectivity: We want to show that $(b_{ij}) \sim (b'_{ij}) \implies M \cong M'$. But, by construction, we can see that $(b_{ij}) \sim (b'_{ij}) \iff p_M \sim p_{M'}$ in the sense that $p_M = FG$ and $p_{M'} = GF$ for $G, F : A^d \longrightarrow A^d$. Now, $G|_M : M \longrightarrow A^d$ and $F|_{M'} : M' \longrightarrow A^d$. Next, we check that $G(M) \subseteq M'$ and $F(M') \subseteq M$. For the first containment,

$$\begin{aligned} p_M = FG &\implies M = p_M(M) = FG(M) \\ &\implies G(M) = GFG(M) = p_{M'}G(M) \\ &\implies G(M) \subseteq M'. \end{aligned}$$

The argument is similar for $F(M') \subseteq M$.

Therefore, $F|_{M'} \circ G|_M = (F \circ G)|_M = p_M|_M = id_M$ and $G|_M \circ F|_{M'} = id_{M'}$, which shows that $M \cong M'$.

Surjectivity: Let $(b_{ij}) \in M_\infty(A)$ be an idempotent. There exists d such that $(b_{ij}) \in M_d(A)$. Define $p : A^d \longrightarrow A^d$ by

$$p \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} \sum a_j b_{1j} \\ \vdots \\ \sum a_j b_{dj} \end{pmatrix}.$$

By idempotence, $A^d = \text{Ran } p \oplus \ker p$. Now, Ran p is a module since

$$ap \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} = p \left(a \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} \right) = p \begin{pmatrix} aa_1 \\ \vdots \\ aa_d \end{pmatrix} \in \text{Ran } p.$$

For $\ker p$. $p \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} = 0 \implies p \begin{pmatrix} aa_1 \\ \vdots \\ aa_d \end{pmatrix} = ap \begin{pmatrix} a_1 \\ \vdots \\ a_d \end{pmatrix} = 0$. Hence, Ran p is a finitely generated projective module such that $p_{\text{Ran } p} = p$ and Ran $p \longmapsto (b_{ij})$.

(2) $\iff$ (3) We start with 2 steps.

Step 1. *Each idempotent q is equivalent (as idempotents) to some projection p.*

Step 2. *If p_1, p_2 are projections that are equivalent as idempotents, then they are equivalent as projections.*

We will prove Step 1 now and leave Step 2 as an exercise, which is exercise 5B(f) from Wegge-Olsen.

Proof of Step 1. Assume that $q^2 = q$. Define $s := q^*q + (1-q)^*(1-q)$. One can show that

$$q^*s = q^*q = sq. \tag{11.1}$$

Now,

$$s = 1 - q^* - q + 2q^*q = \frac{1}{2} + \frac{1}{2}(1+2q)^*(1+2q).$$

But, $\frac{1}{2}(1+2q)^*(1+2q) \geqslant 0$ is positive. Therefore, by spectral mapping theorem, s is positive such that $0 \notin \operatorname{spec}(s)$ (the spectrum of s), and thus s is invertible. Furthermore, s^{-1} is positive and has a unique positive square root $s^{-1/2}$. Now,

$$\begin{aligned} \text{Equation (11.1)} &\implies s^{-1/2}(q^*s)s^{-1/2} = s^{-1/2}(sq)s^{-1/2} \\ &\implies s^{-1/2}q^*s^{1/2} = s^{1/2}qs^{-1/2} \\ &\implies (s^{1/2}qs^{-1/2})^* = s^{1/2}qs^{-1/2} \end{aligned} \tag{11.2}$$

Next, define $p = s^{1/2}qs^{-1/2}$. Note that $q = s^{-1/2}ps^{1/2}$. By Expression (11.2) and idempotence of q, we have $p^* = p = p^2$, which is a projection. Furthermore,

$$p = p^2 = (ps^{1/2})(qs^{-1/2}) = (a)(b)$$
$$q = q^2 = q(s^{-1/2}ps^{1/2}) = (b)(a).$$

Therefore, p is a projection such that $p \sim q$ as idempotents. □

□

12. 4 October 2016

Motivated by Theorem (11.1), we define.

Definition 12.1. Let $\mathfrak{A}$ be a unital C*-algebra. Define

$$K_0(\mathfrak{A}) := \sigma\left(V_3(\mathfrak{A})\right),$$

the Grothendieck group of the semigroup $V_3(A)$.

Thus, in particular, we have that $K_0(C(X)) = K^0(X)$, where X is compact Hausdorff. We present some other examples.

Example 12.2. For the first 3 examples, we note that projections in $B(H)$ (C*-algebra of bounded operators on a Hilbert space) are equivalent if they have the same rank.

(1) $K_0(\mathbb{C}) = \sigma(\mathbb{N}_0) = \mathbb{Z}$.
(2) $K_0(M_n(\mathbb{C})) = \mathbb{Z}$, which follows from the first example since $M_\infty(\mathbb{C}) = M_\infty(M_n(\mathbb{C}))$.
(3) If H is infinite dimensional then there exist projections with infinite rank and $K_0(B(H)) = \sigma(\mathbb{N}_0 \cup \{\infty\}) = 0$.
(4) This example considers von Neumann algebras. In a II_1-factor, prjections are equivalent when the trace are the same. Thus, $K_0(\mathrm{II}_1\text{-factor}) = \sigma(\mathbb{R}_+) = \mathbb{R}$.

    In a II_∞-factor, semi-finite traces can take infinite value, and thus, $K_0(\mathrm{II}_\infty\text{-factor}) = \sigma(\mathbb{R}_+ \cup \{\infty\}) = 0$.

In a III-factor, all non-zero projections are equivalent. Hence, $K_0(\text{III-factor}) = \sigma(0 \cup \{\infty\}) = 0$.

(5) (Cuntz) $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$, where $\mathcal{O}_n$ is the Cuntz algebra.

Example (4) shows that K-theory is not useful in the setting of von Neumann algebras.

Theorem 12.3. *K_0 is a covariant functor.*

Proof. Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a *-homomorphism. We may extend φ to a map from $M_n(\mathfrak{A})$ to $M_n(\mathfrak{B})$ by defining $(a_{ij}) \in M_n(\mathfrak{A}) \longmapsto (\varphi(a_{ij}) \in M_n(\mathfrak{B})$. Thus, for $p \in M_n(\mathfrak{A})$ a projection, we have that $\varphi(p) \in M_n(\mathfrak{B})$ is a projection. Hence, $[p] \in V_3(\mathfrak{A}) \longmapsto [\varphi(p)] \in V_3(\mathfrak{B})$ is well-defined. And, by a previous lecture, we may uniquely extend this map to the associated Grothendieck groups. Denote this map by

$$K_0(\mathfrak{A}) \xrightarrow{K_0(\varphi)} K_0(\mathfrak{B}).$$

$\square$

Next, we provide K_0 for nonunital C*-algebras. Assume that $\mathfrak{A}$ is a non-unital C*-algebra. Let $\mathfrak{A}^+$ denote its unitlization (with multiplication $(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$). Consider

$$\mathbb{C} \overset{i}{\hookrightarrow} \mathfrak{A}^+ \overset{p}{\twoheadrightarrow} \mathbb{C},$$

where $i(\lambda) = (0, \lambda)$ and $p(a, \lambda) = \lambda$ and note that $p \circ i = id_{\mathbb{C}}$. By Theorem (12.3) and $K_0(\mathbb{C}) = \mathbb{Z}$, we have

$$\mathbb{Z} \overset{K_0(i)}{\hookrightarrow} K_0(\mathfrak{A}^+) \overset{K_0(p)}{\twoheadrightarrow} \mathbb{Z},$$

and thus, $K_0(p) \circ K_0(i) = id_{\mathbb{Z}}$. Hence,

$$K_0(\mathfrak{A}^+) = \mathbb{Z} \oplus \ker K_0(p).$$

This allows us to define K_0 for nonunital C*-algebras.

Definition 12.4. Let $\mathfrak{A}$ be a nonunital C*-algebra. Define $p : \mathfrak{A}^+ \longrightarrow \mathbb{C}$ by $p(a, \lambda) = \lambda$. Define

$$K_0(\mathfrak{A}) := \ker K_0(p).$$

(This works also for unital $\mathfrak{A}$ assuming that $\mathfrak{A}^+ = \mathfrak{A} \oplus \mathbb{C}$.)

Hence, if X is not compact but locally compact, then $C_0(X)$ is nonunital. Also, $C(X^+) = (C_0(X))^+$, where X^+ denotes the one-point compactification of X. Therefore,

$$K_0(C_0(X)) = K^0(X).$$

Proposition 12.5. *K_0 is a covariant functor.*

Proof. Let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$. Extend φ to $\varphi^+ : \mathfrak{A}^+ \longrightarrow \mathfrak{B}^+$ by $\varphi^+(a, \lambda) = (\varphi(a), \lambda)$. Consider

$$K_0(\mathfrak{A}^+) \xrightarrow{K_0(\varphi^+)} K_0(\mathfrak{B}^+).$$

Let $p_{\mathfrak{A}} : (a, \lambda) \in \mathfrak{A}^+ \longmapsto \lambda \in \mathbb{C}$ and similarly for $p_{\mathfrak{B}}$. It remains to check that $K_0(\varphi^+)$ sends $\ker K_0(p_{\mathfrak{A}})$ to $\ker K_0(p_{\mathfrak{B}})$. Let $[(a_{ij}, \lambda_{ij})] - [(b_{ij}, \mu_{ij})] \in K_0(\mathfrak{A}^+)$. Then,

$$K_0(p_{\mathfrak{A}})([(a_{ij}, \lambda_{ij})] - [(b_{ij}, \mu_{ij})]) = [(\lambda_{ij})] - [(\mu_{ij})],$$

which equals 0 if $\lambda_{ij} = \mu_{ij}$. Hence, let $[(a_{ij}, \lambda_{ij})] - [(b_{ij}, \lambda_{ij})] \in \ker(\varphi^+)(p_{\mathfrak{A}})$. Then,

$$K_0(\varphi^+)\left([(a_{ij}, \lambda_{ij})] - [(b_{ij}, \lambda_{ij})]\right) = [(\varphi(a_{ij}), \lambda_{ij})] - [(\varphi(b_{ij}), \lambda_{ij})] \in \ker K_0(p_{\mathfrak{B}}).$$

□

Properties of the functor K_0:

(1) Homotopy invariance: Let $f, g : \mathfrak{A} \longrightarrow \mathfrak{B}$ be homotopic. That is, for all $t \in [0,1]$ there exist $f_t : \mathfrak{A} \longrightarrow \mathfrak{B}$ such that f_t is a continuous path with $f_0 = f$ and $f_1 = g$. (Idea of proof: fix a projection $p \in M_n(\mathfrak{A})$. Extend the f_t to $M_n(\mathfrak{A})$ we have that $f_t(p)$ forms a continuous path of projections. But, if 2 projections are close, then they are equivalent. Thus, $[f(p)] = [g(p)]$ and $K_0(f) = K_0(g)$.)

(2) Continuity: $K_0(\varinjlim \mathfrak{A}_n) = \varinjlim K_0(\mathfrak{A}_n)$. (Idea of proof: let $p \in M_d(\mathfrak{A}) = \overline{\cup_n M_d(\mathfrak{A}_n)}$. One can prove there exists $p_n \in M_d(\mathfrak{A}_n)$ such that $p - p_n$ is small, which implies equivalence. Thus, $[p] = [p_n]$, where $[p_n] \in K_0(\mathfrak{A}_n)$.)

(3) Half-exactness: Assume that we have a short exact sequence

$$0 \longrightarrow \mathfrak{A} \longrightarrow \mathfrak{B} \longrightarrow \mathfrak{D} \longrightarrow 0,$$

then the following sequence is exact

$$K_0(\mathfrak{A}) \longrightarrow K_0(\mathfrak{B}) \longrightarrow K_0(\mathfrak{D}).$$

(4) Stability: $K_0(\mathfrak{A}) \cong K_0(\mathfrak{A} \otimes K(H))$, where $K(H)$ is the C*-algebra of compact operators. The isomorphism is given by $[p] \longmapsto [p] \otimes [1]$, where 1 denotes a rank 1 projection. (Vague explanation: $K(H)$ is AF and $K(H) = \varinjlim M_n(\mathbb{C})$. Hence,

$$\mathfrak{A} \otimes K(H) = \mathfrak{A} \otimes \varinjlim M_n(\mathbb{C}) = \varinjlim (\mathfrak{A} \otimes M_n(\mathbb{C})) = \varinjlim M_n(\mathfrak{A})).$$

The next theorem shows that the above properties along with values on the C*-algebras $\mathbb{C}$ and $C_0(\mathbb{R})$ completely characterize the functor K_0.

