Singularities of general relativity and their quantum fate
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Singularity resolution in **Wheeler–DeWitt** quantum cosmology

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The Wheeler–DeWitt equation in quantum cosmology

- full Wheeler–DeWitt equation is mathematically difficult to handle

➡ quantization of a symmetry-reduced model of the universe

- consider a spatially flat homogeneous and isotropic universe

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\,\mathrm{d}\Omega_3^2$$

with a minimally coupled scalar field ϕ with potential $V(\phi)$

- infinitely many degrees of freedom of *"superspace"* are reduced to two:

➔ scale factor a and scalar field ϕ ➔ ***<u>mini</u>superspace***

➡ Wheeler–DeWitt equation: $\kappa^2 := 8\pi G$

$$\frac{\hbar^2}{2}\left(\frac{\kappa^2}{6a^2}\frac{\partial}{\partial a}\left(a\,\frac{\partial}{\partial a}\right) - \frac{1}{a^3}\,\frac{\partial^2}{\partial \phi^2}\right)\Psi(a,\phi) + a^3\,V(\phi)\,\Psi(a,\phi) = 0$$

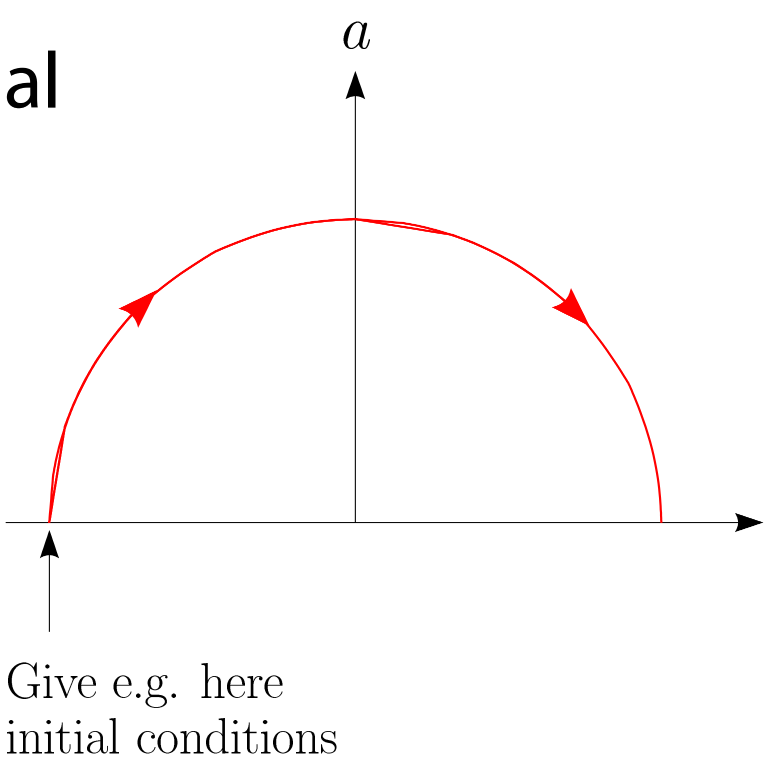
- clearly avoids mathematical problems of the functional WDW equation

