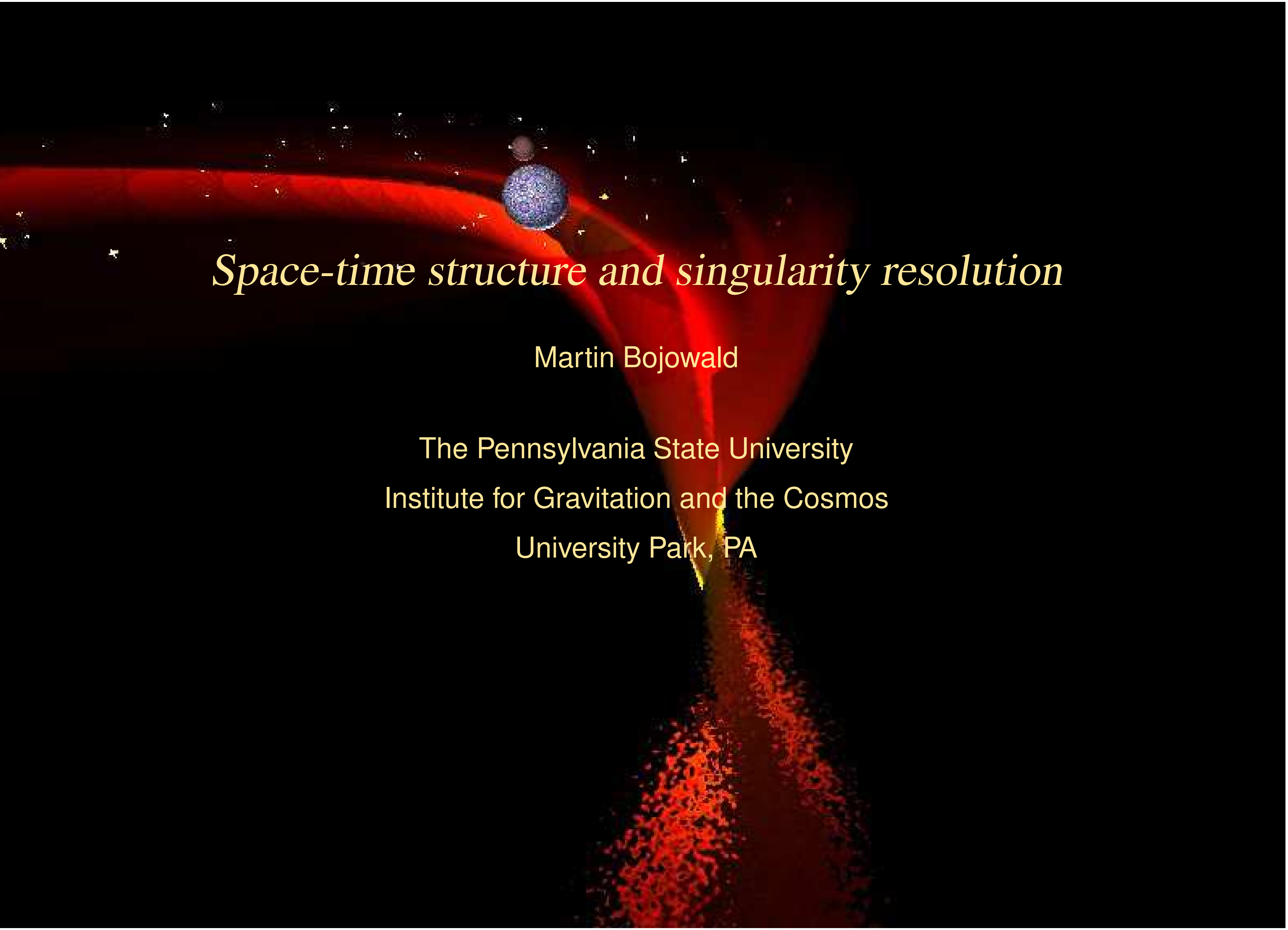

Space-time structure and singularity resolution

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Space-time structure

Easy to avoid divergences by introducing discreteness/bounded functions, in particular in minisuperspace models.

- → Consistent with covariance? If not, how can one avoid low-energy problems? [Polchinski: arXiv:1106.6346]
- → How exactly are singularity theorems evaded?
  Example: "Bounce" in some models of loop quantum cosmology without violating energy conditions.

Many open questions at different levels.