Theorem 12.6 (Cuntz). *If K is a covariant functor from from the category of C*-algebras to the category of abelian groups that satisfies properties (1)-(4) above and $K(\mathbb{C}) = \mathbb{Z}, K(C_0(\mathbb{R})) = 0$, then $K(\mathfrak{A}) = K_0(\mathfrak{A})$ for all C*-algebras $\mathfrak{A}$ from a bootstrap class.*

Definition 12.7. $K_1(\mathfrak{A}) := K_0(\mathfrak{A} \otimes C_0(\mathbb{R}))$ and $K_n(\mathfrak{A}) = (\mathfrak{A} \otimes C_0(\mathbb{R}^n))$, but Bott periodicity for C*-algebras gives that $K_2(\mathfrak{A}) = K_0(\mathfrak{A})$.

13. 5 October 2016

Today we present some standard results for Fredholm operators in preparation for Brown-Douglas-Filmore theory.

Notation 13.1. Let $\mathcal{H}$ be a Hilbert space.

$\mathcal{B}(\mathcal{H}) = \{T : H \longrightarrow H : T \text{ is linear and bounded }\}$.

Let $O_1(H) := \{h \in H : \|h\|_H \leqslant 1\}$ denote the unit ball of H. Define

$$\mathfrak{K}(\mathcal{H}) = \left\{T \in \mathcal{B}(\mathcal{H}) : \overline{T\left(O_1(H)\right)}^{\|\cdot\|_{\mathcal{B}(\mathcal{H})}} \text{ is compact in norm}\right\}.$$

$\mathfrak{K}(\mathcal{H})$ is an ideal (norm-closed two-sided ideal) of $\mathcal{B}(\mathcal{H})$. Then, $\mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ is a C*-algebra called the Calkin algebra denoted by $\mathcal{Q}(\mathcal{H})$.

Let $\pi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ denote the quotien surjection. If $T \in \mathcal{B}(\mathcal{H})$, then instead of $\pi(T)$ we write $\dot{T}$.

Definition 13.2. $T \in \mathcal{B}(\mathcal{H})$ is *Fredholm* if $\dim\ker T < \infty$, $\dim\ker T^* < \infty$, and Ran T is closed.

Denote set of all Fredholm operators by $\mathfrak{F}(\mathcal{H})$.

Index of a Freholm operator T is

$$\begin{aligned} j(T) &:= \dim\ker T - \dim\ker T^* \\ &= \dim\ker T - \text{codim Ran } T \end{aligned}$$

since $\ker T^* = (\text{Ran } T)^\perp$.

Let's note some examples and observations.

Observation 13.3.

(1) If A is invertible, then $j(A) = 0$.

(2) Let $U_+ \in \mathcal{B}(\mathcal{H})$ denote the unilateral shift. That is,

$$U_+(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

Hence, $\ker U_+ = 0$ and it's range is closed of codimension 1. Thus, $j(U_+) = -1$.

(3) $j(T^*) = -j(T)$.

(4) $j\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} = j(T) + j(S)$.

(5) If T is Fredholm and S is invertible, then

$$j(TS) = j(ST) = j(T).$$

(6) If $T : \mathbb{C}^n \longrightarrow \mathbb{C}^m$, then

$$\begin{aligned} j(T) &= \dim\ker T - \text{codim Ran } T \\ &= \dim\ker T - (m - \dim \text{Ran } T) \\ &= \dim\ker T + \dim \text{Ran } T - m \\ &= n - m \end{aligned}$$

since $\mathbb{C}^n/\ker T = \text{Ran } T$. Thus, the finite dimensional case is not of interest.

Theorem 13.4. *T is Fredholm if and only if $\dot{T}$ is invertible.*

Proof. <u>If:</u> Assume that $\dot{T}$ is invertible. So, there exists $S \in \mathcal{B}(\mathcal{H})$ such that $\dot{T}\dot{S} = 1_{\mathcal{Q}(\mathcal{H})} = \dot{S}\dot{T}$, and thus $(TS)^{\cdot} = 1_{\mathcal{Q}(\mathcal{H})} = (ST)^{\cdot}$. Hence, there exist $K, K' \in \mathfrak{K}(\mathcal{H})$ such that

$$TS = 1_{\mathcal{B}(\mathcal{H})} + K \text{ and } ST = 1_{\mathcal{B}(\mathcal{H})} + K'.$$

Claim 13.5. $\ker T$ *is finite dimensional.*

Proof of claim. Suppose $\ker T$ is not finite dimensional. Thus, there exists an infinite orthonormal basis $\{x_n\}_{n=1}^{\infty}$ for $\ker T$. Therefore, for each $n \in \mathbb{N}$, we have

$$Tx_n = 0 \implies STx_n = 0 \implies (1_{\mathcal{B}(\mathcal{H})} + K')x_n = 0.$$

However, by definition of compact operator, there exists a subsequence $\{x_{n_m}\}_{m=1}^{\infty}$ such that $K'x_{n_m} \xrightarrow[m\to\infty]{} x$. But, the last in the above string of implications implies that $x_{n_m} + K'x_{n_m} \xrightarrow[m\to\infty]{} 0$.

Therefore, $x_{n_m} \xrightarrow[m\to\infty]{} x$. But, no subsequence of an orthonormal basis can converge as no subsequence is Cauchy. Indeed, by Pythagoreans theorem, $\|x_k - x_l\|_H = \sqrt{2}$ for all $k, l \in \mathbb{N}, k \neq l$. Thus, we have reached a contradiction. $\square$

The proof that $\ker T^*$ is finite dimensional is similar as one would consider $S^*T^* = 1_{\mathcal{B}(\mathcal{H})} + K^*$ and so on.

Claim 13.6. Ran T *is closed.*

Proof of claim. Assume that $Ty_n \xrightarrow[n\to\infty]{} y$. We want to show that $y \in \text{Ran } T$.

$$Ty_n \xrightarrow[n\to\infty]{} y \implies STy_n \xrightarrow[n\to\infty]{} Sy \implies (1_{\mathcal{B}(\mathcal{H})} + K')y_n \xrightarrow[n\to\infty]{} Sy.$$

As above, there exists some subsequence such that $K'y_{n_m} \xrightarrow[m\to\infty]{} z$. Therefore, $y_{n_m} \xrightarrow[m\to\infty]{} Sy - z \implies Ty_{n_m} \xrightarrow[m\to\infty]{} T(Sy - z)$. But, $Ty_{n_m} \xrightarrow[m\to\infty]{} y$. Hence, $y = T(Sy - z) \in \text{Ran } T$. $\square$

<u>Only if:</u> Assume T is Fredholm. We may decompose $\mathcal{H} = (\ker T)^{\perp} \oplus \ker T$, and since Ran T is closed by Fredholm, $\mathcal{H} = \text{Ran } T \oplus (\text{Ran } T)^{\perp}$. Therefore, we may decompose T in the following way

$$T = \begin{array}{c|c|c|} & (\ker T)^{\perp} & \ker T \\ \hline \text{Ran } T & A & 0 \\ \hline (\text{Ran } T)^{\perp} & 0 & 0 \\ \hline \end{array},$$

where A is invertible as it is surjective and injective. Define and operator S with respect to the same decomposition as

$$S := \begin{array}{|c|c|} \hline A^{-1} & 0 \\ \hline 0 & 0 \\ \hline \end{array}$$

Therefore,

$$TS = ST = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array} = 1 + R,$$

where R is finite rank and therefore compact. Hence, in $\mathcal{Q}(\mathcal{H})$, we have

$$\dot{T}\dot{S} = \dot{S}\dot{T} = 1.$$

$\square$

Corollary 13.7. *$\mathfrak{F}(\mathcal{H})$ is closed under compact perturbations and small perturbations.*

Proof. If $T \in \mathfrak{F}(\mathcal{H}), K \in \mathfrak{K}(\mathcal{H})$, then by definition $\dot{T} = (T+K)^{\cdot}$. Hence, invertibility of $\dot{T}$ implies invertibility of $(T+K)^{\cdot}$.

Next, assume that S is small in norm. Then, $\dot{S}$ is small in norm. Now, if T is Fredholm, then $\dot{T}$ is invertible. But, the set of invertible elements is open, so we would could take $\dot{S}$ sufficiently small such that $\dot{T} + \dot{S} = (T+S)^{\cdot}$ is invertible. ☐

Theorem 13.8. *If $T_0 \in \mathfrak{F}(\mathcal{H})$, then $\exists \varepsilon > 0$ such that $\forall T$ with $\|T - T_0\|_{\mathcal{B}(\mathcal{H})} \leqslant \varepsilon$, we have $T \in \mathfrak{F}(\mathcal{H})$ and $j(T) = j(T_0)$.*

Proof. We decompose T as

$$T = \begin{array}{c|c|c|} & (\ker T_0)^{\perp} & \ker T_0 \\ \hline \text{Ran } T_0 & A & B \\ \hline (\text{Ran } T_0)^{\perp} & C & D \\ \hline \end{array}.$$

and under the same decomposition, as in the proof of the above theorem,

$$T_0 := \begin{array}{|c|c|} \hline A_0 & 0 \\ \hline 0 & 0 \\ \hline \end{array},$$

where A_0 is invertible. Since T is close to T_0, then A is close to A_0, which implies that A is invertible since the set of invertibles is open.

Thus, we define invertible operators R, S.

$$R = \begin{array}{c|c|c|} & \text{Ran } T_0 & (\text{Ran } T_0)^{\perp} \\ \hline \text{Ran } T_0 & 1 & 0 \\ \hline (\text{Ran } T_0)^{\perp} & -CA^{-1} & 1 \\ \hline \end{array}$$

and

$$S = \begin{array}{c|c|c|} & (\ker\ T_0)^{\perp} & \ker T_0 \\ \hline (\ker\ T_0)^{\perp} & 1 & -A^{-1}B \\ \hline \ker T_0 & 0 & 1 \\ \hline \end{array}$$

Hence, by (5) of Observation (13.3) and matrix multiplication

$$\begin{aligned} j(T) &= j(RTS) \\ &= j\left(\begin{pmatrix} A & B \\ 0 & -CA^{-1}B + D \end{pmatrix} S\right) \\ &= j\left(\begin{pmatrix} A & 0 \\ 0 & -CA^{-1}B + D \end{pmatrix}\right) \\ &= j(A) + j(-CA^{-1}B + D), \end{aligned}$$

where the last equality is given by (4) of Observation (13.3). But, by (1) of Observation (13.3), we have $j(A) = 0$ since A is invertible and $-CA^{-1}B + D : \ker T_0 \longrightarrow (\text{Ran } T_0)^{\perp}$, which are finite dimensional space. Hence, by (6) of Obseravation (13.3), we have $j(-CA^{-1}B+D) = \dim \ker T_0 - \operatorname{codim} \text{Ran } T_0 = j(T_0)$. Therefore, $j(T) = j(T_0)$. ☐

Corollary 13.9. *If A, B are Fredholm and A_t is a path of Fredholm operators with $A_0 = A$ and $A_1 = B$, then $j(A) = j(B)$.*

The idea of this proof is that we may find $s \in (0,1]$ such that A_s is close enough to A and apply the previous theorem to achieve $j(A) = j(A_s)$. We can do this again for $r \in (s,1]$, and continue until we reach B. In other words, Index is a locally constant function.

Corollary 13.10. *If T is Fredholm and K is compact, then $j(T+K) = j(T)$.*

The idea of the proof of this is to consider the path $T + tK$ from T to $T+K$.

Corollary 13.11. *If $a \in \mathcal{Q}(\mathcal{H})$ is invertible, then we may define $j(a) := j(T)$ such that $\dot{T} = a$.*

The previous Corollary shows that this is well-defined by definition of the Calkin algebra.

14. 06 October 2016

Brown-Douglas-Filmore theory came about for the classification of essentially normal operators. Today we will discuss how the classification of essentially normal operators became a problem about C*-algebras.

Definition 14.1. $T \in \mathcal{B}(\mathcal{H})$ is *essentially normal* if $T^*T - TT^* \in \mathfrak{K}(\mathcal{H})$. But, this is equivalent to $(TT^* - T^*T)^{\cdot} = 0$, where $^{\cdot}$ denotes the image of an operator in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H}) =: \mathcal{Q}(\mathcal{H})$. Also,

$$(TT^* - T^*T)^{\cdot} = 0 \iff \dot{T}^*\dot{T} = \dot{T}\dot{T}^* \iff \dot{T} \text{ is normal.}$$

Example 14.2.
(1) If N is normal and K is compact, then $N+K$ is essentially normal.
(2) Recall the unilateral shift U_+. Now,
$$U_+^*(x_1, x_2, \ldots) = (x_2, x_3, \ldots) \implies U_+^*U_+(x_1, x_2, \ldots) = (x_1, x_2, \ldots)$$
$$\implies U_+^*U_+ = 1.$$
But, $U_+U_+^*(x_1, x_2, \ldots) = (0, x_2, x_3, \ldots)$. Hence, $U_+U_+^* = 1 - P_1$, where P_1 is a rank 1 projection. Therefore, U_+ is unitary in the Calkin algebra and thus normal in the Calkin algebra. Hence, U_+ is essentially normal.

We note that the example (2) is different than (1) since if we could write $U_+ = N+K$, then the index $-1 = j(U_+) = j(N+K) = j(N) = 0$ by the results of yesterday along with an exercise from yesterday. This provides a contradiction. This leads to the natural question:

<u>Question:</u> When is an essentially normal operator equal to normal + compact? The answer to this uses Brown-Douglas-Filmore theory. This also motivates the following equivalence.