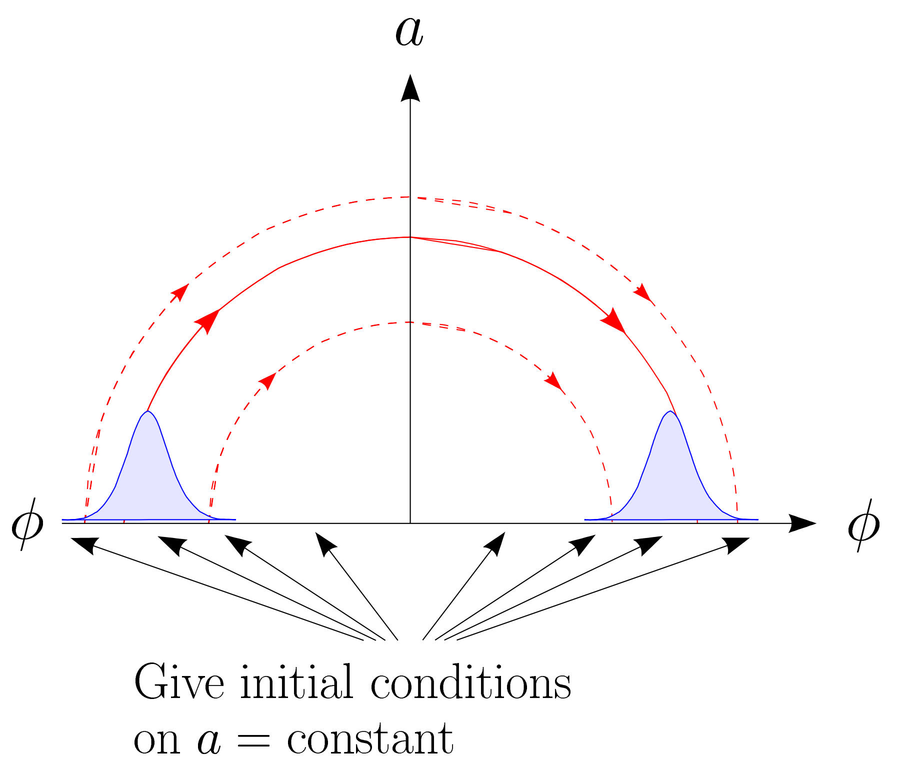
Solving the Wheeler–DeWitt equation

- to simplify the equation set: $a \equiv a_0 \exp(\alpha)$

$$\frac{\hbar^2}{2}\left(\frac{\kappa^2}{6}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2}\right)\Psi(\alpha,\phi) + a_0^6\,\mathrm{e}^{6\alpha}\,V(\phi)\,\Psi(\alpha,\phi) = 0$$

- evaluate this equation for a cosmological model of your choice
  - choose a potential
  - give boundary conditions
  - ***example:*** *cyclic model*





  - apply Born–Oppenheimer approximation: $\Psi(\alpha,\phi) = \varphi(\alpha,\phi)\,C(\alpha)$

➡ *wave function of the Universe*

Classical singularities in cosmology

- most prominent singularity: *Big Bang* $a \to 0\,,\ \rho \to \infty\,,\ |P| \to \infty$

  energy density (ρ) pressure ($|P|$)

- depending on what kind of matter/energy is present in the universe, a variety of **future** singularities can occur:

  Nojiri, Odintsov and Tsujikawa, Phys. Rev. D **71**, 063004 (2005).

  - Type I – "Big Rip": for $t \to t_{\rm sing}$: $a \to \infty\,,\ \rho \to \infty\,,\ |P| \to \infty$
  - Type II – "Big Brake": for $t \to t_{\rm sing}$: $a \to a_{\rm sing}\,,\ \rho \to \rho_{\rm sing}\,,\ |P| \to \infty$
  - Type III – "Big Freeze": for $t \to t_{\rm sing}$: $a \to a_{\rm sing}\,,\ \rho \to \infty\,,\ |P| \to \infty$
  - Type IV – ("Big Separation"):

    for $t \to t_{\rm sing}$: $a \to a_{\rm sing}\,,\ \rho \to \rho_{\rm sing}\,,\ |P| \to P_{\rm sing}$,

    2nd and higher time derivatives of H diverge
  - Type V: divergence of the barotropic index $w(t)$

➡ *Can these singularities be resolved in quantum cosmology?*

Construction of a quantum-cosmological model

- universe filled with ideal fluid, equation of state: $P = f(\rho)$
  - conventional: $P = w\rho$, Chaplygin gas: $P = -\dfrac{A}{\rho}$
- convert this to a scalar field ϕ with potential $V(\phi)$

$$\rho \stackrel{!}{=} \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad\qquad P \stackrel{!}{=} \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

➡ **Wheeler–DeWitt equation:**

$$\frac{\hbar^2}{2}\left(\frac{\kappa^2}{6}\frac{\partial^2}{\partial\alpha^2} - \frac{\partial^2}{\partial\phi^2}\right)\Psi(\alpha,\phi) + a_0^6\,\mathrm{e}^{6\alpha}\,V(\phi)\,\Psi(\alpha,\phi) = 0$$

- use Born–Oppenheimer approximation: $\Psi(\alpha,\phi) = \varphi(\alpha,\phi)\,C(\alpha)$
- require φ to satisfy: $-\dfrac{\hbar^2}{2}\dfrac{\partial^2\varphi}{\partial\phi^2} + a_0^6\,\mathrm{e}^{6\alpha}V(\phi)\,\varphi = E(\alpha)\,\varphi$

