Resolving gravitational singularities with affine coherent states

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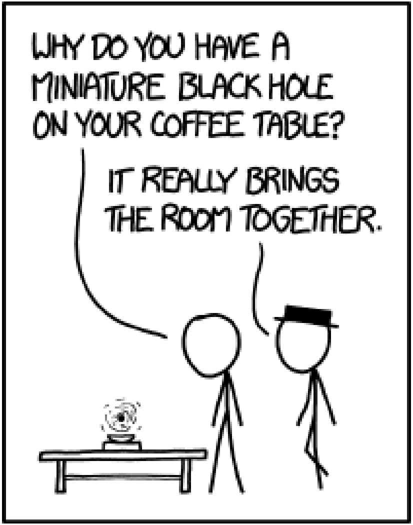
Singularities of general relativity and their quantum fate
Warsaw, 28 June 2016

The importance of resolving singularities



xkcd.com

Classical singularities & canonical variables

Classical singularities

$$q \to 0$$

Canonical classical variables + physical condition

$$\{q, p\} = 1 \qquad q > 0$$

Issues with quantum operators:

- can define $[Q, P] = i\hbar$ with $Q > 0$
- P not self-adjoint
- Translation operator $exp\ (\mathrm{i}qP/\hbar)$ not unitary

$$[Q, D] = \mathrm{i}\hbar Q \qquad D = \frac{1}{2}(PQ + QP)$$

Functional representation for operators

$$\begin{aligned} Df(q) &= -\mathrm{i}\hbar\, q^{1/2}\partial_q(q^{1/2}f(q)) \\ Qf(q) &= qf(q) \end{aligned}$$

D acts as a dilation operator
D can be shown to be self-adjoint

We can implement $Q > 0$ consistently

Affine coherent states

$$|p, q\rangle = e^{\mathrm{i}pQ/\hbar} e^{-\mathrm{i}\ln(q/\mu)D/\hbar} |\eta\rangle$$

with polarisation condition

$$\left[\frac{Q}{\mu} - 1 + \mathrm{i}\frac{D}{\beta\hbar}\right] |\eta\rangle = 0$$

with a functional representation

$$\langle x|p, q\rangle = N e^{\mathrm{i}px/\hbar} \left(\frac{x}{q}\right)^{\beta} x^{-\frac{1}{2}} e^{-\frac{\beta x}{q}}$$

which yields

$$\begin{aligned} \langle p, q|Q|p, q\rangle &= q \\ \langle p, q|D|p, q\rangle &= p\ q \end{aligned}$$

can be used for a resolution of identity $\mathbb{I} \sim \int |p, q\rangle\langle p, q|$

The Weak Correspondence Principle

- Correspondence between classical and quantum systems
- Quantum dynamics incorporated via (affine) coherent states
- Semi-classical dynamics in Hamiltonian form

A quick recipe:

1. A classical Hamiltonian $H\left(p_c, q_c\right)$
2. Its quantum counterpart $\mathcal{H} =: H\left(P, Q\right)$:
3. Unitary translation operators $U[p, q]$
4. Unit vectors $|p, q\rangle := U|\eta\rangle$ w/ unique polarisation condition

Enhanced Hamiltonian

$$h(p, q) := \langle p, q|\mathcal{H}\left(P, Q\right)|p, q\rangle = H(p, q) + \mathcal{O}(\hbar, p, q)$$

Resolving Black Hole singularities

SZ, Phys. Rev. D 90, 064046 (2014) - arXiv:1409.1761

Black Holes in 1+1d dilaton gravity

Standard 2dGDT action: metric + dilaton X + potentials U, V:

$$S_{dg} = -\frac{1}{2}\int dx^2\sqrt{-g}\left(XR - U(X)\,(\nabla X)^2 - 2V(X)\right)$$

Gauge fixing:

$$\begin{aligned} ds^2 &= -\xi(r)dt^2 + \frac{1}{\xi(r)}dr^2 \\ X &= X(r) \end{aligned}$$

General solutions for BH:

$$\begin{aligned} X_r &= e^{-Q(X)} \\ \xi &= e^{Q(X)}\,(w(X) - 2M) \end{aligned}$$

Horizons:

$$X = X_h \qquad w\,(X_h) = 2M \qquad \xi \to 0$$

Singularities:

$$X \to 0 \qquad U(X) \to \infty$$

Classical Black Holes

Classical geometry described by $\xi(X) = e^{Q(X)}\,(w(X) - 2M)$

$$ds^2 = -\xi(X)dt^2 + \frac{1}{\xi(X)}dr^2$$

Classical physical constraint

$$X > 0$$

Affine coherent state quantisation for BH 1/2

Classical background $\xi(X)$ + Quantum dynamics of X

Effective action for BH solutions

$$\mathcal{L}_{eff} = N(r)^{-1} g(r) X_r^2 + N(r) e^{-2Q(X)}$$

$N(r)$ Lagrange multiplier
$g(r)$ background "metric"
gauge invariant for $r \to \epsilon(r)$