Definition 14.3. $T \sim S$ if $T = U^*SU + K$ if U is unitary and K is compact.

This is equivalent to $\dot{T} = \dot{U}^*\dot{S}\dot{U}$ where U is unitary in $\mathcal{B}(\mathcal{H})$, which is stronger than assume that $\dot{U}$ is unitary in $\mathcal{Q}(\mathcal{H})$.

Our main problem is:

(14.1)

Main Problem: to classify essentially normal operators up to the above equivalence.

Definition 14.4. The *essential spectrum* of an operator T is $\sigma_{ess}(T) := \sigma(\dot{T})$, which is the spectrum of $\dot{T}$ in the C*-algebra $\mathcal{Q}(\mathcal{H})$.

Note that since the quotient map is a *-homomorphism, we have $\sigma(T) \supseteq \sigma(\dot{T}) = \sigma_{ess}(T)$. This containment can be strict since $\sigma(U_+) = \{z \in \mathbb{C} : |z| \leqslant 1\}$, the unit disc, and $\sigma_{ess}(U_+) = \{z \in \mathbb{C} : |z| = 1$, the unit circle.

Next, note that $T \sim S \iff \dot{T} = \dot{U}^*\dot{S}\dot{U}$ implies that $\sigma_{ess}(T) = \sigma_{ess}(S)$ since spectrum is preserved under conjugtion by unitary. Therefore, Main Problem (14.1) can be translated to:
to classify essentially normal operators with given essential spectrum $X \subset \mathbb{C}$, where X is compact.

Next, we establish that (up to certain equivalences) there is a one-to-one correspondence between the set of essentially normal operators, T, with a fixed essential spectrum $\sigma_{ess}(T) = X$ and injective *-homomorphisms $\tau : C(X) \longrightarrow \mathcal{Q}(\mathcal{H})$.

Fix essentially normal T such that $\sigma_{ess}(T) = X$. For $f \in C(X)$, by functional calculus, we define $\tau(f) := f(\dot{T})$. To show that τ is injective, assume that $f \in C(X)$ such that $f \neq 0$. Then, by functional calculus,

$$\sigma(\tau(f)) = \sigma(f(\dot{T})) = f(\sigma(T)) = f(\sigma_{ess}(T)) = f(X) \neq \{0\}.$$

Hence, $\tau(f) \neq 0$.

Next, assume that $\tau : C(X) \longrightarrow \mathcal{Q}(\mathcal{H})$ is an injective *-homomorphism. Define T to be any pre-image of $\tau(id_X)$ under the quotient map onto the Calkin algebra $\mathcal{Q}(\mathcal{H})$. Hence, T is esstially normal. Let $f \in C(X)$ be a polynomial, then since τ is an injective *-homomorphism, we have

$$f(\dot{T}) = f(\tau(id_X)) = \tau(f \circ id_X) = \tau(f).$$

And, this can be extended to all of $C(X)$ by continuity.

Also, again by injective *-homomorphism, we have

$$\sigma_{ess}(T) = \sigma(\dot{T}) = \sigma(\tau(id_X) = \sigma(id_X) = X.$$

Now, consider the following equivalence for injective *-homomorphisms from $C(X) \longrightarrow \mathcal{Q}(\mathcal{H})$.

(14.2)

$\tau \sim \tau'$ if there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\tau(f) = \dot{U}^*\tau'(f)\dot{U}, \forall f \in C(X)$.

And, we note that this can be extended to a C*-algebra $\mathfrak{A}$ in place of $C(X)$.

So, we show that our correspondence is injective up to this equivalence and the equivalence for essentially normal operators. Indeed, assume that $T \sim T'$. Then, there exists unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\dot{T} = \dot{U}^*\dot{T}'\dot{U}$. Note that if $f \in C(X)$, then $f(\dot{U}^*\dot{T}'\dot{U}) = \dot{U}^* f(\dot{T}')\dot{U}$ since U is unitary and one can easily check that this holds on monomials and thus polynomials, which are dense. Therefore,

$$\tau(f) = f(\dot{T}) = f(\dot{U}^*\dot{T}'\dot{U}) = \dot{U}^* f(\dot{T}')\dot{U} = \dot{U}^*\tau'(f)\dot{U}.$$

Thus, $\tau \sim \tau'$. Hence, our correspondence is one-to-one up to these equivalences. Hence, our Main Problem (14.1) translates to:

to classify injective *-homomorphisms $C(X) \longrightarrow \mathcal{Q}(\mathcal{H})$ up to $\sim$ where $X \subset \mathbb{C}$ is compact.

The next definition will help extend this to arbitrary C*-algebras.

Definition 14.5. Let $\mathfrak{A}$ be a C*-algebra. A C*-subalgebra $E \subset \mathcal{B}(\mathcal{H})$ is and *extension* of $\mathfrak{K}(\mathcal{H})$ by $\mathfrak{A}$ if $\mathfrak{K}(\mathcal{H}) \subseteq E$ and $E/\mathfrak{K}(\mathcal{H}) \cong \mathfrak{A}$.

Now, consider

$$\begin{tikzcd} & \mathcal{B}(\mathcal{H}) \arrow[d, "\pi"] \\ \mathfrak{A} \arrow[r, "\tau"] & \mathcal{Q})(\mathcal{H}) \end{tikzcd}$$

where τ is an injective *-homomorphism and π is the quotient *-homomorphism. We can construct an extension by $E := \pi^{-1}(\tau(\mathfrak{A})) \supset \mathfrak{K}(\mathcal{H})$. Therefore, $E/\mathfrak{K}(\mathcal{H}) = \tau(\mathfrak{A}) \cong \mathfrak{A}$. Thus, any injective *-homomorphism produces an extension.

Next, assume that $\mathfrak{K}(\mathcal{H}) \subseteq E \subseteq \mathcal{B}(\mathcal{H})$ is an extension of $\mathfrak{K}(\mathcal{H})$ by $\mathfrak{A}$.

Thus, $\mathfrak{A} \underset{\cong}{\xrightarrow{\psi}} E/\mathfrak{K}(\mathcal{H}) \xrightarrow{\iota} \mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H}) = \mathcal{Q}(\mathcal{H})$, where ι is inclusion. Hence, $\tau := \iota \circ \psi : \mathfrak{A} \longrightarrow \mathcal{Q}(\mathcal{H})$ is an injective *-homomorphism. Thus, any extension produces an injective *-homomorphism.

Definition 14.6. Let $\mathfrak{A}$ be a C*-algebra.

$$\mathrm{Ext}(\mathfrak{A}) := \{\text{injective *-homomorphisms } \mathfrak{A} \longrightarrow \mathcal{Q}(\mathcal{H})\} / \sim,$$

where $\sim$ is given by Equivalence (14.2).

Therefore, Main Problem (14.1) translates to:

to compute $\mathrm{Ext}(C(X))$ where $X \subseteq \mathbb{C}$ is compact.

But, in order to compute $\mathrm{Ext}(C(X))$, we need Abelian group structure.

Group structure on $\mathrm{Ext}(\mathfrak{A})$:

Addition: $[\tau]+[\tau'] := \left[\begin{pmatrix} \tau & \\ & \tau' \end{pmatrix}\right]$, which is possible since $\mathcal{H} = \mathcal{H}\oplus\mathcal{H} \implies \mathcal{B}(\mathcal{H}) \cong M_2(\mathcal{B}(\mathcal{H})) \implies \mathfrak{K}(\mathcal{H}) \cong M_2(\mathfrak{K}(\mathcal{H})) \implies \mathcal{Q}(\mathcal{H}) \cong M_2(\mathcal{Q}(\mathcal{H}))$. It is left as an exercise to show that addition is well-defined.

Abelian: Consider τ, τ'. A simple calculation shows that

$$\begin{pmatrix} \tau & \\ & \tau' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \tau' & \\ & \tau \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence, $[\tau] + [\tau'] = [\tau'] + [\tau]$.

Neutral element: This is non-trivial and due to Voiculescu. We will start towards finding this neutral element today.

Definition 14.7. τ is a *trivial extension* if $\exists$ an injective *-homomorphism $\rho : \mathfrak{A} \longrightarrow \mathcal{B}(\mathcal{H})$ such that $\tau = \pi \circ \rho$, where $\pi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{Q}(\mathcal{H})$ is the quotient map.

Now, for a C*-algebra $\mathfrak{A}$, since every abstract C*-algebra can be realized as a concrete C*-algebra as a consequence of GNS, there always exists some Hilbert space $\mathcal{H}$ and an injective *-homomorphism $\rho : \mathfrak{A} \longrightarrow \mathcal{B}(\mathcal{H})$. One might think that the natural choice is to define $\tau = \pi \circ \rho$. But, it may be the case that $\rho(\mathfrak{A}) \cap \mathfrak{K}(\mathcal{H}) \neq \{0\}$. Hence, τ would not be injective in this case. The next proposition remedies this. But, first, a lemma.

Lemma 14.8. *If $A \in \mathcal{B}(\mathcal{H})$ and $A \neq 0$, then* $\begin{pmatrix} A & & \\ & A & \\ & & \ddots \end{pmatrix} \notin \mathfrak{K}(\mathcal{H})$, *where we view $\mathcal{H} \cong \mathcal{H} \oplus \mathcal{H} \oplus \cdots$ by separability.*

Proof. Exercise. □

Proposition 14.9. *Trivial extensions exist.*

Proof. As discussed before the lemma, let $\rho : \mathfrak{A} \longrightarrow \mathcal{B}(\mathcal{H})$ be an injective *-homomorphism. Define $\rho^\infty = \begin{pmatrix} \rho & & \\ & \rho & \\ & & \ddots \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H} \oplus \cdots) \cong \mathcal{B}(\mathcal{H})$. Define $\tau := \pi \circ \rho^\infty : \mathfrak{A} \longrightarrow \mathcal{Q}(\mathcal{H})$, where $\pi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{Q}(\mathcal{H})$ is the quotient map. By the previous lemma, τ is an injective *-homomorphism and therefore a trivial extension. □

It remains to show that trivial extensions form 1 equivlance class and that this is a neutral element, which is due to Voiculescu.

15. 07 October 2016, TeX'd by Kaveh Mousavand

In the previous lecture, we introduced

$$\mathcal{E}xt(A) = \{\tau : A \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) | \tau \text{ is an injective } *\text{-homomorphism}\}/\sim,$$

where $\tau_1 \sim \tau_2$ if there exists a unitary element $u \in \mathcal{B}(\mathcal{H})$ such that for every $a \in A$, we have

$$\tau_1(a) = \dot{u}^* \tau_2(a) \dot{u}.$$

We remark that our main goal in the following is computing $\mathcal{E}xt(C(X))$, since it is equivalent to the classification of essential normal operators with essential spectrum X.

Recall that for τ_1 and $\tau_2 \in \mathcal{E}xt(A)$, we previously defined

$$[\tau_1] + [\tau_2] := \left[\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}\right] \equiv \tau_1 \oplus \tau_2,$$

and $\tau : A \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ was called trivial if there exists a $*$-homomorphism $\rho : A \to \mathcal{B}(\mathcal{H})$ such that $\tau = \dot{\rho} = \pi \circ \rho$. i.e., the following diagram commutes:

$$\begin{tikzcd} & \mathcal{B}(\mathcal{H}) \arrow[d, "\pi"] \\ A \arrow[ur, dotted, "\exists\rho"] \arrow[r, "\tau"'] & \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \end{tikzcd}$$

Definition 15.1. Two representations $\rho_1, \rho_2 : A \to \mathcal{B}(\mathcal{H})$ are called *approximately unitary equivalent* and denoted by $\rho_1 \sim_a \rho_2$ if there exist unitaries $u_n \in \mathcal{B}(\mathcal{H})$ such that for every $a \in A$ and every n,

$$\rho_1(a) - u_n^* \rho_2(a) u_n \in \mathcal{K}(\mathcal{H}),$$

and furthermore,

$$\|\rho_1(a) - u_n^* \rho_2(a) u_n\| \to 0,$$

as $n \to \infty$.

Theorem 15.2. *(Voiculescu)*
If ρ_1 and ρ_2 are two representations of a C^*-algebra A, then $\rho_1 \sim_a \rho_2$ if and only if $rank \rho_1(a) = rank \rho_2(a)$, for every $a \in A$.

Remark 15.3. We do not prove the Voiculescu's theorem here. However, we remark that the "only if" side of the theorem is easy to prove, whereas the "if" side is very hard.