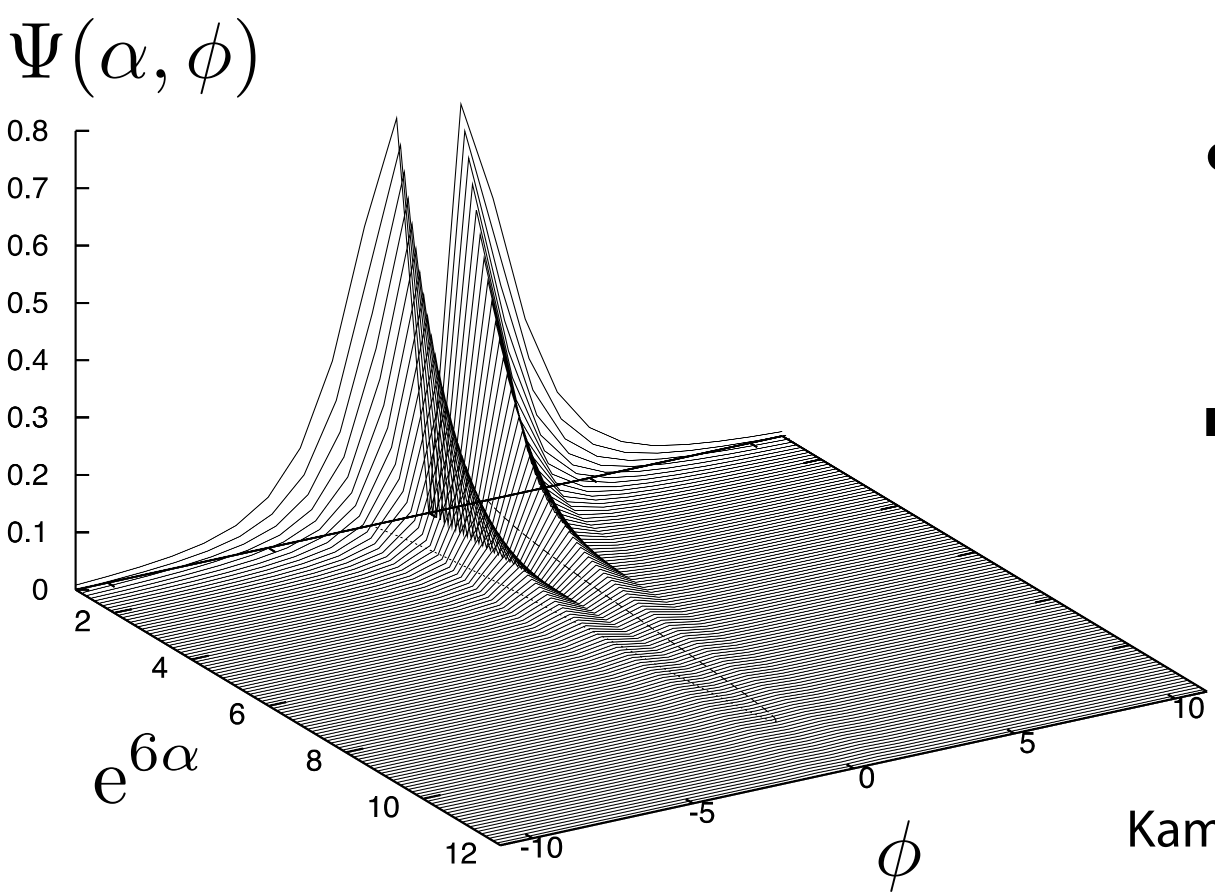
➡ gravitational part: $\dfrac{\partial C}{\partial\alpha}\dfrac{\partial\varphi}{\partial\alpha} + C\dfrac{\partial^2\varphi}{\partial\alpha^2} + \left(\dfrac{\kappa^2}{6}\dfrac{\partial^2 C}{\partial\alpha^2} + 2E(\alpha)\,C\right)\varphi = 0$

Example 1: Big Brake

- expansion of the universe comes to a halt in the future, $t \to t_{\rm sing}$:

$$a \to a_{\rm sing}\ ,\ \ \rho \to \rho_{\rm sing}\ ,\ \ |P| \to \infty\ ,\ \ \dot{H} \to \infty$$