- → I. Lessons from quantum mechanics.
- → II. Lessons from quantum field theory.
- → III. Lessons from Dirac.

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Reduced action

$$
\begin{aligned}
S_{\mathrm{EH}}[g] &= \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-\det g}\,R \\
&= \frac{3V_0}{8\pi G}\int \mathrm{d}t\, Na^3\left(\frac{\ddot{a}}{N^2a}+\frac{\dot{a}^2}{N^2a^2}-\frac{\dot{a}\dot{N}}{aN^3}+\frac{k}{a^2}\right) \\
&= -\frac{3V_0}{8\pi G}\int \mathrm{d}t\left(\frac{a\dot{a}^2}{N}-kaN\right)
\end{aligned}
$$

Coordinate volume $V_0=\int \mathrm{d}^3x$ of some finite spatial region.

Canonical variables a and

$$p_a = -\frac{3V_0}{4\pi G}\frac{a\dot{a}}{N}$$

Lagrange multiplier N, $p_N=0$.

Hamiltonian formulation of isotropic models

Hamiltonian constraint in canonical variables

$$C = -\frac{2\pi G}{3V_0}\frac{p_a^2}{a} - \frac{3V_0}{8\pi G}ka + V_0 H_{\mathrm{matter}} = 0$$

Deparameterized Wheeler–DeWitt equation with dust

$$\hat{p}_T \Psi(q,T) = \frac{3\pi G}{2V_0}\hat{p}_q^2 \Psi(q,T) + \frac{3kV_0}{8\pi G} q^{2/3}\Psi(q,T)$$

for $q = a^{3/2}$. (Constant $H_{\mathrm{matter}} = p_T$ conjugate to T.)

Physical Hilbert space $L^2(\mathbb{R}, \mathrm{d}q)$.

Evolution in T with Hamiltonian $\hat{p}_T$: relational observables.

[Dirac: Can. J. Math. 2 (1950) 129; Blyth, Isham: PRD 11 (1975) 768]

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Questions

- → Quantum corrections depend on arbitrary V_0.

  Is quantum cosmology coordinate dependent?

- → Does choice of T as time affect predictions?

- → How do we choose initial states for T-evolution?

- → Classical constraint strongly restricted by reduction from covariant theory.

  Restrictions on quantum corrections in minisuperspace model?

Loop quantum cosmology

Isotropic connection $A_a^i = c\delta_a^i$, $E_j^b = p\delta_j^b$ with $c = \gamma\dot{a}$, $|p| = a^2$
Friedmann equation (flat space)

$$-\frac{c^2}{\gamma^2|p|} + \frac{8\pi G}{3}\rho = 0$$

modified by using "holonomies"

$$\frac{c^2}{|p|} \mapsto \frac{\sin(\ell c/\sqrt{|p|})^2}{\ell^2} \sim \frac{c^2}{|p|}\left(1 - \frac{1}{3}\ell^2\frac{c^2}{|p|} + \cdots\right)$$

If $\ell \sim \ell_{\rm P}$, corrections $\ell_{\rm P}^2 c^2/|p| \sim \rho/\rho_{\rm P}$.

Taken *in isolation,* holonomy modifications imply a "bounce" of isotropic models:

$$\frac{\sin(\ell c/\sqrt{|p|})^2}{\gamma^2\ell^2} = \frac{8\pi G}{3}\rho$$

I. Lessons from quantum mechanics

Questions:

- → Higher-curvature/higher-derivative contributions?
- → Dynamical quantum effects? (Fluctuations, ...)
- → Generic quantum behavior?