Lemma 15.4. *All trivial extensions are equivalent to each other.*

Proof. Let $\dot{\rho_1}$ and $\dot{\rho_2}$ be two trivial extensions. We want to show that $\dot{\rho_1} \sim \dot{\rho_2}$.
We claim $\dot{\rho_1} \sim_a \dot{\rho_2}$, because $\dot{\rho_1}$ and $\dot{\rho_2}$ are injective and therefore for every $a \in A$,

$$rank \rho_1(a) = \infty = rank \rho_2(a).$$

Hence, there exist unitaries u_n such that for every $a \in A$ and every n,

$$\rho_1(a) - u_n^* \rho_2(a) u_n \in \mathcal{K}(\mathcal{H}).$$

In particular, for $n = 1$ we have $\rho_1(a) - u_1^* \rho_2(a) u_1 \in \mathcal{K}(\mathcal{H})$, which implies

$$\dot{\rho}_1(a) = \dot{u_1}^* \dot{\rho}_2(a) \dot{u_1}.$$

Thus, $\dot{\rho_1} \sim \dot{\rho_2}$ and we are done. $\square$

Lemma 15.5. *Every extension equivalent to a trivial extension is trivial itself.*

Proof. If $\tau \sim \dot{\rho}$, there exists u such that $\tau(a) = \dot{u}^* \dot{\rho}(a) \dot{u}$, for every $a \in A$. Define a $*$-homomorphism ρ_1 by $\rho_1(a) := u^* \rho(a) u$, for every $a \in A$. Hence, $\tau(a) = \dot{\rho}(a)$ for every $a \in A$, and consequently $\tau = \dot{\rho}_1$ is trivial. $\square$

Corollary 15.6. *Trivial extensions form a single equivalence class.*

Theorem 15.7. *The class of all trivial extensions is the neutral element in $\mathcal{E}xt(A)$.*

Proof. We want to show that $[\tau] \oplus [trivial] = [\tau]$, for every arbitrary extension τ. Equivalently, we must show $\tau \oplus \dot{\rho} \sim \tau$.

We have the following diagram (note that we do not claim commutativity of the diagram!):

$$\begin{tikzcd} & \mathcal{B}(\mathcal{H}) \arrow[d, "\pi"] \\ A \arrow[ur, "\rho"] \arrow[r, "\tau"'] & \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}) \end{tikzcd}$$

Let's assume $E := \pi^{-1}(\tau(A))$, and consider $\rho \circ \tau^{-1} \circ \pi_{|_E} \oplus id_{|_E}$ and id_E as two representations of E.

Claim: $\rho \circ \tau^{-1} \circ \pi_{|_E} \oplus id_{|_E}$ and id_E are approximately equivalent.

For $T \in E$ and we should show that

$$rank\rho(\tau^{-1}(\pi(T))) \oplus T = rankT.$$

We consider two cases:

Case 1: If $T \in \mathcal{K}(\mathcal{H})$, then $\rho(\tau^{-1}(\pi(T)) = 0$, and there is nothing to show.

Case 2: If $T \notin \mathcal{K}(\mathcal{H})$, then we get $\infty = \infty$ and we are done.

Thus, there exist unitaries u_n such that $\rho(\tau^{-1}(\pi(T)) \oplus T - u_n^* T u_n \in \mathcal{K}(\mathcal{H})$. In particular, for $n = 1$,

$$\begin{gathered} \rho(\tau^{-1}(\pi(T)) \oplus T - u_1^* T u_1 \in \mathcal{K}(\mathcal{H}) \\ \dot{\rho}(\tau^{-1}(\pi(T)) \oplus \dot{T} = \dot{u}_1^* \dot{T} \dot{u}_1 \\ \dot{\rho}(a) \oplus \tau(a) = \dot{u}_1^* \tau(a) \dot{u}_1 \\ \dot{\rho} \oplus \tau \sim \tau, \end{gathered}$$

where we used the fact that $T \in E = \pi^{-1}(\tau(A))$ implies $\pi(T) = \tau(a)$ for some $a \in A$, such that $\dot{\rho}(\tau^{-1}(\pi(T)) \oplus \dot{T} = \dot{\rho}(a) \oplus \tau(a)$.

□

Theorem 15.8. $[\tau] \in \mathcal{E}xt(A)$ *is invertible if and only if* τ *lifts to a completely positive map, i.e.,* $\tau = \dot{\phi}$, *for some* $\phi : A \to \mathcal{B}(\mathcal{H})$.

Proof. We postpone the proof until we have the required tools which will be presented in the following. □

Definition 15.9. A linear map between C^*-algebras $\varphi : A \to B$ is called *positive* if for every $a \in A_+$, we have $\varphi(a) \in B_+$.

Moreover, the *n-th inflation* of φ is denoted by $\varphi^{(n)} : M_n(A) \to M_n(B)$ such that φ acts entry-wise:

$$\varphi^{(n)} \begin{pmatrix} a_{11} & ... & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & ... & a_{nn} \end{pmatrix} = \begin{pmatrix} \varphi(a_{11}) & ... & \varphi(a_{1n}) \\ \vdots & \vdots & \vdots \\ \varphi(a_{n1}) & ... & \varphi(a_{nn}) \end{pmatrix},$$

and φ is called *n-positive* if $\varphi^{(n)}$ is positive.

Example 15.10. If we consider $\varphi : M_2 \to M_2$, given by $\varphi(T) = T^t$, it is clear that φ is positive. However, $\varphi^{(2)}$ is not a 2-positive map. To see that, consider the following non-example:

$$\varphi^{(2)} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array} \right) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right),$$

where the central (2×2)-block of the matrix on the right takes $\lambda = \pm 1$ as the eigenvalues, thus the image is not positive!

Definition 15.11. φ is called *completely positive* if $\varphi^{(n)}$ is positive, for every n.

Exercise 15.12. Show that the following maps are completely positive:

- $*$-homomorphisms;
- Let $\rho : A \to B$ be a $*$-homomorphism and for a fixed $b_0 \in B$, define $\varphi : A \to B$, such that $\varphi(a) = b_0^* \rho(a) b_0$;
- Each positive functional $\varphi : A \to \mathbb{C}$.

Theorem 15.13. *(Stinespring)*
Let A be a unital C^*-algebra and $\varphi : A \to \mathcal{B}(\mathcal{H})$ is a unital completely positive map. There exists a Hilbert space K with an isometry $V : \mathcal{H} \to K$ and a $*$-homomorphism $f : A \to \mathcal{B}(K)$ such that for every $a \in A$, we have

$$\varphi(a) = V^* f(a) V.$$

Remark 15.14. Informally speaking, the previous theorem implies that for a unital completely positive map $\varphi : A \to \mathcal{B}(\mathcal{H})$, we can always find a representation f such that:

$$f(a) = \left(\begin{array}{c|c} \varphi(a) & \vdots \\ \hline \cdots & \cdots \end{array} \right).$$

16. 10 October 2016, Tex'd by Kaveh Mousavand

We finished the last lecture by the following theorem:

Theorem 16.1. *(Stinespring)*
Let A be a unital C^*-algebra and $\varphi : A \to \mathcal{B}(\mathcal{H})$ a unital completely positive map. There exists a Hilbert space K with an isometry $V : \mathcal{H} \to K$ and a $*$-homomorphism $\rho : A \to \mathcal{B}(K)$ such that $\varphi(a) = V^* \rho(a) V$, for every $a \in A$.

Proof. The proof is given via the following four steps:

Construction of K:

Let's consider $A \otimes \mathcal{H}$, equipped with the following inner product:

$$\langle \sum_{i=1}^n a_i \otimes \zeta_i, \sum_{j=1}^m b_j \otimes \eta_j \rangle := \sum_{i=1}^n \sum_{j=1}^m \big(\varphi(b_j^* a_i) \zeta_i, \eta_j \big),$$

where the inner product on the right-hand side of the equation comes from $\mathcal{H}$.

To show that the new inner product is semi-positive definite, we must prove that for every $\sum_{i=1}^{n} a_i \otimes \zeta_i$ in $A \otimes \mathcal{H}$, the following is non-negative:

$$
\begin{aligned}
\langle \sum_{i=1}^{n} a_i \otimes \zeta_i, \sum_{i=1}^{n} a_i \otimes \zeta_i \rangle &= \sum_{i}^{n} \sum_{j=1}^{n} \left(\varphi(a_j^* a_i) \zeta_i, \zeta_j \right) \\
&= \left(\left(\varphi(a_j^* a_i) \right)_{i,j=1}^{n} \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix}, \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix} \right) \\
&= \left(\varphi^{(n)} \left(\left(a_j^* a_i \right)_{i,j=1}^{n} \right) \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix}, \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix} \right),
\end{aligned}
\tag{16.1}
$$

where the fact that $\left(a_j^* a_i \right)_{i,j=1}^{n} \geqslant 0$ implies $\varphi^{(n)} \left(\left(a_j^* a_i \right)_{i,j=1}^{n} \right) \geqslant 0$, hence the above inner product is non-negative. However, we should remark that $\langle -, - \rangle$ is not necessarily positive definite!

Now, consider all elements of $A \otimes \mathcal{H}$ whose paring with themselves with respect to the inner product $\langle -, - \rangle$ is zero, and denote the set of all such elements by

$$N_\varphi := \{ \textstyle\sum_{i=1}^{n} a_i \otimes \zeta_i | \ \langle \sum_{i=1}^{n} a_i \otimes \zeta_i, \sum_{i=1}^{n} a_i \otimes \zeta_i \rangle = 0 \text{ for } a_i \in A, \zeta_i \in \mathcal{H}, n \in \mathbb{N} \}.$$

By Cauchy-Schwartz inequality we have

$$|\langle x, y \rangle|^2 \leqslant \langle x, x \rangle . \langle y, y \rangle,$$

which implies that if $\langle x, x \rangle = 0$, then $\langle x, y \rangle = 0$, for every y. Therefore, we have

$$\{ \textstyle\sum_{i=1}^{n} a_i \otimes \zeta_i | \ \langle \sum_{i=1}^{n} a_i \otimes \zeta_i, \sum_{j=1}^{m} b_j \otimes \eta_j \rangle = 0 \text{ for } b_j \in A, \eta_j \in \mathcal{H}, m \in \mathbb{N} \},$$

which shows that N_φ is a subspace of $A \otimes \mathcal{H}$. Hence, we can consider the quotient space $(A \otimes \mathcal{H})/N_\varphi$, on which the inner product $\langle -, - \rangle$ is positive definite.

Now, we consider K as the completion of $(A \otimes \mathcal{H})/N_\varphi$ with respect to the norm

$$\| - \| = \langle -, - \rangle^{\frac{1}{2}}.$$

Construction of V:

Consider $V : \mathcal{H} \to K$, given by $V(\eta) := [1 \otimes \eta]$. We must show that V is an isometry. i.e, $\|V\eta\| = \|\eta\|$, for every $\eta \in \mathcal{H}$.

To show the desired equality, we note that

$$\langle V\eta, V\eta \rangle = \langle 1 \otimes \eta, 1 \otimes \eta \rangle = (\varphi(1), \eta) = (\eta, \eta) = \|\eta\|^2.$$

Construction of $\rho : A \to \mathcal{B}(K)$:

Define

$$\rho(a)\left([\sum a_i \otimes \eta_i]\right) := [\sum aa_i \otimes \eta_i].$$

In order to prove that ρ is a well-defined map, we must show $\sum a_i \otimes \eta_i \in N_\varphi$ implies $\sum aa_i \otimes \eta_i \in N_\varphi$.
It is enough to notice that we have the following:

$$\begin{aligned} \langle \sum_{i=1}^n aa_i \otimes \zeta_i, \sum_{i=1}^n b_i \otimes \eta_i \rangle &= \sum_{i,j} \left(\varphi(b_j^* aa_i)\zeta_i, \eta_j\right) \\ &= \langle \sum_{i=1}^n a_i \otimes \zeta_i, \sum_{i=1}^n a^* b_j \otimes \eta_j \rangle = 0, \end{aligned} \tag{16.2}$$

where we used $b_j^* aa_i = (a^* b_j)^* a_i$.

Exercise 16.2. Show that $\|\rho(a)\| \leqslant \|a\|$.

Now, we can extend $\rho(a)$ on K, for every $a \in A$. We also have ρ is linear and multiplicative.

Exercise 16.3. Show that $\rho(a^*) = \rho(a)^*$, for every $a \in A$.

Checking that $\varphi(a) = V^* \rho(a) V$:

We have $V : \mathcal{H} \to K$ and $V^* : K \to \mathcal{H}$. Moreover, from

$$\begin{aligned} \left(V^* \left[\sum a_i \otimes \zeta_i\right], \eta\right) &= \langle \left[\sum a_i \otimes \zeta_i\right], V\eta \rangle \\ &= \langle \left[\sum a_i \otimes \zeta_i\right], [1 \otimes \eta] \rangle \\ &= \left(\sum_i \varphi(a_i)\zeta_i, \eta\right), \end{aligned} \tag{16.3}$$

we get

$$V^*\left([\textstyle\sum_i a_i \otimes \zeta_i]\right) = \textstyle\sum_i \varphi(a_i)\zeta_i.$$

Hence,

$$V^* \rho(a) V\eta = V^* \rho(a) [1 \otimes \eta] = V^* [a \otimes \eta] = \varphi(a)\eta.$$

This shows the desired result and finishes the proof. □

Remark 16.4. If $\varphi : A \to \mathbb{C}$ is a positive functional, then Stienspring's theorem gives the GNS-construction for φ.