  - can be described by an ***anti***-Chaplygin gas: $P = A/\rho$
  - potential for ϕ: $V(\phi) \propto \left[\sinh\left(\sqrt{3}\kappa|\phi|\right) - \sinh^{-1}\left(\sqrt{3}\kappa|\phi|\right)\right]$
  - WDW equation: $\dfrac{\hbar^2}{2}\left(\dfrac{\kappa^2}{6}\dfrac{\partial^2}{\partial\alpha^2} - \dfrac{\partial^2}{\partial\phi^2}\right)\Psi(\alpha,\phi) - \dfrac{\tilde{V}_0}{|\phi|}\,\mathrm{e}^{6\alpha}\,\Psi(\alpha,\phi) = 0$

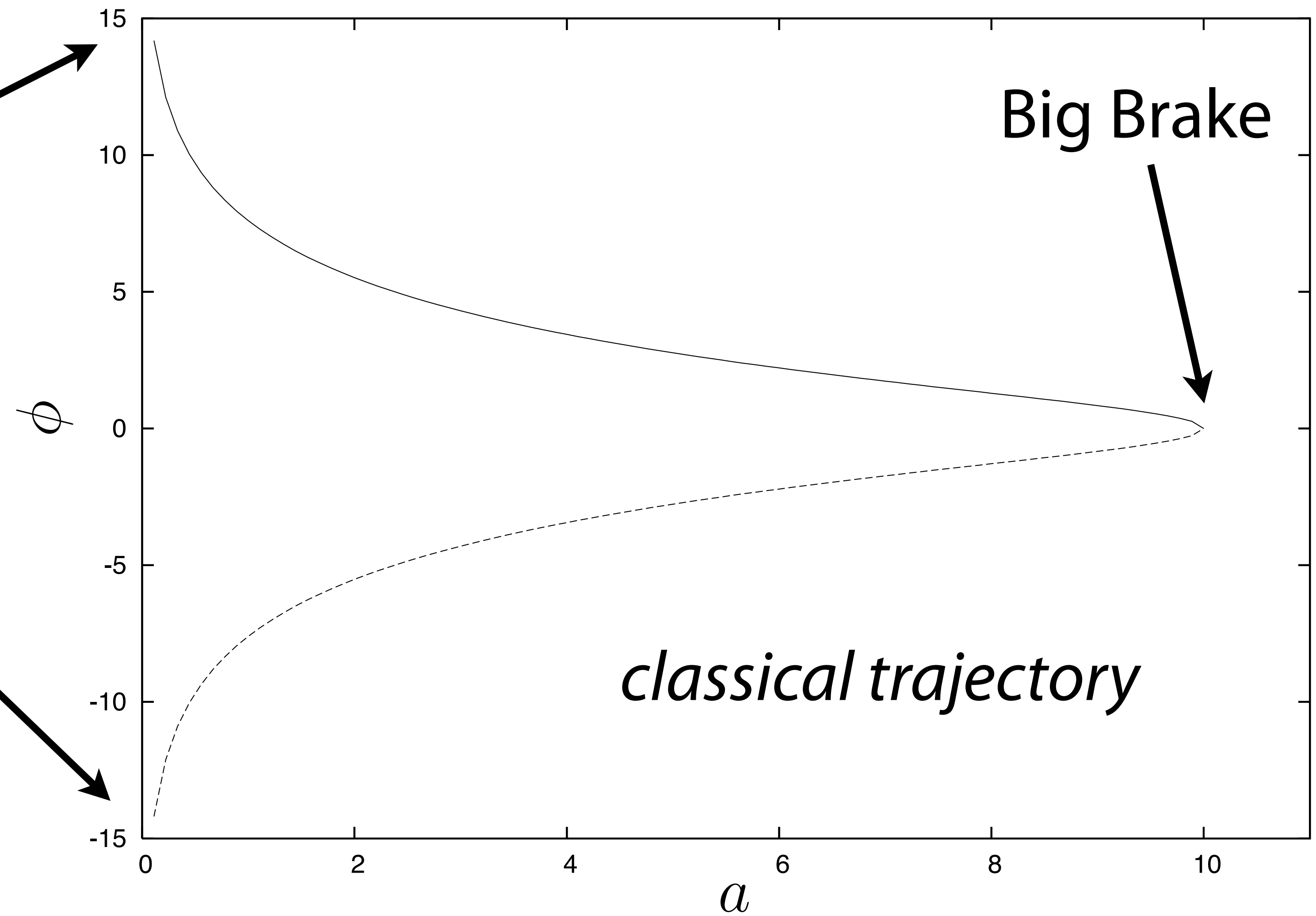


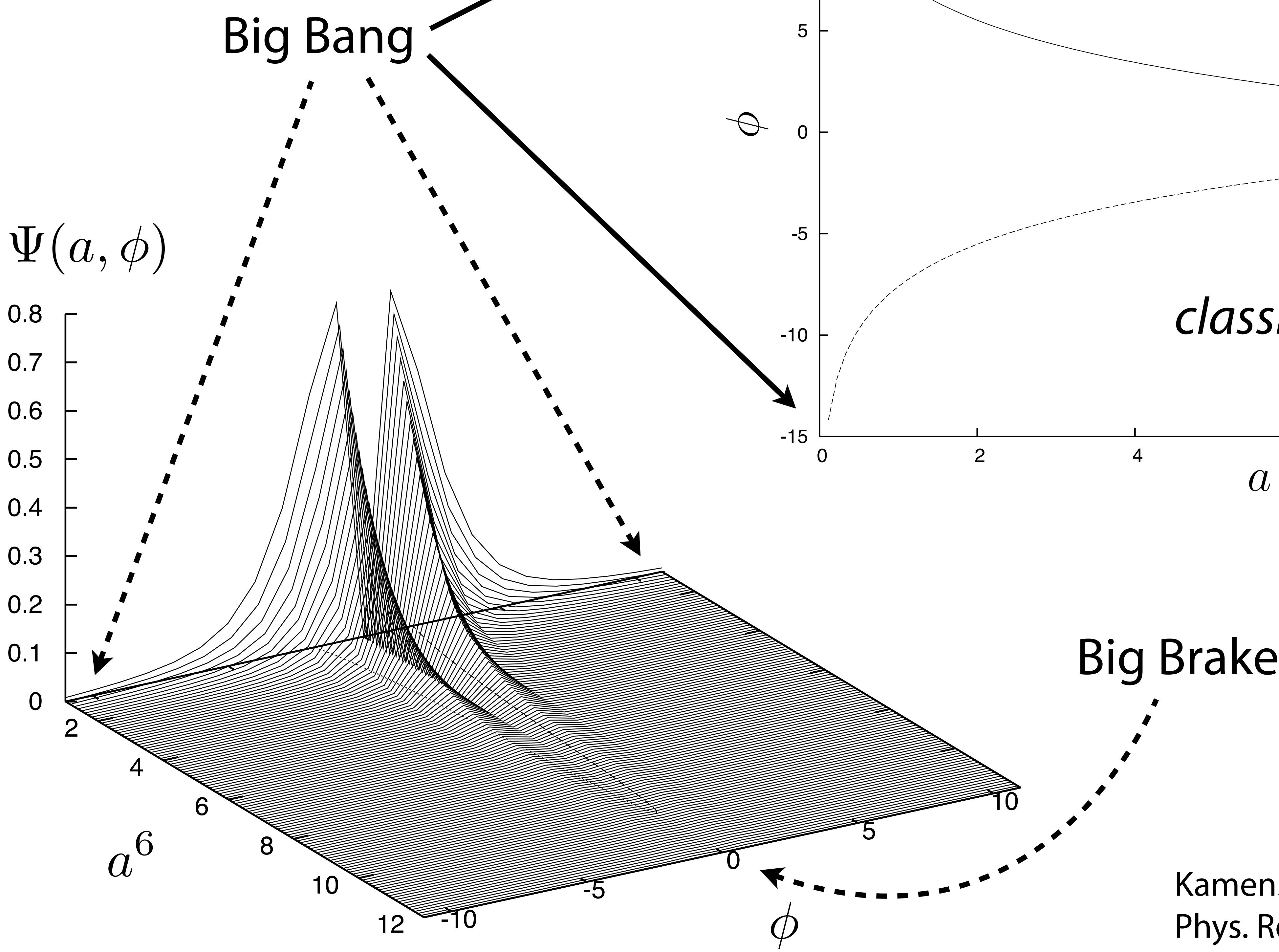
- can be solved analytically after using Born–Oppenheimer approximation

➡ normalizable solutions vanish at the classical singularities (Big Brake *and* Big Bang)

Kamenshchik, Kiefer and Sandhöfer, Phys. Rev. D **76**, 064032 (2007).

Example 1: Big Brake





Kamenshchik, Kiefer and Sandhöfer,
Phys. Rev. D **76**, 064032 (2007).

