Trapped submanifolds and singularity theorems

José M M Senovilla

Department of Theoretical Physics and History of Science
University of the Basque Country UPV/EHU, Bilbao, Spain

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Outline

The Penrose singularity theorem

Theorem (The 1965 Penrose singularity theorem)

If $(\mathcal{V}, g)$ contains a non-compact Cauchy hypersurface Σ and a closed future-trapped surface, and if the null convergence condition holds, then $(\mathcal{V}, g)$ is future null geodesically incomplete.

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The classical Hawking-Penrose theorem

Theorem (Hawking and Penrose)

If the convergence, causality and generic conditions hold and if there is one of the following:

- *a closed achronal set without edge,*
- *a closed trapped surface,*
- *a point with re-converging light cone*

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What about co-dimensions $3, \ldots, n-1$ — for instance, closed spacelike curves?

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We need a unification of the concept of trapping for arbitrary co-dimension: $\Longrightarrow$ The mean curvature vector $\vec{H}$!

Mathematical interlude: trapped submanifolds

Definition (Codimension-m embedded submanifold)

*A **submanifold** is (ζ, Φ), where ζ is an $(n-m)$-dimensional oriented manifold and $\Phi : \zeta \longrightarrow \mathcal{V}$ is an embedding.*

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$$\mu, \nu \dots = 1, \dots, n \quad A, B, \dots = m+1, \dots, n$$

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$$\gamma_{AB} = g_{\mu\nu}(\Phi) e^\mu_A e^\nu_B$$

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Decomposing into tangent and normal parts we have

$$e^\rho_A \nabla_\rho e^\mu_B = \overline{\Gamma}^C_{AB} e^\mu_C - K^\mu_{AB}$$

Notation: extrinsic curvature

K^{μ}_{AB} is called the **shape tensor** or *second fundamental form vector* of ζ in $\mathcal{V}$.

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relative to any normal vector $\vec{n}$ is:

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These are 2-covariant symmetric tensor fields on ζ.

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- If they correspond to (future) null normals, they are called (future) null second fundamental forms.

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The *mean curvature vector:*

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Future-trapped subamanifolds: $\vec{H}$ is future on ζ

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A spacelike submanifold ζ is said to be ***future trapped*** *(f-trapped from now on) if $\vec{H}$ is timelike and future-pointing everywhere on ζ, and similarly for past trapped.*

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$\vec{H}$	Type of submanifold
timelike future	f-trapped
causal and future	weakly f-trapped
consistently null and future	marginally f-trapped
consistently null	marginally outer trapped
zero	stationary or minimal

The case of co-dimension 2

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- Hence, the traditional trapping conditon $\theta^{\pm} < 0$ is obviously equivalent to H^{μ} being timelike and future.
- Similarly, the marginally trapped case corresponds to (say) $\theta^{+} = 0$ and $\theta^{-} < 0$.
- One thus recovers the traditional Penrose definitions.

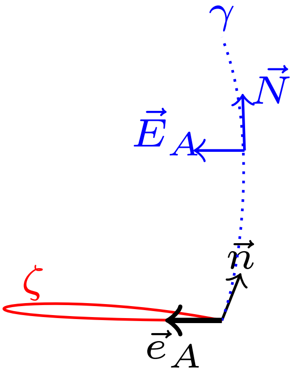
Existence of focal points (Galloway & JMMS (2010))

Notation

- n^μ: *future-pointing* normal to the spacelike submanifold ζ,
- γ: geodesic curve tangent to n^μ at ζ
- u: affine parameter along γ ($u = 0$ at ζ).
- N^μ: geodesic vector field tangent to γ $(N_\mu|_{u=0} = n_\mu)$.
- E^μ_A: vector fields defined by parallelly propagating e^μ_A along γ $(E^\mu_A|_{u=0} = e^\mu_A)$
- By construction $g_{\mu\nu}E^\mu_A E^\nu_B$ is independent of u, so that $g_{\mu\nu}E^\mu_A E^\nu_B = g_{\mu\nu}e^\mu_A e^\nu_B = \gamma_{AB}$
- $P^{\nu\sigma} \equiv \gamma^{AB}E^\nu_A E^\sigma_B$ (at $u = 0$ this is the projector to ζ).

Note that $N_\nu P^{\nu\sigma} = 0$ and $N^\mu \nabla_\mu P^{\nu\sigma} = 0$ all along γ.

Notation on a picture



Existence of focal points

Proposition

Let ζ be a spacelike submanifold of co-dimension m, and let n_μ be a future-pointing normal to ζ. If $\theta(\vec{n}) < 0$ and the curvature tensor satisfies the inequality

$$\boxed{R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} \geq 0} \tag{1}$$

along γ, then there is a point focal to ζ along γ at or before $u = (m-n)/\theta(\vec{n})$, provided γ is defined up to that point.