Remark 16.5. If A and $\mathcal{H}$ are separable, then so is K, and furthermore, $K \simeq \mathcal{H}$.

Remark 16.6. $K = V(\mathcal{H}) \oplus V(\mathcal{H})^\perp$. With respect to this decomposition, we have

$$\rho(a) = \left(\begin{array}{c|c} \varphi(a) & \vdots \\ \hline \cdots & \cdots \end{array} \right).$$

Convention: From now on, all algebras are assumed to be unital.

Theorem 16.7. $[\tau] \in \mathcal{E}xt(A)$ *has an inverse element if and only if there exists a unital completely positive map* $\varphi : A \to \mathcal{B}(\mathcal{H})$*, such that* $\tau = \pi \circ \varphi = \dot{\varphi}$*.*

Proof. **(If):** Let's assume $\tau = \dot{\varphi}$ and φ is completely positive. By Stinespring's theorem, there exists a unital $*$-homomorphism $\rho : A \to \mathcal{B}(\mathcal{H})$ such that

$$\rho = \begin{pmatrix} \varphi & \psi_2 \\ \psi_3 & \psi_4 \end{pmatrix}.$$

Therefore,

$$\dot{\rho} = \begin{pmatrix} \varphi & \dot{\psi}_2 \\ \dot{\psi}_3 & \dot{\psi}_4 \end{pmatrix}.$$

Claim: $\dot{\psi}_2 = 0 = \dot{\psi}_3$ and $\dot{\psi}_4$ is a $*$-homomorphism.
proof of the claim: ρ is a $*$-homomorphism. Then, it preserves involution, i.e., $\rho(a^*) = \rho(a)^*$. Therefore,

$$\begin{pmatrix} \varphi(a^*) & \psi_2(a^*) \\ \psi_3(a^*) & \psi_4(a^*) \end{pmatrix} = \begin{pmatrix} \varphi(a)^* & \psi_3(a)^* \\ \psi_2(a)^* & \psi_4(a)^* \end{pmatrix},$$

which implies $\psi_4(a^*) = \psi_4(a)^*$ and $\psi_2(a^*) = \psi_3(a)^*$.

Since $\rho(ab) = \rho(a)\rho(b)$, we get

$$\begin{pmatrix} \varphi(ab) & \psi_2(ab) \\ \psi_3(ab) & \psi_4(ab) \end{pmatrix} = \begin{pmatrix} \varphi(a) & \psi_2(a) \\ \psi_3(a) & \psi_1(a) \end{pmatrix} \begin{pmatrix} \varphi(b) & \psi_2(b) \\ \psi_3(b) & \psi_4(b) \end{pmatrix}$$
$$= \begin{pmatrix} \varphi(a)\varphi(b) + \psi_2(a)\psi_3(b) & \cdots \\ \cdots & \psi_3(a)\psi_2(b) + \psi_1(a)\psi_4(b) \end{pmatrix}.$$

Hence, $\dot{\varphi}(ab) = \dot{\varphi}(a)\dot{\varphi}(b) + \dot{\psi}_2(a)\dot{\psi}_3(b)$.

But, $\dot{\varphi} = \tau$ implies $\tau(ab) = \tau(a)\tau(b) + \dot{\psi}_2(a)\dot{\psi}_3(b)$, which simplifies to

$$\dot{\psi}_2(a)\dot{\psi}_3(b) = 0, \text{ for } \forall a, b \in A.$$

Therefore, assuming $a = b^*$, we get

$$\dot{\psi}_2(b^*)\dot{\psi}_3(b) = 0, \text{ for } \forall b \in A.$$

Since we previously showed that $\psi_2(a^*) = \psi_3(a)^*$, we have $\dot{\psi}_2(a^*) = \dot{\psi}_3(a)^*$, and consequently $\dot{\psi}_3(b)^*\dot{\psi}_3(b) = 0$, which implies $\dot{\psi}_3(b) = 0$, $\forall b \in A$, hence $\dot{\psi}_3 = 0$. Since $\dot{\psi}_2(a^*) = \dot{\psi}_3(a^*)$, we conclude that $\dot{\psi}_2 = 0$. Since $\dot{\psi}_4(ab) = \dot{\psi}_3(a)\dot{\psi}_2(b) + \dot{\psi}_4(a)\dot{\psi}_4(b)$, we conclude that $\dot{\psi}_4$ is a *-homomorphism.

Therefore,

$$\dot{\rho} = \begin{pmatrix} \tau & 0 \\ 0 & \dot{\psi}_4 \end{pmatrix},$$

where $\dot{\psi}_4$ is a $*$-homomorphism, but it may not be injective!

$\square$

17. 11 October 2016, TeX'd by Kaveh Mousavand

At the end of the previous lecture we wanted to prove the following theorem:

Theorem 17.1. $[\tau] \in \mathcal{E}xt(A)$ *is invertible if and only if there exists a unital completely positive map* $\varphi : A \to \mathcal{B}(\mathcal{H})$*, such that* $\tau = \pi \circ \varphi = \dot{\varphi}$.

We already showed that

$$\dot{\rho} = \begin{pmatrix} \tau & 0 \\ 0 & \dot{\psi}_4 \end{pmatrix},$$

where $\dot{\psi}_4$ is a $*$-homomorphism. However, we pointed out that it may not be injective!

In order to tackle this issue, we consider

$$\left(\begin{array}{c|cc} \tau & 0 & 0 \\ \hline 0 & \dot{\psi}_4 & 0 \\ 0 & 0 & \dot{\rho}_1 \end{array}\right) = \begin{pmatrix} \dot{\rho} & 0 \\ 0 & \dot{\rho}_1 \end{pmatrix} = \begin{pmatrix} \rho & 0 \\ 0 & \rho_1 \end{pmatrix}^{\cdot},$$

such that the bottom right block is injective and we get

$$[\tau] \oplus \left[\dot{\psi}_4 \oplus \dot{\rho}_1\right] = [trivial].$$

Namely, $\left[\dot{\psi}_4 \oplus \dot{\rho}_1\right]$ is the inverse for $[\tau]$.

We complete the proof by showing the other side of the assertion, as following:

Proof. **($\Leftarrow$ Only if)**
Suppose $[\tau] \oplus [\sigma] = [trivial]$. Then

$$\begin{pmatrix} \tau & 0 \\ 0 & \sigma \end{pmatrix} = \dot{\rho} = \begin{pmatrix} \dot{\psi}_1 & \dot{\psi}_2 \\ \dot{\psi}_3 & \dot{\psi}_4 \end{pmatrix},$$

where

$$\rho = \begin{pmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{pmatrix}.$$

Then $\tau = \dot{\psi}_1$, where ψ is unital and completely positive. $\square$

In the following, we show that if A is a nuclear and separable algebra, then every unital completely positive map $A \to B/I$ lifts to a unital completely positive map $A \to B$, here B is any unital C^*-algebra and I an arbitrary ideal of B.

Before we get to the details, we recall some definitions.

Definition 17.2. A is called a *nuclear* C^*-algebra if there exist unital completely positive maps $\delta_n : A \to M_n$ and $\gamma_n : M_n \to A$, such that for every $a \in A$, we have $\lim_{n\to\infty} \gamma_n \circ \delta_n(a) = a$.

$$\begin{tikzcd} A \arrow[rr, "id"] \arrow[dr, "\delta_n"'] & & A \\ & M_n \arrow[ur, "\gamma_n"'] & \end{tikzcd}$$

Exercise 17.3. Show that $C(X)$ is nuclear.

Definition 17.4. If $\varphi : A \to B/I$ is a unital completely positive map, it is called *liftable* if there exists a unital completely positive $\psi : A \to B$ such that $\varphi = \pi \circ \psi \equiv \dot{\psi}$. Thus we have the following diagram commutes:

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ A \arrow[ur, dotted, "\exists\psi"] \arrow[r, "\varphi"'] & B/I \end{tikzcd}$$

Theorem 17.5. *(Choi-Effros)*
Let A be a unital nuclear and separable C^*-algebra. If B is a unital C^*-algebra and I is an ideal of B, then every unital completely positive map $A \to B/I$ is liftable to $A \to B$.

The following proof is given by Arveson:

Proof. The proof of this theorem requires the following steps:

Step 1: The set of all liftable unital completely positive maps from A to B/I is a closed set in the pointwise convergence topology. i.e., if $\varphi_n(a) \to \varphi(a)$ for every $a \in A$, and φ_n are liftable, then so is φ.

Step 2: Every unital completely positive $M_n \to B/I$ is liftable.

Before we prove the above-mentioned assertions, let's check that they will finish the proof of the theorem.

Let's assume the statements of 1 and 2 both hold. Let $\varphi : A \longrightarrow B/I$ be a unital completely positive map. Consider the following diagram.

$$\begin{tikzcd} & & B \arrow[d, "\pi"] \\ A \arrow[r, "id_A"] \arrow[d, "\delta_n"'] & A \arrow[ur, dotted] \arrow[r, "\varphi"'] & B/I \\ M_n \arrow[ur, "\gamma_n"'] & & \end{tikzcd}$$

By nuclearity, id_A is the pointwose limit of $\gamma_n \circ \delta_n$. Hence, φ is the pointwise limit of $\varphi \circ \gamma_n \circ \delta_n$. Note that $\varphi \circ \gamma_n : M_n \longrightarrow B/I$. Hence by the 2^{nd} assumption, there exist unital c.p. maps $\psi_n : M_n \longrightarrow B$ such that $\dot{\psi}_n = \varphi \circ \gamma_n$. Thus, $\varphi \circ \gamma_n \circ \delta_n = (\psi_n \circ \gamma_n)^{\cdot}$. Therefore, φ is the pointwise limit of liftable unital c.p. maps. By the 1^{st} assumption, φ is liftable. $\square$

Now we prove each of the assertions separately.

*Proof of **Step 2**.* We need to use *Choi's Criterion*, which is the following:

Let $\varphi : M_n \to D$ be a linear map, then φ is completely positive if and only if $(\varphi(e_{ij}))_{i,j=1}^n \geqslant 0$, where e_{ij} is the elementary matrix with 1 as the (i,j)-th entry and 0 elsewhere.

Now, if we suppose $\varphi_n : M_n \to B/I$ is a unital completely positive map, by Choi's criterion we have $(\varphi(e_{ij}))_{i,j=1}^n \geqslant 0$ in $M_n(B/I) = M_n(B)/M_n(I)$.

Lemma 17.6. *If $x \in (B/I)_+$, there exists $b \in B_+$ such that $\dot{b} = x$.*

Proof of Lemma. Suppose $x = y^*y$ and take b_1 such that $\dot{b_1} = y$. Then, $(b_1^* b_1)^{\cdot} = x$. $\square$

By the previous lemma, there exists $(b_{i,j})_{i,j=1}^n \in M_n(B)_+$ such that

$$\left((b_{ij})_{i,j=1}^n\right)^{\bullet} = (\varphi(e_{ij}))_{i,j=1}^n.$$

Define $\psi : M_n(A) \to B$ by $\psi(e_{ij}) = b_{ij}$.

By Choi's criterion ψ is completely positive, because

$$(\psi(e_{ij}))_{i,j=1}^n = (b_{ij})_{i,j=1}^n \geqslant 0.$$

Thus, ψ is c.p. and $\dot{\psi} = \varphi$ since $\dot{\psi}(e_{ij}) = \varphi(e_{ij}), \forall i,j$. However, ψ might not be unital. We note that $\psi(1) = 1 + k$ for $k = k^*$, and use the following fact:

Fact: If $k = k^*$, then $k = k_1 - k_2$, for some $k_1, k_2 \geqslant 0$. In fact, $k_1 = f(k)$, and $k_2 = g(k)$, where f, g are continuous functions and $f(0) = g(0) = 0$.

Therefore, if $\psi(1) = 1 + k$, for $k = k^*$, then $k = k_1 - k_2$, where $k_1, k_2 \geqslant 0$. Fix a state σ on B and consider $\widetilde{\psi} : A \longrightarrow B$ defined by

$$\widetilde{\psi}(a) := (1 + k_1)^{-\frac{1}{2}}(\psi(a) + \sigma(a)k_2)(1 + k_1)^{-\frac{1}{2}}.$$

Thus,

$$\widetilde{\psi}(1) := (1 + k_1)^{-\frac{1}{2}}(1 + k_1 - k_2 + k_2)(1 + k_1)^{-\frac{1}{2}} = 1.$$

We want to show that $\widetilde{\psi}$ is a lift of φ. Since $\dot{\widetilde{\psi}}(1) = 1$, we get $\dot{k} = 0$ and by the above Fact, $\dot{k_1} = 0$ and $\dot{k_2} = 1$. Consequently, $\dot{\widetilde{\psi}}(a) = \varphi(a)$, as desired. $\widetilde{\psi}$ is c.p. by the following fact:

Exercise 17.7. For a state G and $b \geqslant 0$, the map given by $a \mapsto G(a)b$ is completely positive.