Example 2: Big Rip

- from observations (SN Type Ia): expansion of universe is accelerating
- one way to model this acceleration: **dark energy** ➔ negative pressure

$$P = w\rho \qquad\qquad w < -\frac{1}{3}$$

- observationally even $w \lesssim -1$ cannot be excluded
  - phantom field, violates the null energy condition $\rho + P > 0$
  - induces Big Rip singularity: $a \to \infty\,,\ \rho \to \infty\,,\ |P| \to \infty$ for $t \to t_{\text{sing}}$
  - WDW equation becomes elliptic

$$\frac{\hbar^2}{2}\left(\frac{\kappa^2}{6}\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\phi^2}\right)\Psi(\alpha,\phi) + \mathrm{e}^{6\alpha}\left(V_0\,\cosh^2\left(\frac{\phi}{F}\right) + \frac{\Lambda}{6}\right)\Psi(\alpha,\phi) = 0$$

  - wave-packet solutions *disperse* at the classical singularity

➡ time & classical evolution come to an end, stationary quantum state left

Dąbrowski, Kiefer and Sandhöfer, Phys. Rev. D **74**, 044022 (2006).

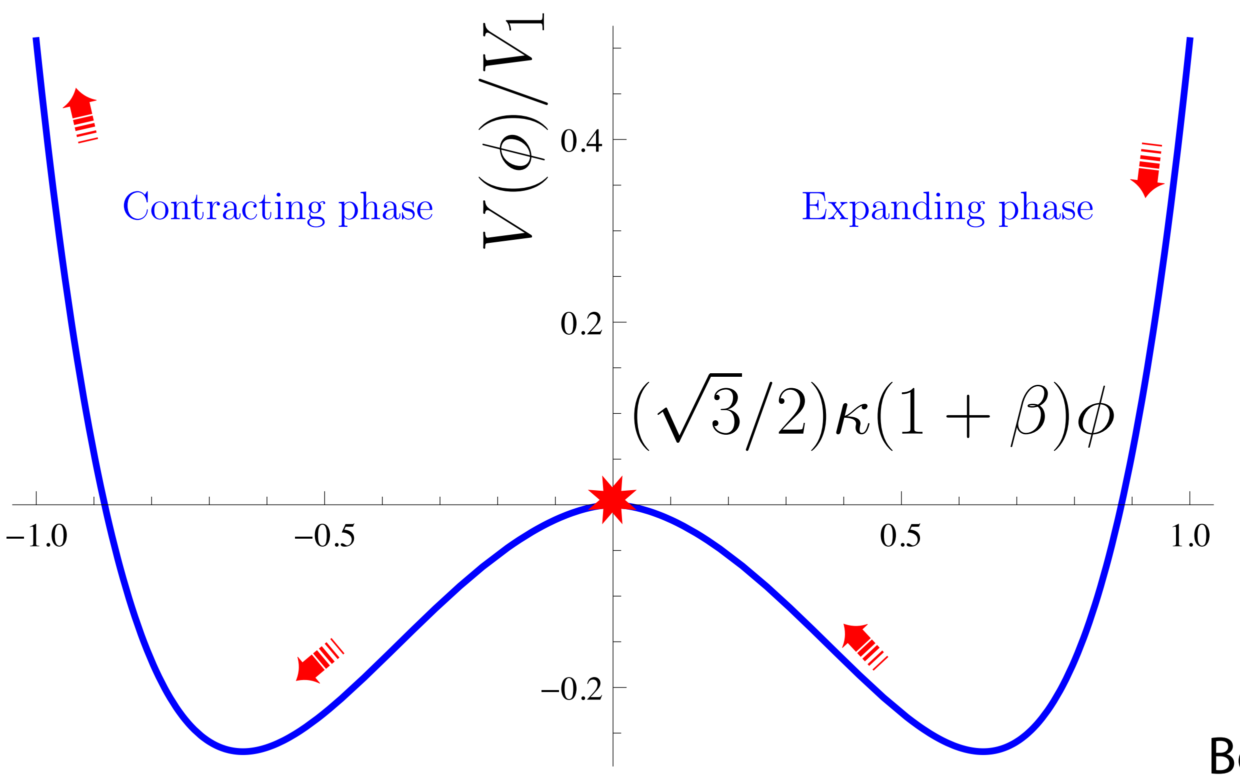
Example 3: Type IV singularity

- expansion of the universe comes to a halt in the future, $t \to t_{\rm sing}$:

$$a \to a_{\rm sing}\ ,\ \rho \to \rho_{\rm sing}\ ,\ P \to P_{\rm sing}\ ,\ \ddot{H}, \dddot{H}, \ldots \to \infty$$