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Instead of using a typical Raychaudhuri equation, in order to prove this result one uses the energy index form.

Remarks:

1. Spacelike hypersurfaces: $m = 1$, there is a unique timelike orthogonal direction n^μ. Then $P^{\mu\nu} = g^{\mu\nu} - (N_\rho N^\rho)^{-1} N^\mu N^\nu$ and (1) reduces to

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2. Spacelike 'surfaces': $m = 2$, there are two independent null normals on ζ, say n^μ and ℓ^μ. (Define L^μ parallelly propagating ℓ^μ on γ). Then, $P^{\mu\nu} = g^{\mu\nu} - (N_\rho L^\rho)^{-1}(N^\mu L^\nu + N^\nu L^\mu)$ and again (1) reduces to

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(the *null convergence condition* along γ).

The curvature condition

For co-dimension $m > 2$, the interpretation of condition (1) can be given physically in terms of tidal forces, or geometrically in terms of sectional curvatures.

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Timelike *unit* normal n_μ

Sectional curvature $\mathcal{K}(n, e)$ relative to the plane $\langle \vec{n}, \vec{e} \rangle$ $(n_\mu e^\mu = 0)$

$$R_{\mu\nu\rho\sigma} n^\mu e^\nu n^\rho e^\sigma = \mathcal{K}(n, e)(n_\rho n^\rho)(e_\rho e^\rho) = -\mathcal{K}(n, e)(e_\rho e^\rho)$$

Hence (1): *the sum of the $n - m$ sectional curvatures relative to a set of independent and mutually orthogonal timelike planes aligned with n^μ is non-positive, and remains so along γ.*

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Hence (1): *the sum of the $n - m$ sectional curvatures relative to a set of independent and mutually orthogonal timelike planes aligned with n^μ is non-positive, and remains so along γ.*

In physical terms, this is a statement about the attractiveness of the gravitational field on average. **The tidal force in directions initially tangent to ζ is attractive on average.**

The curvature condition

Null normal n^μ

For a null normal n^μ one may consider analogously,

$$R_{\mu\nu\rho\sigma}n^\mu e^\nu n^\rho e^\sigma = -\mathcal{K}(n,e)(e_\rho e^\rho)$$

where $n_\mu e^\mu = 0$, and $\mathcal{K}(n,e)$ is called the *null* sectional curvature relative to the plane spanned by $\vec{n}$ and $\vec{e}$.

Hence (1): *the sum of the $n-m$ null sectional curvatures relative to a set of independent and mutually orthogonal null planes aligned with n^μ is non-positive, and remains so along γ.*

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Proposition (Intermediate result)

Let ζ be a closed f-trapped submanifold of co-dimension $m > 1$, and assume that

$$R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} \geq 0 \tag{1}$$

for any future-pointing null normal n^μ. Then, either $E^+(\zeta)$ is compact, or the spacetime is future null geodesically incomplete, or both.

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Remark: The case with $m = 1$ is not included here because it is trivial. If ζ is a spacelike hypersurface, then $E^+(\zeta) \subset \zeta$ —and actually $E^+(\zeta) = \zeta$ if ζ is achronal—, and the compactness of $E^+(\zeta)$ follows readily without any further assumptions.

The generalized Penrose singularity theorem

Theorem (Generalized Penrose singularity theorem)

If $(\mathcal{V}, g)$ contains a non-compact Cauchy hypersurface Σ and a closed f-trapped submanifold ζ of arbitrary co-dimension, and if

$$R_{\mu\nu\rho\sigma} N^{\mu} N^{\rho} P^{\nu\sigma} \geq 0 \tag{1}$$

holds along every future-directed null geodesic emanating orthogonally from ζ, then $(\mathcal{V}, g)$ is future null geodesically incomplete.

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- Then the canonical projection on Σ of the compact achronal $E^+(\zeta)$ would have to have a boundary, ergo the contradiction.

The Hawking-Penrose singularity theorem

Proposition (Intermediate result)

If $(\mathcal{V}, g)$ is strongly causal and there is a closed f-trapped submanifold ζ of arbitrary co-dimension $m > 1$ such that

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holds along every null geodesic emanating orthogonally from ζ, then either $E^+(E^+(\zeta) \cap \zeta)$ is compact, or the spacetime is null geodesically incomplete, or both.