Therefore, due to the above exercise, $\dot{\widetilde{\psi}}$ is a unital completely positive lift of φ and we are done. $\square$

18. 12 October 2016

Today we finish the proof of the Choi-Effros Theorem (17.5). To do this, all that remains is to prove **Step 1** in the proof of Choi-Effros Theorem (17.5). We restate this as a Theorem since it is the main tool for the Choi-Effros Theorem and it is most of the focus of today.

Theorem 18.1. *Let $\mathfrak{A}$ be a unital separable C^*-algebra and $\mathfrak{B}$ be a C^*-algebra. If I is and ideal of $\mathfrak{B}$, then the set of liftable u.c.p. (unital completely positive) $\mathfrak{A} \longrightarrow \mathfrak{B}/I$ is closed in the point-norm topology.*

The proof of this Theorem relies on the notion of quasi-central approximate units.

Definition 18.2. Let I be an ideal in a C*-algebra $\mathfrak{B}$. Let $\{u_\lambda\} \subseteq I$ be a net of positive elements. An *approximate unit* $\{u_\lambda\} \subseteq I$ is an increasing net such that $\|u_\lambda\| \leqslant 1$ and $\lim_\lambda iu_\lambda = \lim_\lambda u_\lambda i = i$ for all $i \in I$.

If in addition, $\forall b \in \mathfrak{B}, \|[u_\lambda, b]\| \longrightarrow 0$, then $\{u_\lambda\}$ is *quasi-central*.

Theorem 18.3. *[Arveson; Ackemann and Pedersen] Every ideal in every C^*-algebra has a quasi-central approximate unit (q.a.u).*

Before we prove our main theorem, we prove a lemma that is useful in general, which helps us compute norms in quotients. In fact, the following lemma can be used to show that the quotient norm induced by an ideal of a C*-algebra satisfies the C*-identity, and therefore quotients are C*-algebras.

Lemma 18.4. *Let $\mathfrak{B}$ be a C^*-algebra and I an ideal. For $b \in \mathfrak{B}$, let $\dot{b} \in \mathfrak{B}/I$ denote the image of b under the quotient map.*

If $\{u_\lambda\}$ is an approximate unit in I and $b \in \mathfrak{B}$, then

$$\|\dot{b}\|_{\mathfrak{B}/I} = \lim_\lambda \|b(1-u_\lambda)\|_{\mathfrak{B}}.$$

Proof. It is enough to prove

$$\limsup_\lambda \|b(1-u_\lambda)\|_{\mathfrak{B}} \leqslant \|\dot{b}\|_{\mathfrak{B}/I} \leqslant \liminf_\lambda \|b(1-u_\lambda)\|_{\mathfrak{B}}$$

since clearly the right hand side is less than or equal to the left hand side in general.

By definition, $\|\dot{b}\|_{\mathfrak{B}/I} = \inf_{i\in I} \|b+i\|_{\mathfrak{B}}$.

Now, fix $i \in I$. Note that $1 - u_\lambda$ is a contraction. Thus,

$$\|b+i\|_{\mathfrak{B}} \geqslant \|(b+i)(1-u_\lambda)\|_{\mathfrak{B}} = \|b(1-u_\lambda) + i(1-u_\lambda)\|_{\mathfrak{B}}.$$

Hence, the net on the right hand side is bounded above and by the definition of approx. unit $\lim_\lambda i(1-u_\lambda) = \lim i - iu_\lambda = 0$. These together imply that

$$\|b+i\|_{\mathfrak{B}} \geqslant \limsup_\lambda \|b(1-u_\lambda)\|_{\mathfrak{B}}, \quad \forall i \in I.$$

Thus, since the right hand side is a lower bound for the left hand side for all $i \in I$, the infimum is

$$\|\dot{b}\|_{\mathfrak{B}/I} \geqslant \limsup_{\lambda} \|b(1-u_\lambda)\|_{\mathfrak{B}}.$$

Next, since I is an ideal and $\{u_\lambda\} \subseteq I$, we have that $bu_\lambda \in I$ for all λ. Therefore,

$$\|\dot{b}\|_{\mathfrak{B}/I} = \inf_{i \in I} \|b+i\|_{\mathfrak{B}} \leqslant \|b - bu_\lambda\|_{\mathfrak{B}} = \|b(1-u_\lambda)\|_{\mathfrak{B}}, \quad \forall \lambda$$

and $\|\dot{b}\|_{\mathfrak{B}/I} \leqslant \liminf_\lambda \|b(1-u_\lambda)\|_{\mathfrak{B}}$. □

Now, we are in a position to prove Theorem (18.1).

Proof of Theorem (18.1). For $n \in \mathbb{N}$, let $\varphi_n : \mathfrak{A} \longrightarrow \mathfrak{B}/I$ be liftable u.c.p. maps such that there exists $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}/I$ with $\varphi_n(a) \to \varphi(a)$ for all $a \in \mathfrak{A}$.

Recall the following general fact from analysis. If $x_n \to x$ then there exists a subsequence x_{n_k} such that $\|x_{n_k} - x_{n_{k+1}}\| \leqslant 1/2^k$.

Therefore, by separability let $a_1, a_2, \ldots \in \mathfrak{A}$ be a dense subset of $\mathfrak{A}$ and passing to subsequences we can assume for each $n \in \mathbb{N}$, we have

$$\|\varphi_n(a_i) - \varphi_{n+1}(a_i)\| \leqslant 1/2^n, \quad i = 1, \ldots, n.$$

By assumption, for each $n \in \mathbb{N}$, let $\psi_n : \mathfrak{A} \longrightarrow \mathfrak{B}$ be the lift of φ_n. We will construct new lifts $\widetilde{\psi}_n$ that converge in point-norm. Namely, we will find $\widetilde{\psi}_n$'s such that for all $n \in \mathbb{N}$ and $i = 1, \ldots, n$

$$\|\widetilde{\psi}_n(a_i) - \widetilde{\psi}_{n+1}(a_i)\| \leqslant 1/2^{n-1}.$$

By Theorem (18.3), let $\{u_\lambda\}$ be a q.a.u. for I. Define
$\widetilde{\psi}_1 := \psi_1$.

For the next map, fix λ. We will specify later which λ.
$\widetilde{\psi}_2(a) := (1-u_\lambda)^{1/2}\psi_2(a)(1-u_\lambda)^{1/2} + u_\lambda^{1/2}\widetilde{\psi}_1(a)u_\lambda^{1/2}$.

Before we estimate, $\|\widetilde{\psi}_2(a) - \widetilde{\psi}_1(a)\|$, we verify some basic limits.

Claim 18.5. *If* $b \in \mathfrak{B}$, *then* $\lim_\lambda \|b(1-u_\lambda) - (1-u_\lambda)^{1/2}b(1-u_\lambda)^{1/2}\| = 0$ *and* $\lim_\lambda \|bu_\lambda - u_\lambda^{1/2}bu_\lambda^{1/2}\| = 0$.

Proof of claim. By q.a.u., $\|[u_\lambda, b]\| \to 0$ implies that $\|[(1-u_\lambda)^{1/2}, b]\| \to 0$ since $(1-u_\lambda)^{1/2}$ is a continuous function of u_λ. Now, note that $(1-u_\lambda)^{1/2}$ is contractive since by the C*-identity $\|(1-u_\lambda)^{1/2}\|^2 = \|1-u_\lambda\| \leqslant 1$. Therefore,

$$\begin{aligned}
\|b(1-u_\lambda) - (1-u_\lambda)^{1/2}b(1-u_\lambda)^{1/2}\| &= \|\left(b(1-u_\lambda)^{1/2} - (1-u_\lambda)^{1/2}b\right)(1-u_\lambda)^{1/2}\| \\
&= \|[(1-u_\lambda)^{1/2}, b](1-u_\lambda)^{1/2}\| \\
&\leqslant \|[(1-u_\lambda)^{1/2}, b]\| \cdot \|(1-u_\lambda)^{1/2}\| \\
&\leqslant \|[(1-u_\lambda)^{1/2}, b]\| \cdot 1 \to 0.
\end{aligned}$$

The proof of the second limit is similar. □

Let $\varepsilon > 0$. The $\approx_{2\varepsilon}$ in the following expression is provided by the Claim and the $\approx_{\varepsilon}$ is provied by Lemma (18.4).

$$\begin{aligned}
\|\widetilde{\psi}_2(a) - \widetilde{\psi}_1(a)\| &= \|(1-u_\lambda)^{1/2}\psi_2(a)(1-u_\lambda)^{1/2} + u_\lambda^{1/2}\widetilde{\psi}_1(a)u_\lambda^{1/2} - \widetilde{\psi}_1(a)\| \\
&\approx_{2\varepsilon} \|\psi_2(a_1)(1-u_\lambda) + \widetilde{\psi}_1(a_1)u_\lambda - \widetilde{\psi}_1(a_1)\| \\
&= \|\psi_2(a_1)(1-u_\lambda) - \widetilde{\psi}_1(a_1)(1-u_\lambda)\| \\
&= \|(\psi_2(a_1) - \widetilde{\psi}(a_1))(1-u_\lambda)\| \\
&\approx_{\varepsilon} \|\dot{\psi}(a_1) - \dot{\widetilde{\psi}}_1(a_1)\| \\
&= \|\varphi_2(a_1) - \varphi_1(a_1)\| < 1/2
\end{aligned}$$

So, fix $\varepsilon = 1/6$ and choose λ for $\widetilde{\psi}_2$ dependent on this ε. Thus, by the above expression

$$\|\widetilde{\psi}_2(a) - \widetilde{\psi}_1(a)\| < 1/2 + 3\varepsilon = 1/2 + 3(1/6) = 1.$$

Next, fix η and define
$\widetilde{\psi}_3 := (1-u_\eta)^{1/2}\psi_3(1-u_\eta)^{1/2} + u_\eta^{1/2}\widetilde{\psi}_2 u_\eta^{1/2}$. And, similarly define η such that

$$\|\widetilde{\psi}_3(a_i) - \widetilde{\psi}_2(a_i)\| < 1/2, \quad i = 1, 2.$$

Continue in this manner to define $\widetilde{\psi}_n$ for all $n \in \mathbb{N}$.

Recalling another basic analysis fact, if $\|x_n - x_{n+1}\| < 1/2^{n-1}$ for all $n \in \mathbb{N}$, then x_n converges. This is an easy exercise.

Thus, define ψ on the dense subset $(a_i) \subset \mathfrak{A}$ by $\widetilde{\psi}_n(a_i) \xrightarrow[n\to\infty]{} \psi(a_i)$ for all $i \in \mathbb{N}$. Now, an application of Stinespring's Theorem (15.13) is that the norm of u.c.p. maps are bounded by 1. Thus, this uniform bound and the density of $(a_i) \subset \mathfrak{A}$ together with the above imply that limit that ψ can be extended to all of $\mathfrak{A}$ with $\|\psi\| \leqslant 1$. Now, $\widetilde{\psi}_n(a) \to \psi(a)$ for all $a \in \mathfrak{A}$ implies that ψ is unital and positive since positive elements form a closed set. Also, as ψ_n are completely positive, one could show that ψ is completely positive. Finally,

$$\dot{\psi}(a) = \lim \dot{\widetilde{\psi}}(a) = \lim \varphi_n(a) = \varphi(a).$$

Thus, ψ is a u.c.p lift of φ. □

Theorem 18.6 (1977)**.** *If* $\mathfrak{A}$ *is a nuclear separable C*-algebra, then* $\mathrm{Ext}(\mathfrak{A})$ *is a group.*

What about non-nuclear? In 1978, J. Anderson proved that there exists a C*-subalgerba $\mathfrak{A} \subset C_r^*(\mathbb{F}_2)$ such that $\mathrm{Ext}(\mathfrak{A})$ is not a group, where $C_r^*(\mathbb{F}_2)$ is the reduced group C*-algebra of the free group in 2 generators.

In 2005, Haagerup and Thorbjornsen showed that $\mathrm{Ext}(C_r^*(\mathbb{F}_2))$ is not a group.

Proposition 18.7. $\mathrm{Ext} : \mathfrak{A} \longrightarrow \mathrm{Ext}(\mathfrak{A})$ *is a contravariant functor from the category of nuclear and separable C*-algebras to the category of abelian groups.*

Proof. Let $\mathfrak{A} \xrightarrow{f} \mathfrak{B}$. We define $\mathrm{Ext}(\mathfrak{B}) \xrightarrow{f^*} \mathrm{Ext}(\mathfrak{A})$ in the following way. Let $\tau : \mathfrak{B} \longrightarrow \mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ be an injective *-homomorphism. Now, $\tau \circ f : \mathfrak{A} \longrightarrow \mathcal{B}(\mathcal{H})/\mathfrak{K}(\mathcal{H})$ which is a *-homomorphism that is not necessarily injective. Thus, define $f^*([\tau]) := [\tau \circ f \oplus \dot{\rho}]$ where $\dot{\rho}$ is any trivial extension of $\mathfrak{A}$.

Claim 18.8. *f^* is well-defined*

Proof of claim. This proof involves 2 steps that are left as exercises.