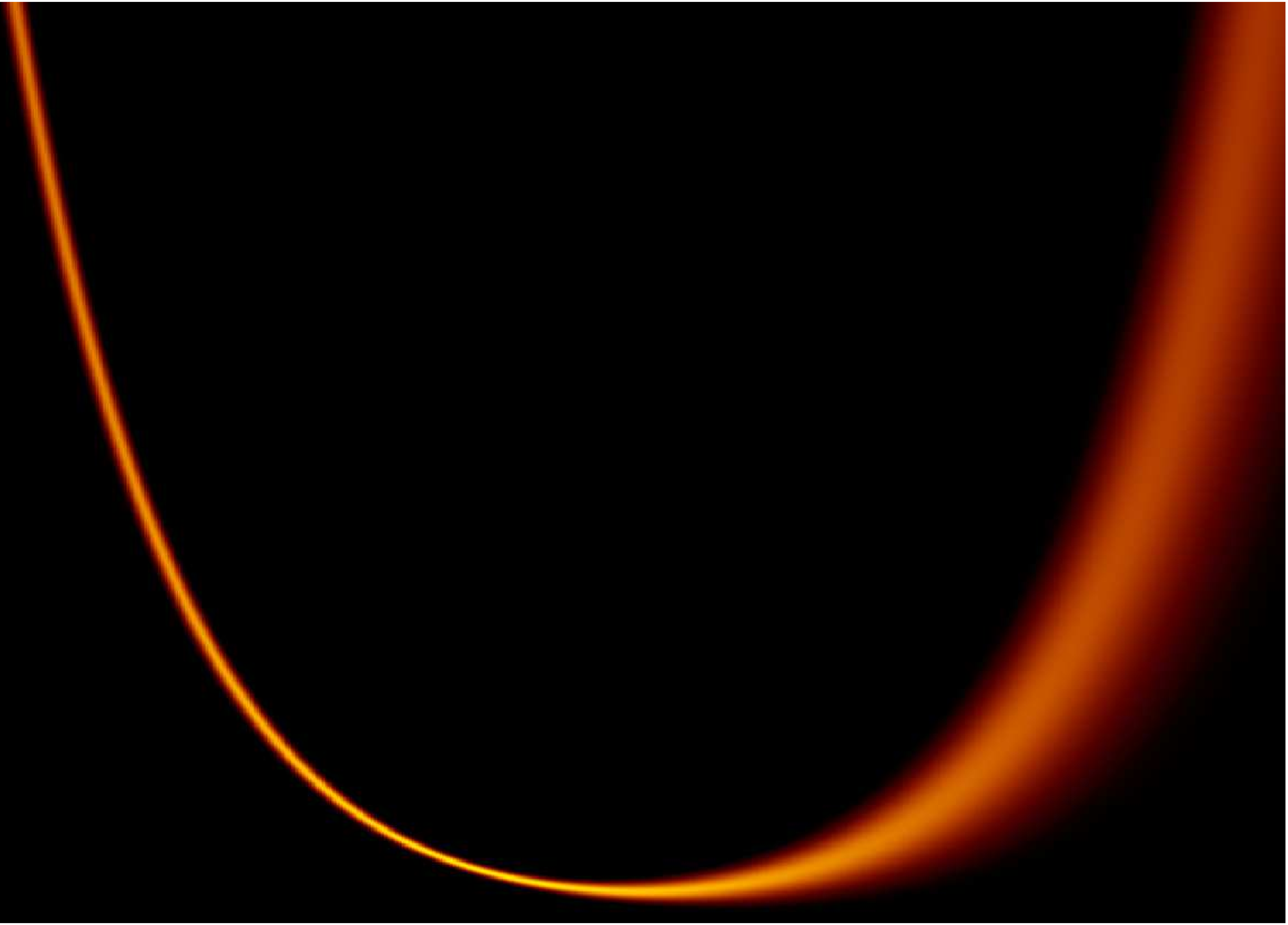
Detailed analysis possible in solvable (harmonic) model:
sourced by free massless scalar (flat space, $\Lambda = 0$).

$$p_\phi \propto |ap_a| \propto |cp| \propto |\mathcal{H}V|$$

with $\mathcal{H} = \dot{a}/a$ and $V \propto a^3$.
Holonomy-modified (ℓ constant):

$$p_\phi \propto |\mathrm{Im}J| \quad , \quad J := V\exp(i\ell\mathcal{H}) \quad , \quad \{J, V\} \propto \ell J$$

$\mathrm{sl}(2,\mathbb{R})$ algebra generated by $(V, J, \bar{J})$.



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General models

- → Perturbation theory around harmonic model: Quantum dynamics approximated *within* non-solvable model.
- → Quantum back-reaction of fluctuations and higher moments on expectation values. (Equivalent to low-energy effective action in standard systems.)
- → Have to know state to estimate moments. *Problem of states.*

Relation to space-time structure:

- → Adiabatic approximation (slow moments): higher time derivatives. Expected if related to higher-curvature terms.
- → Corrections should then be relevant near “bounce.” Holonomy modification and higher-curvature corrections add terms of same order ρ/ρ_{P}.
- → Not clear whether “bounce” is generic.