  - can be described by a ***generalized*** anti-Chaplygin gas: $P = A/\rho^{\beta}$
  - type IV singularity for: $-\frac{1}{2} < \beta < 0$ for $\beta \neq \frac{1}{2p} - \frac{1}{2}$, where $p \in \mathbb{N}$

$$V(\phi) \propto \left[\sinh^{\frac{2}{1+\beta}} \left(\frac{\sqrt{3}}{2} \kappa |1+\beta||\phi| \right) - \frac{1}{\sinh^{\frac{2\beta}{1+\beta}} \left(\frac{\sqrt{3}}{2} \kappa |1+\beta||\phi| \right)} \right]$$



- double-well potential (➔ *tunnelling?*)
- quantum analysis: $\beta = -\frac{1}{2}$

➡ not exactly a type IV singularity, but "close enough" (form of potential does not change)

Bouhmadi-López, Kiefer and M.K., Phys. Rev. D **89**, 064016 (2014).

Example 3: Type IV singularity

- after Born–Oppenheimer decomposition $\Psi(\alpha,\phi)=\varphi(\alpha,\phi)\,C(\alpha)$, matter part of the WDW equation reads

$$(1+x^2)\,\frac{\partial^2\varphi}{\partial x^2}+x\frac{\partial\varphi}{\partial x}-\xi x^2(x^2-1)\varphi=-\epsilon\varphi \qquad x:=\sin\left(\frac{\sqrt{3}}{4}\kappa\phi\right) \qquad \xi:=\frac{32V_1a_0^6\mathrm{e}^{6\alpha}}{3\kappa^2} \qquad \epsilon:=\frac{32E_k(\alpha)}{3\kappa^2}$$

➡ can be solved in terms of confluent Heun functions

$$\varphi(x)=c_1\,\mathrm{e}^{-\frac{\sqrt{\xi}}{2}x^2}\,\mathcal{H}_\mathrm{c}\left(\cdot,-\frac{1}{2},\cdot,\cdot,\cdot;-x^2\right)+c_2\,x\,\mathrm{e}^{-\frac{\sqrt{\xi}}{2}x^2}\,\mathcal{H}_\mathrm{c}\left(\cdot,\frac{1}{2},\cdot,\cdot,\cdot;-x^2\right)$$

- regular at the origin $\mathcal{H}_\mathrm{c}(\cdot,\cdot,\cdot,\cdot,\cdot;0)=1$, increase as power for large x
- left side symmetric; right side antisymmetric ➔ vanishes at $x=\phi=0$
- gravitational part $C(\alpha)$ does not spoil this result