(1) Let $\dot{\rho}_1$ be another trivial extension of $\mathfrak{A}$. Check that $\tau \circ f \oplus \dot{\rho}_1 \sim \tau \circ f \oplus \dot{\rho}$.
(2) If $\tau \sim \tau'$, then $\tau \circ f \oplus \dot{\rho} \sim \tau' \circ f \oplus \dot{\rho}$.

□

Claim 18.9. *f^* is a group homomorphism.*

Proof of claim. Check that $f^*([\tau] \oplus [\sigma]) = f^*([\tau]) \oplus f^*([\sigma])$, which is left as an exercise.

□

The remaining properties of a contravariant functor are easy to check. □

19. 13 October 2016

Today, we discuss how we may compute extension groups. Before we continue, we need one more thing about index.

Proposition 19.1. *If T, S are Fredholm operators, then $j(TS) = j(T) + j(S)$.*

Proof. Recall Corollary (13.9), which states that a family of Fredholm operators on a continuous path have the same index. Now, if $\begin{pmatrix} TS & \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} S & \\ & T \end{pmatrix}$ can be conntected by continuous path of Fredholm operators, then we would be done since Corollary (13.9) along with Observation (13.3) would imply

$$j(TS) = j(TS) + j(1) = j\begin{pmatrix} TS & \\ & 1 \end{pmatrix} = j\begin{pmatrix} S & \\ & T \end{pmatrix} = j(T) + j(S)$$

since 1 is invertible. For the continuous path, for $t \in [0,1]$ define Fredholm operators

$$F_t := \begin{pmatrix} \cos(2\pi t)1 & \sin(2\pi t)1 \\ -\sin(2\pi t)1 & \cos(2\pi t)1 \end{pmatrix} \begin{pmatrix} 1 & \\ & T \end{pmatrix} \begin{pmatrix} \cos(2\pi t)1 & -\sin(2\pi t)1 \\ \sin(2\pi t)1 & \cos(2\pi t)1 \end{pmatrix} \begin{pmatrix} S & \\ & 1 \end{pmatrix}.$$

$$F_0 = \begin{pmatrix} 1 & \\ & T \end{pmatrix} \begin{pmatrix} S & \\ & 1 \end{pmatrix} = \begin{pmatrix} S & \\ & T \end{pmatrix}$$

and

$$\begin{aligned} F_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ & T \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S & \\ & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & T \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ S & 1 \end{pmatrix} \\ &= \begin{pmatrix} TS & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

□

The last tool that we need from index theory for today is the following. By Corollary (13.11), we were able to define index for invertibles in the Calkin algebra. Indeed, if $x \in \mathcal{Q}(\mathcal{H})$ is invertible, then $j(x) := j(T)$ such that $T \in \mathcal{B}(\mathcal{H})$ and $\dot{T} = x$. Next, we move on to our main tool for calculating certian extension groups.

19.1. **Fredholm index map.** Let $\mathfrak{A}$ be a unital C*-algebra.

$\mathrm{Inv}(\mathfrak{A})$ is a group under multiplication.

$\mathrm{Inv}(\mathfrak{A})_0$ is defined to be the connected component of 1. It is a normal subgroup of $\mathrm{Inv}(\mathfrak{A})$. Indeed, let $a, b \in \mathrm{Inv}(\mathfrak{A})_0$, then there are continuouts paths a_t from a to 1 and b_t from b to 1. However, $a_t b_t$ is a continuous path from ab to 1. Now, let $c \in \mathrm{Inv}(\mathfrak{A})$, then ca_tc^{-1} is a continuous path from cac^{-1} to 1, which provides that $\mathrm{Inv}(\mathfrak{A})_0$ is normal. Hence, we may define.

Definition 19.2. $\pi_1(\mathfrak{A}) := \mathrm{Inv}(\mathfrak{A})/\mathrm{Inv}(\mathfrak{A})_0$ is a group since $\mathrm{Inv}(\mathfrak{A})_0$ is a normal subgroup of $\mathrm{Inv}(\mathfrak{A})$. For $a \in \mathrm{Inv}(\mathfrak{A})$, let $[a] \in \mathrm{Inv}(\mathfrak{A})/\mathrm{Inv}(\mathfrak{A})_0$ denote its equivalence class.

Fredholm index map: Fix $[\tau] \in \mathrm{Ext}(\mathfrak{A})$. Fix $a \in \mathrm{Inv}(\mathfrak{A})$. Since $\tau : \mathfrak{A} \longrightarrow \mathcal{Q}(\mathcal{H})$ is an injective *-isomorphism, $\tau(a) \in \mathcal{Q}(\mathcal{H})$ is invertible and by the comments above, $j(\tau(a))$ is defined. We may define

$$a \in \mathrm{Inv}(\mathfrak{A}) \longmapsto j(\tau(a)) \in \mathbb{Z}.$$

However, we would like to replace $\mathrm{Inv}(\mathfrak{A})$ with $\pi_1(\mathfrak{A})$. We can do this if the above map vanishes on $\mathrm{Inv}(\mathfrak{A})_0$. Let $a \in \mathrm{Inv}(\mathfrak{A})_0$. Then, there is a continuous path a_t from a to $1_{\mathfrak{A}}$. Therefore, $\tau(a_t)$ is a continuous path from $\tau(a)$ to $1_{\mathcal{Q}(\mathcal{H})}$. Hence, $j(\tau(a)) = j(1_{\mathcal{Q}(\mathcal{H})}) = 0$. Therefore, we may define

$\gamma_\tau : [a] \in \pi_1(\mathfrak{A}) \longmapsto j(\tau(a)) \in \mathbb{Z}$.

Next, we prove some important properties of this map.

Proposition 19.3. *γ_τ is a group homomorphism.*

Proof. By Proposition (19.1),

$$\gamma_\tau([a][b]) = \gamma_\tau([ab]) = j(\tau(ab)) = j(\tau(a)\tau(b)) = j(\tau(a)) + j(\tau(b)) = \gamma_\tau([a]) + \gamma_\tau([b]).$$

□

Proposition 19.4. *If $\tau \sim \tau'$, then $\gamma_\tau = \gamma_{\tau'}$.*

Proof. There exists unitary $U \in \mathcal{B}(\mathcal{H})$ such that $\tau(\cdot) = \dot{U}^*\tau'(\cdot)\dot{U}$.

Let $a \in \mathrm{Inv}(\mathfrak{A})$. By Proposition (19.1) and Observation (13.3),

$$\begin{aligned}\gamma_\tau([a]) &= j(\tau(a)) \\ &= j(\dot{U}^*\tau'(a)\dot{U}) \\ &= j(\dot{U}^*) + j(\tau'(a)) + j(\dot{U}) \\ &= -j(\dot{U}) + j(\tau'(a)) + j(\dot{U}) \\ &= j(\tau'(a)) = \gamma_{\tau'}([a]).\end{aligned}$$

□

By these last two propositions, the following map is well-defined

$$[\tau] \in \mathrm{Ext}(\mathfrak{A}) \overset{\gamma}{\longmapsto} \gamma([\tau]) := \gamma_\tau \in \mathrm{Hom}(\pi_1(\mathfrak{A}), \mathbb{Z})$$

in which

$$\gamma([\tau])([a]) = \gamma_\tau(a) = j(\tau(a)). \tag{19.1}$$

γ is called the *Fredholm index map.*

In general, if G is a (semi)group, then $\mathrm{Hom}(G, \mathbb{Z})$ is an abelian group with addition defined by: if $f_1, f_2 \in \mathrm{Hom}(G, \mathbb{Z})$ then $(f_1 + f_2)(g) := f_1(g) + f_2(g)$. Thus, it is natural to ask if γ is a (semi)group homomorphism. The answer is yes, and if $\mathfrak{A}$ is nuclear and separable, then $\mathrm{Ext}(\mathfrak{A})$ is a group and γ would be a group homomorphism. This is the following proposition.

Proposition 19.5. *γ is a (semi)group homomorphism.*

Proof. We approach the proof through 3 Claims.

Claim 19.6. $\gamma([\tau] \oplus [\sigma]) = \gamma([\tau]) + \gamma([\sigma])$.

Proof of claim. By Observation (13.3),

$$\begin{aligned}
\gamma([\tau] \oplus [\sigma])([a]) &= \gamma\left(\left[\begin{pmatrix} \tau & \\ & \sigma \end{pmatrix}\right]\right)([a]) \\
&= j\left(\begin{pmatrix} \tau(a) & \\ & \sigma(a) \end{pmatrix}\right) \\
&= j(\tau(a)) + j(\sigma(a)) \\
&= \gamma([\tau])([a]) + \gamma([\sigma])([a]) \\
&= (\gamma([\tau]) + \gamma([\sigma]))([a])
\end{aligned}$$

□

Claim 19.7. $\gamma([\mathit{trivial}]) = 0 \in \mathrm{Hom}(\pi_1(\mathfrak{A}), \mathbb{Z})$.

Proof of claim. Assume that $\dot{\rho} : \mathfrak{A} \longrightarrow \mathcal{Q}(\mathcal{H})$ is a trivial extension, so there exists an injective *-homomorphism $\rho : \mathfrak{A} \longrightarrow \mathcal{B}(\mathcal{H})$ such that $\dot{\rho} = \pi \circ \rho$ where $\pi : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{Q}(\mathcal{H})$ is the quotient map. Note that since ρ is an injective *-homomoprhism, if $a \in \mathrm{Inv}(\mathfrak{A})$, then $\rho(a) \in \mathrm{Inv}(\mathcal{B}(\mathcal{H})$ and thus has 0 index. Thus, since all trivial extensions are in the same equivalence class by Corollary (15.6),

$$\gamma([\text{trivial}])([a]) = j(\dot{\rho}(a)) = j(\rho(a)) = 0, \quad \forall a \in a \in \mathrm{Inv}(\mathfrak{A}).$$

□

Claim 19.8. $\gamma([\tau]^{-1}) = -\gamma([\tau])$.

Proof.
$$0 = \gamma([\text{trivial}]) = \gamma([\tau]^{-1} \oplus [\tau]) = \gamma([\tau]^{-1}) + \gamma([\tau]).$$
□

□

Now, γ is not always an isomorphism (not even for the abelian case in general). However, it is and isomorphism when $\mathfrak{A} = C(X)$, where $X \subset \mathbb{C}$ is compact. But, this was our initial goal. Recall, Main Problem (14.1), which was to classify essentially normal operators with spectrum X. A goal of that section was to show that this problem translated to calculating the group $\mathrm{Ext}(C(X))$, which is stated after Definition (14.6). The fact that γ is an isomorphism in this case is the Brown-Douglas-Fillmore theorem.

Theorem 19.9. *[Brown, Douglas, Fillmore] When X is a compact subset of $\mathbb{C}$,*
$$\gamma : \mathrm{Ext}(C(X)) \longrightarrow \mathrm{Hom}(\pi_1(C(X)), \mathbb{Z})$$
is an isomorphism.

The proof of injectivity is extremely difficult and oustide the scope of this course. However, after introducing some further results, we can tackle surjectivity. We begin with the following, which allows us to identify $\pi_1(C(X))$ in this case.

Theorem 19.10. *When X is a compact subset of $\mathbb{C}$, the group $\pi_1(C(X))$ is the free abelian group generated by $\{[z - \lambda_i]\}$ where z is the identity function on X and each λ_i corresponds to exactly one point in each bounded connected component of $\mathbb{C} \setminus X$.*

Corollary 19.11. *If there are n bounded connected components of $\mathbb{C}\setminus X$, then $\pi_1(C(X)) = \mathbb{Z}^n$.*

On our way to surjectivity, we recall the winding number of a continuous non-zero complex valued function on the circle. Let $f : \mathbb{T} \longrightarrow \mathbb{C} \setminus \{0\}$ be continuous, then the winding number $\mathrm{wind}(f)$ is the number of revolutions of f around $\{0\}$ with respect to 1 revolution around the circle $\mathbb{T}$. For example, if $f(z) = z^2$, then $\mathrm{wind}(f) = 2$, and if $f(z) = z^n$, then $\mathrm{wind}(f) = n$.

We finish today with the following lemma.

Lemma 19.12. *If $f : \mathbb{T} \longrightarrow \mathbb{C} \setminus \{0\}$, then $j(f(\dot{U_+})) = -\mathrm{wind}(f)$, where U_+ is the unilateral shift.*

Proof. Assume that $\mathrm{wind}(f) = n$. Then, f is homotopic to z^n. Indeed, we can write f as $f(z) = |f(z)|e^{i\arg f(z)}$, and define for $t \in [0, 1]$
$$f_t(x) = |f(x)|^{1-t}e^{i((1-t)\arg f(x) + t2\pi i x)}.$$
Thus, $f_0 = f$ and $f_1(\cdot) = e^{i2\pi(\cdot)n} = z^n$.