Effective shortcut

Simple effective equations in loop quantum cosmology:

- → Analyze exact solutions of solvable quantum model. Expectation value $\langle \hat{V} \rangle$ follows modified Friedmann equation.
- → Numerical studies of some related models with semiclassical initial states.
- → Call modified Friedmann equation an effective equation in all models.

But: “Bounce” density depends on type of initial state, such as Gaussian in $(\mathcal{H}, V)$ or (ϕ, p_ϕ). (Problem of states.)

[Diener, Gupt, Singh: arXiv:1310.4795]

Solvable model: Semiclassical state at large volume stays semiclassical.
General models: States spread out, change form.

II. Lessons from quantum-field theory

Wheeler–DeWitt equation

$$\hat{p}_T \Psi(q,T) = \frac{3\pi G}{2V_0} \hat{p}_q^2 \Psi(q,T) + \frac{3kV_0}{8\pi G} q^{2/3} \Psi(q,T)$$

or related version.

Quantum corrections depend on arbitrary coordinate volume V_0.

Minisuperspace *approximation* possible in self-interacting scalar field theory.

→ Relates V_0 to infrared scale (not a cut-off).

→ Minisuperspace models inconclusive about magnitude of quantum corrections.

Model minisuperspace model

[with S Brahma: arXiv:1509.00640]

Lagrangian $L = \int \mathrm{d}^3x \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\nabla\phi|^2 - W(\phi)\right)$ reduced to

$$L_{\mathrm{mini}} = V_0 \left(\frac{1}{2}\dot{\phi}^2 - W(\phi)\right)$$

with $V_0 = \int \mathrm{d}^3x$, ϕ spatially constant.

Momentum $p = \partial L_{\mathrm{mini}}/\partial\dot{\phi} = V_0\dot{\phi}$.

Hamiltonian

$$H_{\mathrm{mini}} = \frac{1}{2}\frac{p^2}{V_0} + V_0 W(\phi)$$

quantized to

$$\hat{H}_{\mathrm{mini}} = \frac{1}{2}\frac{\hat{p}^2}{V_0} + V_0 W(\hat{\phi})$$

Effective Hamiltonian

$$H_{\rm eff}=\langle\hat{H}_{\rm mini}\rangle=\frac{\langle\hat{p}\rangle^2}{2V_0}+V_0W(\langle\hat{\phi}\rangle)+\frac{\Delta(p^2)}{2V_0}+\frac{1}{2}V_0W''(\langle\hat{\phi}\rangle)\Delta(\phi^2)+\cdots$$

generates equations of motion for fluctuations, covariance:

$$\begin{aligned}\dot{\Delta}(\phi^2) &= \frac{2}{V_0}\Delta(\phi p)\\ \dot{\Delta}(\phi p) &= \frac{1}{V_0}\Delta(p^2)-V_0W''(\langle\hat{\phi}\rangle)\Delta(\phi^2)\\ \dot{\Delta}(p^2) &= -2V_0W''(\langle\hat{\phi}\rangle)\Delta(\phi p)\end{aligned}$$

Zeroth-order adiabatic, saturate uncertainty relation:

$$\Delta_0(\phi p)=0 \quad , \quad \Delta_0(\phi^2)=\frac{1}{2}\frac{\hbar}{V_0\sqrt{W''(\langle\hat{\phi}\rangle)}} \quad , \quad \Delta_0(p^2)=\frac{1}{2}\hbar V_0\sqrt{W''(\langle\hat{\phi}\rangle)}$$

Effective potential

$$\begin{aligned} V_{\mathrm{eff}}(\phi) &= H_{\mathrm{eff}}|_{\langle\hat{p}\rangle=0,\langle\hat{\phi}\rangle=\phi} \\ &= V_0 W(\phi) + \frac{1}{2V_0}\Delta_0(p^2) + \frac{1}{2}V_0 W''(\phi)\Delta_0(\phi^2) \\ &= V_0 W(\phi) + \frac{1}{2}\hbar\sqrt{W''(\phi)} \\ &= V_0 W_{\mathrm{eff}}(\phi) \end{aligned}$$

V_0-dependent quantum correction in

$$W_{\mathrm{eff}}(\phi) = W(\phi) + \frac{1}{2V_0}\hbar\sqrt{W''(\phi)}$$

Can make quantum corrections small by choosing large V_0.