➡ only a ***subset*** of wave functions $\Psi(\alpha,\phi)$ vanishes at $\phi=0$

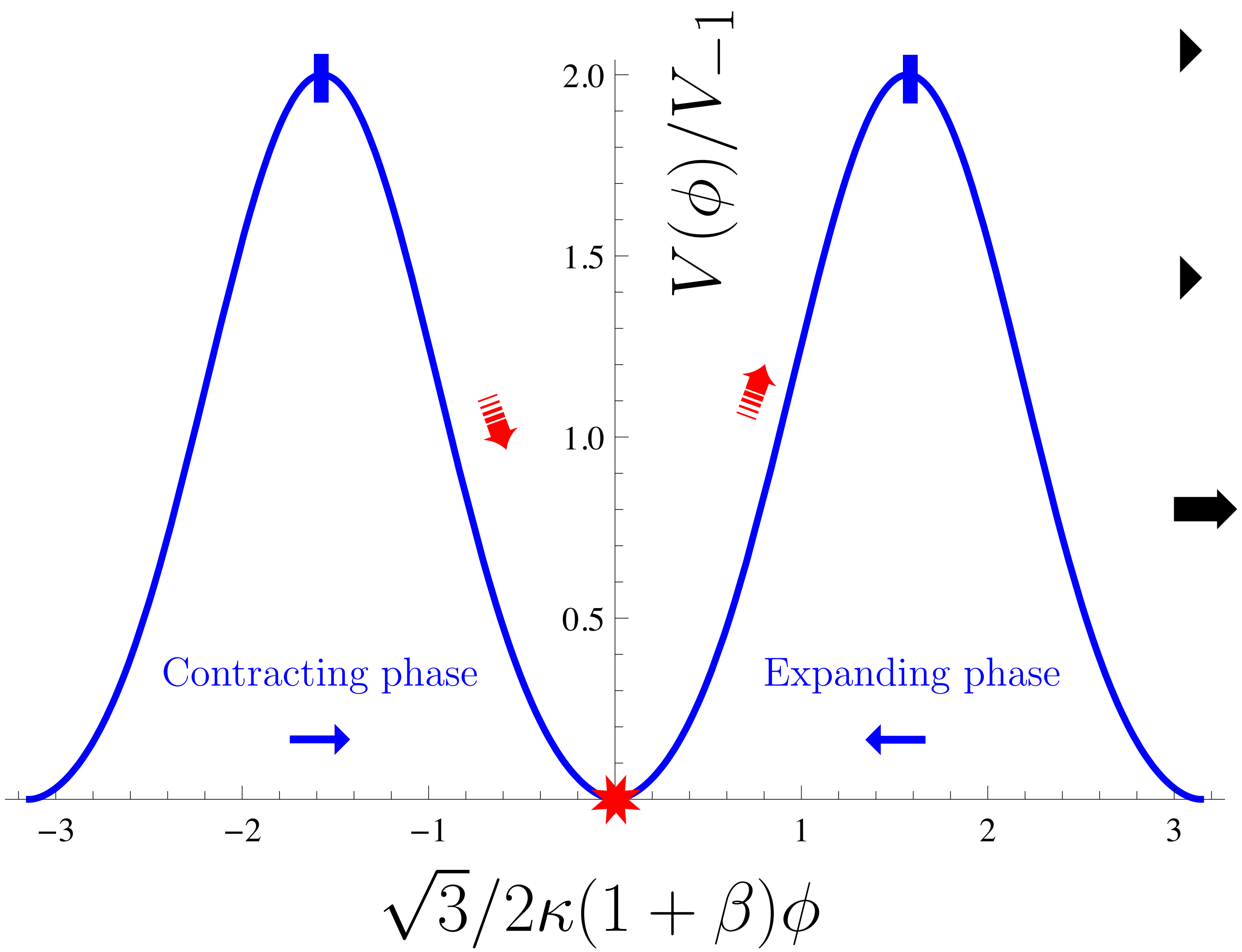
➡ from the structure of the double-well potential, one can conclude that this result is also valid for $\beta=-\frac{1}{2}+\varepsilon$, i.e. in the case of a type IV sing.

Example 3bis: Type IV singularity – phantom field

- analysis can be repeated for a phantom field

➡ leads to a periodic potential:

$$V(\phi) = V_{-1}\left[\sin^{-\frac{2\beta}{1+\beta}}\left(\frac{\sqrt{3}}{2}\kappa|1+\beta||\phi|\right) + \sin^{\frac{2}{1+\beta}}\left(\frac{\sqrt{3}}{2}\kappa|1+\beta||\phi|\right)\right]$$



- evolution of the universe goes through a type IV singularity
- asymptotically de Sitter

➡ again, singularity resolution only for a ***subset*** of the wave functions

Bouhmadi-López, Kiefer and M.K.,
Phys. Rev. D **89**, 064016 (2014).

Summary

- *How are classical singularities resolved in quantum-cosmological models?*
  - **Big Rip** (I): wave packets disperse at the classical singularity
    $a, \rho, P \to \infty$
    Dąbrowski, Kiefer, Sandhöfer, PRD **74**, 044022 ('06), `hep-th/0605229`.
  - **Big Brake** (II): normalizable solutions vanish at Big Brake ***and*** Big Bang
    $P \to \infty$
    Kamenshchik, Kiefer, Sandhöfer, PRD **76**, 064032 ('07), `0705.1688`.
  - **Big Freeze** (III): *boundary condition:*
    $\rho, P \to \infty$
    let wave function vanish in classically forbidden region
    ➔ wave function vanishes also at the singularity
    Bouhmadi-López, Kiefer, Sandhöfer, Moniz, PRD **79**, 124035 ('09), `0905.2421`.
  - **Type IV**: only a ***subset*** of solutions vanishes at the singularity
    $\ddot{H}, \dddot{H}, \ldots \to \infty$
    Bouhmadi-López, Kiefer, M.K., PRD **89**, 064016 ('14), `1312.5976`.
  - **Little Rip**: wave function vanishes for late times
    Big Rip for $t \to \infty$
    Albarran, Bouhmadi-López, Kiefer, Marto, Moniz, `1604.08365`.

➡ *Is a generalization of these results possible?*