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Theorem (Generalized Hawking-Penrose singularity theorem)

If the chronology, generic and convergence conditions hold and there is a closed f-trapped submanifold ζ of arbitrary co-dimension such that

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- Spacelike 'surfaces' $m = 2$: Condition (1) is actually included in the convergence condition.

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- Spacelike hypersurfaces $m = 1$: no null geodesics orthogonal to ζ ergo no need to assume (1) (nor anything concerning $\vec{H}$)
- Spacelike 'surfaces' $m = 2$: Condition (1) is actually included in the convergence condition.
- Points $m = n$: The 'same' happens.

These three cases cover the original Hawking-Penrose theorem.

Focal points with a milder curvature condition

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Proposition

Let ζ be a spacelike submanifold of co-dimension m and n^μ a future-pointing normal to ζ. If, along γ (assumed to be future complete) the curvature tensor satisfies,

$$\int_0^\infty R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} du > \theta(\vec{n}) \,,$$

then there is a point focal to ζ along γ.

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- Observe that there is no restriction on the sign of $\theta(\vec{n})$.
- Note also that, unlike before, this proposition does not restrict the location of the focal point, but this turns out to be irrelevant to prove singularity theorems

2nd generalized Penrose singularity theorem

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along each future inextensible null geodesic $\gamma : [0, a) \to \mathcal{V}$ emanating orthogonally from ζ with initial tangent n^μ, then $(\mathcal{V}, g)$ is future null geodesically incomplete.

2nd generalized Penrose singularity theorem

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Thus, for example, even if ζ is only weakly or marginally f-trapped, or minimal, the future null geodesic incompleteness still follows, provided the inequality (1) is strict at least at one point on each future directed null geodesic γ emanating orthogonally from ζ.

Selected applications

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- In dimension 11, say, there are now 10 different possibilities for the boundary condition in the theorems
- **As a (provocative) example I will discuss the possible instability of *compact* extra-dimensions.**

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$$ds^2 = -A^2 dt^2 + B^2 d\rho^2 + F^2 d\varphi^2 + E^2 dz^2,$$

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- The cylinders with constant t and ρ are geometrically preferred; however, they are *not* compact in general

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- Thereby, many results on incompleteness of geodesics can be found.
- Moreover, there arises a new hypersurface, defined as the set of points where dF is null, which is a new type of horizon, being a boundary separating the trapped from the untrapped circles, and containing marginally trapped circles.

Application: asymptotically de Sitter cosmologies

Theorem

Let $(\mathcal{V}, g)$ have all null sectional curvatures non-positive. Suppose Σ is a compact Cauchy hypersurface for $(\mathcal{V}, g)$ which is expanding to the future in all directions, i.e., which has positive definite second fundamental form with respect to the future pointing normal. Then, if $\pi_1(\Sigma)$ has non-finite cardinality, $(\mathcal{V}, g)$ is past null geodesically incomplete.

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- On the other hand, our condition (1) on tidal forces is satisfied strictly in the FLRW models, as well as in sufficiently small perturbations of those models.

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- However, passing to a covering spacetime one can get the result.
- Of course, this theorem has a dual version to the future, if the compact Cauchy hypersurface is contracting.

Example: instability of spatial extra dimensions?

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- His argument: take the typical (super-)string classical spacetime $\mathcal{V} \times \mathcal{Y}$ with the product metric

  $$ds^2 = g_{ab}dx^a dx^b + \gamma_{AB}dx^A dx^B$$

  where $(\mathcal{Y}, \gamma_{AB})$ is a Calabi-Yau 6-dimensional manifold and g_{ab} the metric of the large visible 4-dimensional spacetime.

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- He then splits $\mathcal{V} = \mathbb{R} \times V_3$ and considers the 7-dimensional spacetime given by $\mathbb{R} \times \mathcal{Y}$, with metric $-dt^2 \oplus \gamma_{AB}$.
- He can then apply the Hawking-Penrose theorem using the compact hypersurface given by any $t =$const. in this 7-dimensional spacetime.

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- Thus, one would only need that the condition $\int_0^a R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} du > 0$ held along the null geodesics orthogonal to $\mathcal{Y}$.
- Actually, $R_{\mu\nu\rho\sigma} N^\mu N^\rho P^{\nu\sigma} = 0$, but one sees that the slightest perturbation will destroy this fine tuned equality, and lead to geodesic incompleteness.

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- For instance, for a 5-dimensional submanifold the last term is simply $\bar{R}_{AC}N^{A}N^{C}$, and in principle one can choose submanifolds such that the integrated condition is satisfied.
- Hence, the basic argument of Penrose acquires a wider applicability and requires less restrictions.

References, and thanks

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Thank you for your attention

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