Effective quantum-field theory

Coleman–Weinberg potential: [S Coleman, E Weinberg: PRD 7 (1973) 1888]

$$W_{\rm eff}(\phi) = W(\phi) + \frac{1}{2} i\hbar \int \frac{{\rm d}^4k}{(2\pi)^4} \log\left(1 + \frac{W''(\phi)}{||\mathbf{k}||^2}\right)$$

From moments or k^0-integration: [with S Brahma: arXiv:1411.3636]

$$W_{\rm eff}(\phi) = W(\phi) + \frac{1}{2}\hbar \int \frac{{\rm d}^3k}{(2\pi)^3} \left(\sqrt{|\vec{k}|^2 + W''(\phi)} - |\vec{k}|\right)$$

Infrared-contribution: Integration over $|\vec{k}| \leq k_{\rm max} = 2\pi/V_0^{1/3}$

$$W_{\rm eff}(\phi) \approx W(\phi) + \frac{\hbar}{12\pi^2} k_{\rm max}^3 \sqrt{W''(\phi)} = W(\phi) + \frac{2\pi}{3V_0}\hbar\sqrt{W''(\phi)}$$

in qualitative agreement with minisuperspace effective potential.

Minisuperspace approximation

Quantum corrections depend on V_0 via infrared scale $k_{\max}$ of field theory:
Averaging with larger V_0 leaves fewer modes, smaller $k_{\max}$.

Small quantum corrections for large V_0 or small $k_{\max}$.
But minisuperspace truncation becomes less accurate.
Quantum corrections ignored if $V_0 \to \infty$.

Minisuperspace approximation by expansion of square root in $k_{\max}^2/W''(\phi) \propto W''(\phi)^{-1}V_0^{-2/3}$.

→ Need some information about full theory for expansion.

→ Covariance in minisuperspace models?
Fixed infrared scale breaks local Poincaré transformations.

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III. Lessons from Dirac

⟶ Lagrangian density and *measure* in

$$S[g] = \frac{1}{16\pi G}\int \mathrm{d}^4x\, \sqrt{|\det g|}\, (R[g] + \cdots)$$

may be subject to quantum corrections.

⟶ Quantum-field theory on curved space-time different from quantum gravity.

⟶ Covariance in canonical quantum gravity:
Quantum version of Dirac's hypersurface deformations.
Anomaly problem.

QFT vs. QG

Perturbative inhomogeneity in $h[A] := A(x)^2$. $A(x) = \bar{A} + \delta A(x)$.

Quantum effects in background dynamics by $\bar{A} \longrightarrow \ell^{-1}\sin(\ell\bar{A})$.

→ *Classical:* $h[\bar{A},\delta A] = \bar{A}^2 + 2\bar{A}\delta A(x) + \delta A(x)^2$

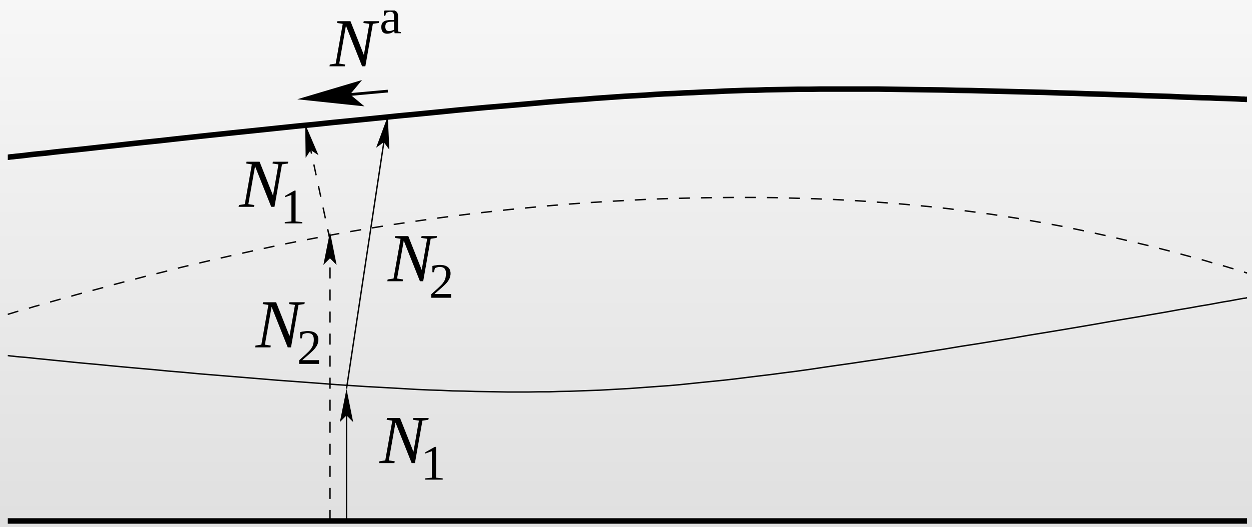
→ *QFT on modified space-time:*
$h_\ell^{\rm QFT}[\bar{A},\delta A] = \ell^{-2}\sin(\ell\bar{A})^2 + 2\bar{A}\delta A(x) + \delta A(x)^2$
[Agulló, Ashtekar, Nelson]