In particular, $f_t(\dot{U_+})$ forms a continuous path from $f(\dot{U_+})$ to $\dot{U_+}^n$. Therefore, by Proposition (19.1), Observation (13.3), and Corollary (13.9), we have

$$j(f(\dot{U_+}) = j(\dot{U_+}^n) = nj(\dot{U_+}) = nj(U_+) = n \cdot -1 = -n.$$

□

20. 14 October 2016

Recall from yesterday, we defined the Fredholm Index map

$$\gamma : \mathrm{Ext}(\mathfrak{A}) \longrightarrow \mathrm{Hom}(\pi_1(\mathfrak{A}), \mathbb{Z}),$$

where $\mathfrak{A}$ is a C*-algebra. This map is not an isomorphism in general, but it is an isomorphism when we need it to be, which is the case $\mathfrak{A} = C(X)$ for $X \subset \mathbb{C}$ compact, for the purpose of classifying essentially normal operators. This is Brown-Douglas-Fillmore Theorem (19.9). We restate it here for convenience.

Theorem (19.9) [Brown-Douglas-Fillmore]: If $X \subset \mathbb{C}$ is compact, then

$$\gamma : \mathrm{Ext}(\mathfrak{A}) \longrightarrow \mathrm{Hom}(\pi_1(\mathfrak{A}), \mathbb{Z})$$

is an isomorphism.

We only prove surjectitivity. We do not have time to prove injectivity.

Proof of surjectivity in Theorem (19.9). Let $X \subset \mathbb{C}$ be compact. Let $O_1, \ldots, O_n$ be the bounded connected components of $\mathbb{C} \setminus X$. For each $i = 1, \ldots n$, fix $\lambda_i \in O_i$. By Theorem (19.10), $[z - \lambda_i]$ are generators for $\pi_1(C(X))$ where z is the identity function on X.

Now, there are n-generators of $\mathrm{Hom}(\pi_1(C(X)), \mathfrak{A})$, which we denote h_i and are determined by

$$h_i([z - \lambda_k]) = \begin{cases} 1 & : i = k \\ 0 & i \neq k. \end{cases}$$

Thus, for surjectivity, without loss of generality, it is enough to show that there exists $[\tau] \in \mathrm{Ext}(C(X))$ such that $\gamma([\tau]) = h_1$ or that

$$\gamma([\tau])([z - \lambda_i]) = h_i([z - \lambda_k]) = \begin{cases} 1 & : i = 1 \\ 0 & i \neq 1. \end{cases}$$

We can translate this to a problem about essentially normal operators by the one-to-one correspndence (up to certain equivalences) from 06 October lecture. We now recall this correspendence.

Let $\tau : C(X) \longrightarrow \mathcal{Q}(\mathcal{H})$ be an injective *-homomorphism. If z is the identity function on X, then let T be any preimage of $\tau(z)$ under the quotient map. Note that $\sigma_{ess}(T) = X$. Also, if $T \in \mathcal{Q}(\mathcal{H})$ is normal, then define $\tau(f) := f(\dot{T})$.

Thus, if T corresponds to τ by the above correspndence, then $\dot{T} - \lambda_i$ is invertible in $\mathcal{Q}(\mathcal{H})$ and by Expression (19.1), we have the index

$$\gamma([\tau])([z - \lambda_i]) = j(\tau(z - \lambda_i)) = j(\dot{T} - \lambda_i).$$

Therefore, our problem of surjectivity reduces to finding an essentially normal operator T with $\sigma_{ess}(T) = X$ such that

$$j(\dot{T} - \lambda_i) = \begin{cases} 1 & : i = 1 \\ 0 & i \neq 1. \end{cases}$$

Let's first look at some cases for X.

Case 1. *Assume $X = \Gamma$, where Γ is a single simple closed curve.*

Proof of Case 1. Need to find essentially normal operator T such that $j(\dot{T} - \lambda_1) = 1$ and $\sigma_{ess}(T) = \Gamma$, where λ_1 lies inside Γ.

Now, since Γ is a simple closed curve, there exists a homeomorphism $f : \mathbb{T} \longrightarrow \Gamma$ in which $\mathbb{T}$ has counter orientation to Γ. Now, consider the unilateral shift, U_+. $\sigma_{ess}(U_+) = \sigma(\dot{U_+}) = \mathbb{T}$. Hence, by continuous functional calculus, we may define T to be any preimage of $f(\dot{U_+})$ under the quotient map. Now, note by construction and counter orientation, $\mathrm{wind}(f - \lambda_1) = -1$. Therefore, by Lemma (19.12),

$$j(\dot{T} - \lambda_1) = j((f - \lambda_1)(\dot{U_+})) = -\mathrm{wind}(f - \lambda_1) = -(-1) = 1.$$

Also, $\sigma_{ess}(T) = \sigma(f(\dot{U_+})) = f(\sigma(\dot{U_+})) = f(\mathbb{T}) = \Gamma$. □

Case 2. *Assume $X = E$ is a simple closed curve union its interior. In particular, $\mathbb{C} \setminus X$ contains no bounded connected components.*

Proof of Case 2. Let $x_1, x_2, \ldots \in X$ be a countable dense subset of X. Define

$$N = \begin{pmatrix} x_1 & & & & & & \\ & x_1 & & & & & \\ & & \ddots & & & & \\ & & & x_2 & & & \\ & & & & \ddots & & \\ & & & & & x_3 & \\ & & & & & & \ddots \end{pmatrix}.$$

Thus, N is normal and $\sigma_{ess}(N) = X$. □

Case 3. *Assume X is the union of a set from Case 1 and a set from Case 2 in which the two sets are disjoint.*

Proof of Case 3. Assume $X = \Gamma \cup E$. Define T for Γ and N for E from their respective case. Let $M = T \oplus N$. □

Now, consider O_1. Let Γ_0 be a simple cloased curve that contains O_1. Now, the boundary of O_1 is $\Gamma_0 \cup \Gamma_1 \cup \ldots \Gamma_n$ the union of oriented simple closed curves in which Γ_0 has positive orientation and $\Gamma_2, \ldots, \Gamma_n$ have counter orientation to Γ_0. Now, assume the circle $\mathbb{T}$ has positive orientation. Let $f_i : \mathbb{T} \longrightarrow \Gamma_i$ be homeomorphisms and as in Case

1, let T_i be any preimage of $f_i(\dot{U}_+)$. Define

$$T = \begin{pmatrix} T_0 & & & \\ & \ddots & & \\ & & T_n & \\ & & & N \end{pmatrix},$$

where N is normal with $\sigma_{ess}(N) = X$. $\square$

Corollary 20.1 (Classification of essentially normal operators)**.** *If T_1, T_2 are essentially normal operators, then $T_1 \sim T_2$ if and only if $\sigma_{ess}(T_1) = \sigma_{ess}(T_2)$ and the index $j(T_1 - \lambda) = j(T_2 - \lambda)$ for all $\lambda \notin \sigma_{ess}(T_1)$.*

Proof. Only if: Assume that $T_1 \sim T_2$. Thus, there exists unitary $U \in \mathcal{B}(\mathcal{H})$ such that $T_1 = U^*T_2U + K$ where K is compact. We already showed in a previous lecture that $\sigma_{ess}(T_1) = \sigma_{ess}(T_2)$. Now, let $\lambda \notin \sigma_{ess}(T_1)$. Then, $\lambda \notin \sigma(\dot{T}_1) \iff \dot{T}_1 - \lambda \in \mathrm{Inv}(\mathcal{Q}(\mathcal{H})) \implies T_1$ is Fredholm $\implies j(T_1 - \lambda)$ is defined. Thus, since index is unchanged by compact perturbations and index of finite products is sum of index and $j(U^*) = -j(U)$, we have

$$j(T_1 - \lambda) = j(U^*T_2U - \lambda + K) = j(U^*T_2U - U^*\lambda U) = j(U^*(T_2 - \lambda)U) = j(T_2 - \lambda).$$

If: Let T_1, T_2 be essentially normal operators such that $X := \sigma_{ess}(T_1) = \sigma_{ess}(T_2)$ and $j(T_1 - \lambda) = j(T_2 - \lambda)$ for all $\lambda \notin X$. Define injective *-homomorphisms $\tau_1, \tau_2 : C(X) \longrightarrow \mathcal{Q}(\mathcal{H})$ by $\tau_1(f) = f(\dot{T}_1)$ and $\tau_2(f) = f(\dot{T}_2)$, which are well-defined by assumption. Therefore, it is enough to show that $\tau_1 \sim \tau_2$ or equivalently $[\tau_1] = [\tau_2] \in \mathrm{Ext}(C(X))$. Fix one $\lambda_i \notin X$ for each connected component of $\mathbb{C} \setminus X$. Then, by assumption, Expression (19.1), and injectivity of γ by the Brown-Douglas-Fillmore Theorem (19.9),

$$\begin{aligned} j(T_1 - \lambda_i) = j(T_2 - \lambda_i) \;\; \forall i &\implies j(\dot{T}_1 - \lambda_i) = j(\dot{T}_2 - \lambda_i) \;\; \forall i \\ &\implies j(\tau_1(z - \lambda_i)) = j(\tau_2(z - \lambda_i)) \;\; \forall i \\ &\implies \gamma([\tau_1])([z - \lambda_i]) = \gamma([\tau_2])([z - \lambda_i]) \;\; \forall i \\ &\implies \gamma([\tau_1]) = \gamma([\tau_2]) \\ &\implies [\tau_1] = [\tau_2] \in \mathrm{Ext}(C(X)). \end{aligned}$$

$\square$

Recall, that the unilateral shift is essentially normal, but is not of the form "normal + compact." But, for these, we can now classify them.

Corollary 20.2. *An essentially normal operator has the form "normal +compact" if and only if $j(T - \lambda) = 0$ for all $\lambda \notin \sigma_{ess}(T)$.*

Proof. Only if: Assume $T = N + K$, where N is normal and K is compact. Fix $\lambda \notin \sigma_{ess}(T)$ so that $j(T - \lambda)$ is defined and note that $N - \lambda$ is normal and recall that the index is unchanged by compact perturbation. Thus,

$$j(T - \lambda) = j(N - \lambda + K) = j(N - \lambda) = 0.$$

If: Assume $j(T-\lambda)=0$ for all $\lambda \notin \sigma_{ess}(T)$. Let N be normal with $\sigma_{ess}(N)=\sigma_{ess}(T)$. Thus, $N-\lambda$ is Fredholm and normal for all $\lambda \notin \sigma_{ess}(T)$. And, by normality, we have

$$j(N-\lambda)=0=j(T-\lambda)$$

for all $\lambda \notin \sigma_{ess}(T)$. Hence, by previous corollary $T \sim N$ and so there exists unitary $U \in \mathcal{B}(\mathcal{H})$ such that $T=U^*NU+K$ where K is compact. But, U^*NU is normal. □

Assume X is a topological space, then $X \longmapsto \mathrm{Ext}(C(X))$ is a covariant functor and a homotopy invariant (this is difficult). Also, there exist a sequence of covariant homotopy invariant functors for $X \longmapsto \mathrm{Ext}(C(S^nX))$ with long exact sequences that forms generalized homology theory.

On the other hand, K-theory is a generalized cohomology theory (that is $X \longmapsto K_0(X \times R^n)$ is a sequence of contravariant functors which are homotopy invariant and form long exact sequence). It turned out, that the above generalized homology theory constructed by Brown, Douglas and Fillmore is dual to K-theory. Thus Brown, Douglas and Fillmore found a concrete realization of K-homologies (K-homologies is by definition the dual homology theory for K-theory and it was a problem of great interest to find out its concrete realization)

21. Exercises

21.1. **Day 2.**

(1) Let $\varphi : E \longrightarrow F$ be a morphism of vector bundles. Prove that φ is an isomorphism $\iff$ $\varphi|_{E_x}$ is an isomorphism for all $x \in X$.
(2) Prove that TS^1 is trivial.

21.2. **Day 3.**

(1) Finish the proof of the "If:" part of Theorem (3.4).

21.3. **Day 4.**

(1) Prove Proposition (4.3).
(2) Prove Proposition (4.5).

21.4. **Day 5.**

(1) Prove Proposition (5.3).
(2) Prove Proposition (5.4).
(3) Prove Proposition (5.6).
(4) Prove Proposition (5.7).
(5) Prove Proposition (5.8).

21.5. **Day 6.**

(1) Finish the proof of Proposition (6.5).

21.6. **Day 9.**

(1) Prove that Definition (9.2) and Definition (9.3) are equivalent.
(2) Verify (2) of Example (9.5).
(3) Finish the proof of Lemma (9.11).

21.7. **Day 11.**

(1) Prove Step (2) in the proof of Theorem (11.1).

21.8. **Day 13.**

(1) If N is normal and Fredholm, then $j(N) = 0$.
(2) Is the set of all invertible operators dense in $\mathcal{B}(\mathcal{H})$?

21.9. **Day 14.**

(1) Show that if $\tau_1 \sim \tau_2, \tau_1' \sim \tau_2'$, then $[\tau_1] + [\tau_2] = [\tau_1'] + [\tau_2']$.
(2) Prove Lemma (14.8).

21.10. **Day 15.**

(1) Solve Exercise (15.12).

21.11. **Day 16.** Solve Exercises (16.2, 16.3).

21.12. **Day 17.** Solve Exercises (17.3, 17.7)

21.13. **Day 18.** Prove Claims (18.8, 18.9).

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