→ *Effective quantum gravity:*
$h_\ell^{\rm QG}[\bar{A},\delta A] = \ell^{-2}\sin(\ell\bar{A})^2 + F_\ell(\bar{A})\delta A(x) + G_\ell(\bar{A})\delta A(x)^2$ with $\lim_{\ell\to0} F_\ell(\bar{A}) = 2\bar{A}$ and $\lim_{\ell\to0} G_\ell(\bar{A}) = 1$.

Subject to *covariance conditions.*

$F_\ell/\bar{A}$ and G_ℓ of same magnitude as $(\ell\bar{A})^{-2}\sin(\ell\bar{A})^2$.
[MB, Hossain, Kagan, Shankaranarayanan; Barrau, Cailleteau, Grain, Mielczarek; Wilson–Ewing]

Covariance: Hypersurface deformations



Generators $D[N^a]$ (tangential deformations along $N^a(x)$) and $H[N]$ (normal deformations by $N(x)$) obey

$$
\begin{aligned}
[D[N^a], D[M^b]] &= -D[\mathcal{L}_{M^b} N^a] \\
[H[N], D[M^b]] &= -H[\mathcal{L}_{M^b} N] \\
[H[N_1], H[N_2]] &= D[q^{ab}(N_1 \partial_b N_2 - N_2 \partial_b N_1)]
\end{aligned}
$$

with induced metric q_{ab} on spatial slice. (Lie algebroid.)

Generally covariant gauge theory

- → Hypersurface-deformation brackets generalize Poincaré algebra, local version.
- → Covariance in canonical quantum gravity:
  Representation of brackets by operators $\hat{D}$, $\hat{H}$, $\hat{q}$ with

$$\begin{aligned} \{D[N^a], D[M^b]\} &= -D[\mathcal{L}_{M^b} N^a] \\ \{H[N], D[M^b]\} &= -H[\mathcal{L}_{M^b} N] \\ \{H[N_1], H[N_2]\} &= D[q^{ab}(N_1 \partial_b N_2 - N_2 \partial_b N_1)] \end{aligned}$$

as classical limit.

"Off-shell" property.
Stronger than anomaly-free reformulated system.

Examples: $\{H+D, H+D\} = 0$ [Gambini, Pullin]
$\{H, H\} = \{D', D'\}$ [Tomlin, Varadarajan]

Model

Scalar field $\phi(x)$, momentum $p(x)$, one spatial dimension.

$$H[N] = \int \mathrm{d}x N \left(f(p) - \frac{1}{4}(\phi')^2 - \frac{1}{2}\phi\phi'' \right) \quad , \quad D[w] = \int \mathrm{d}x w \phi p'$$

Spatial diffeomorphisms:

$$\delta_w \phi = \{\phi, D[w]\} = -(w\phi)' \quad , \quad \delta_w p = \{p, D[w]\} = -wp'$$

H-bracket:

$$\{H[N], H[M]\} = D[\tfrac{1}{2}(\mathrm{d}^2 f/\mathrm{d}p^2)(N'M - NM')]$$

Lorentzian-type hypersurface deformations for $f(p) = p^2$.

Signature change

"Holonomy" modifications, $f(p)=p_0^2\sin^2(p/p_0)$:

$$\frac{1}{2}\mathrm{d}^2f/\mathrm{d}p^2=\cos(2p/p_0)$$

can be negative. At maximum of $f(p)$:

$$\{H[N],H[M]\}=D[-(N'M-NM')]$$

Euclidean signature:

$\Delta x=-\theta\Delta y$ from commutator of infinitesimal rotation by θ and a spatial shift by Δy.

Opposite sign for infinitesimal boost: $\Delta x=v\Delta t$.

Consistent deformation

$\{H[N], D[w]\}$ does not close in the scalar model, but does so in some gravity versions.

- → No gauge transformations broken.
- → No effective line element on standard manifold: $\mathrm{d}x^a$ in

$$\mathrm{d}s^2_{\mathrm{eff}} = \tilde{q}_{ab}\mathrm{d}x^a\mathrm{d}x^b$$

do not transform by deformed gauge transformations that change $\tilde{q}_{ab}$.

Field redefinition to standard q_{ab} possible as long as β does not change sign. With signature change: New model of non-classical space-time.

- → Evaluate theory using canonical observables of deformed gauge theory.

Holonomies and space-time

[Reyes; Barrau, Cailleteau, Grain, Mielczarek]

All known consistent models (spherical symmetry, cosmological perturbations): $K^2 \longrightarrow f(K)$ modifies bracket

$$\{H[N_1], H[N_2]\} = D[\beta q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)]$$

with

$$\beta(K) = \frac{1}{2}\mathrm{d}^2 f(K)/\mathrm{d}K^2 = \cos(2\ell K)$$

for $f(K) = \ell^{-2}\sin^2(\ell K)$.

- → Covariant: Consistent gauge structure, but deformed.
- → Not undone by quantum back-reaction or higher time derivatives. Distinct from higher-curvature corrections.
- → Signature change: $\beta(K) < 0$ around maximum of $f(K)$. "Bounce" indeterministic.

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Structure functions

Systems with several constraints $\hat{C}_I$: $[\hat{C}_I, \hat{C}_J] = \hat{f}_{IJ}^K \hat{C}_K$.

- → Effective constraints $C_{I,\mathrm{pol}} = \langle \widehat{\mathrm{pol}} \hat{C}_I \rangle$ with $\widehat{\mathrm{pol}}$ polynomial in basic operators.
- → Effective constraint algebra by quantum Poisson brackets.
- → No quantum corrections in structure functions: [arXiv:1407.4444]

$$\{C_{I,1}, C_{J,1}\} = f_{IJ}^K(\langle\cdot\rangle) C_{K,1} + \cdots$$

Consistent with higher-curvature effective actions in gravity.

Holonomy modifications in $\hat{C}_I$ change $\hat{f}_{IJ}^K$.

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Related results

- → Closely related behavior in spherically symmetric models and cosmological perturbations.

  [with Barrau, Calcagni, Grain, Kagan: arXiv:1404.1018]

- → Operator version in spherical symmetry. [Brahma: arXiv:1411.3661]

- → Different operator versions in $2+1$ dimensional models, based on reformulations of constraint algebra.

  [Perez, Pranzetti; Henderson, Laddha, Tomlin, Varadarajan]

- → Partially Abelianized constraints: [Gambini, Pullin]
  After holonomy modifications, can reconstruct hypersurface-deformation brackets only if deformed.

  [with Brahma, Reyes: arXiv:1507.00329]

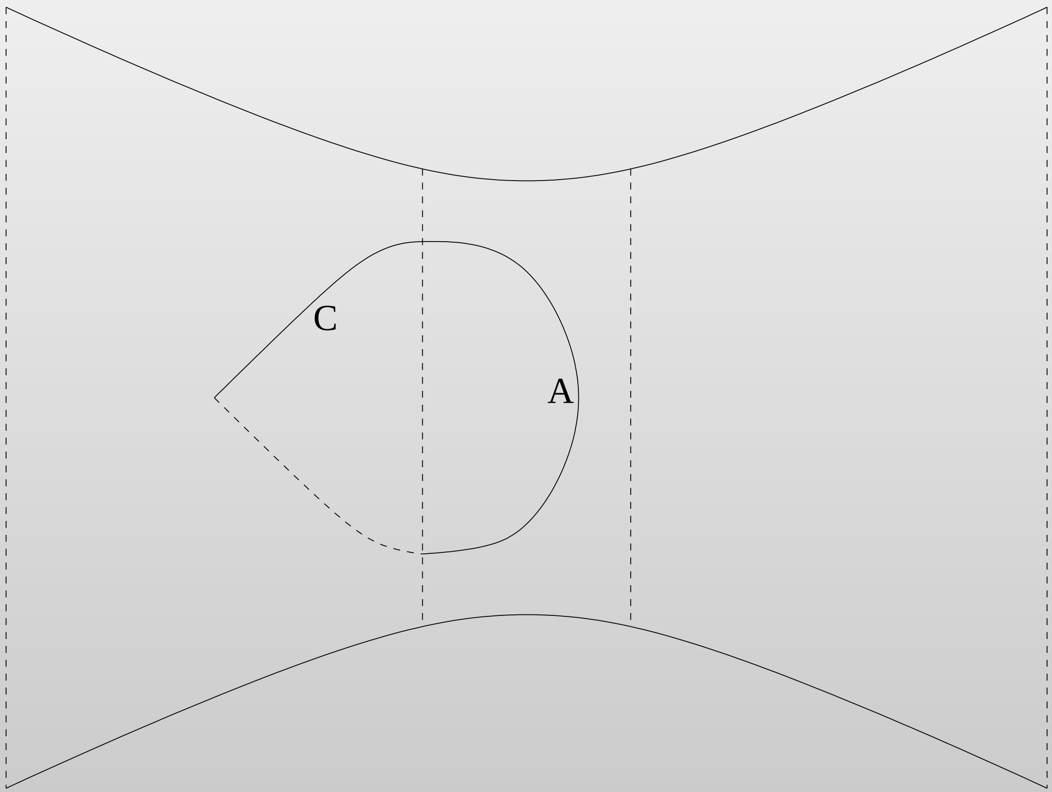
- → Obstructions to anomaly freedom in models with local physical degrees of freedom. [with Brahma: arXiv:1507.00679]

Not much is known about full dynamics of loop quantum gravity.
Modified space-time structures generic.

Tricomi problem and cosmic boom

[with Mielczarek: arXiv:1503.09154]

$-\frac{\partial^2 u}{\partial t^2} + \beta(\mathcal{H})\Delta u = 0$: Characteristic C connected to arc A.



$\rightarrow$ Need future data: No deterministic evolution. Poles generic.

$\rightarrow$ Phenomenology not viable in cyclic interpretation.

[Bolliet, Barrau, Grain, Schander: arXiv:1510.08766]

Information loss

Non-singular black-hole model:

Evolve through classical singularity by quantum evolution of homogeneous interior.

No event horizon. [Ashtekar, MB 2006]

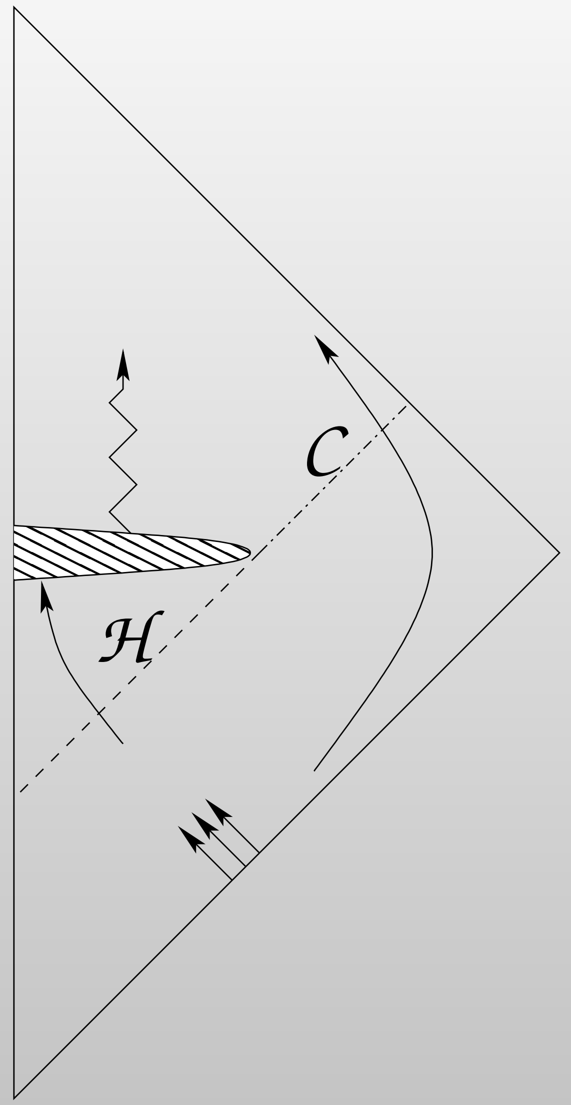
Anomaly-free space-time structure:

High-curvature region Euclidean.

Arbitrary boundary values affect future space-time.

Event horizon $\mathcal{H}$ and Cauchy horizon $\mathcal{C}$.

No-heir theorem?



[arXiv:1409.3157]