Effective Hamiltonian

$$\mathcal{H}_{eff} = N \left(\frac{P_X^2}{4g} - e^{-2Q(X)} \right)$$

Enhanced Hamiltonian $h(X, P_X) = \langle \mathcal{H}(X, P_X) \rangle$:

$$h(X, P_X) = \frac{\delta(2)}{4} P_X^2 + \sum Q_n X^n \delta(-n) + \frac{\gamma(2)}{4} X^{-2}$$

with $\beta \to \infty$:

$$h(X, P_X) = \frac{1}{4} P_X^2 + e^{-2Q(X)} + \frac{\gamma}{4} X^{-2}$$

with $\gamma = \gamma(\hbar) > 0$ and the constraint $h = 0$.

Repulsive potential at $X \to 0$

Comparing classical and semi-classical solutions

1. Calculate analytically the classical solutions $X(r)$, $P_X(r)$.
2. Fix the free parameters to suitable values and calculate the potentials $Q(X)$ and $w(X)$.
3. Calculate initial conditions $X(r_0)$, $P_X(r_0)$ at $r_0 \gg 0$.
4. Numerically solve the semiclassical equations of motion in $r \in [-r_0, r_0]$, using the initial conditions calculate above.
5. Check that the constraint $h = 0$ is enforced.

What we will look at:

* Behaviour at $r \to 0$
* Location/displacement of the horizon
* Kretschmann scalar R^2

Thermodynamics for classical BH in 2dGDT

- Path integral formulation for canonical ensemble
- Temperature fixed by time periodicity
- Thermal reservoir - upper bound $X = X_c$ at che cavity wall
- Partition function $\mathcal{Z} \sim e^{-\Gamma_c} \times$ (quadratic terms)
- Can derive:

$$
\begin{array}{rl}
\text{free energy} & F_c = -T_c \ln \mathcal{Z} \simeq T_c \Gamma_c \\
\text{entropy} & S = -\dfrac{\partial F_c}{\partial T_c} \\
\text{internal energy} & E_c = F_c + T_c S \\
\text{specific heat} & C_c = -\dfrac{\partial E_c}{\partial T_c} = T_c \dfrac{\partial S}{\partial T_c}
\end{array}
$$

Same approach for affine BH
with first order corrections in γ

Thermodynamics for affine BH

Requires calculations of T_c and Γ_c:

1. Calculate first order corrections to T_c
2. Determine the boundary counter-term for vanishing first order variations of the action
3. Calculate the action on-shell
4. Remove the cavity by taking $X_c \to \infty$ when possible

Limits of the analysis:

- No direct comparison of expressions: 1d model VS 1+1d
- Cannot check the entropy-area law $S \sim A$
- Cannot check equivalence of quasi-local energy with internal energy

What can be done:

* Thermodynamical stability and properties in parameter space
* Calculate entropy corrections (for displaced horizons)

$$Q = -a \ln X \qquad w = \frac{B}{(b+1)} X^{b+1}$$

Enhanced Hamiltonian

$$h(X, P_X) = \frac{1}{4} P_X^2 + X^{2a} + \frac{\gamma}{4} X^{-2}$$

On-shell improved action

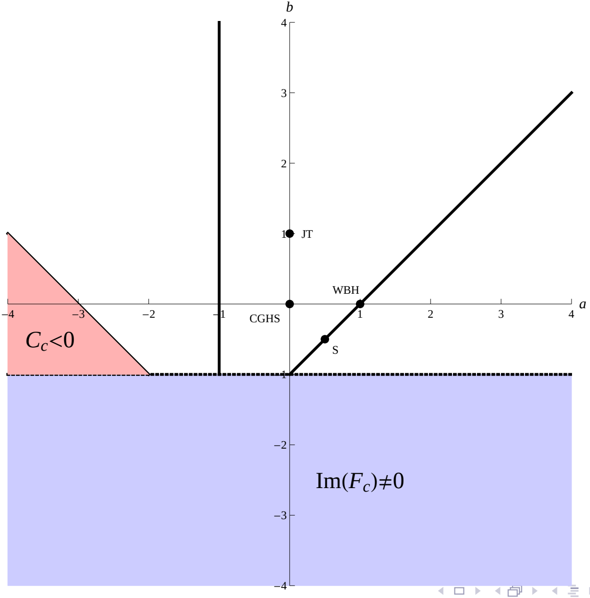
$$\Gamma_c = \frac{\gamma \left(X_h^{-a-1} - X_c^{-a-1} \right)}{4(a+1)}$$

Temperature at the cavity wall

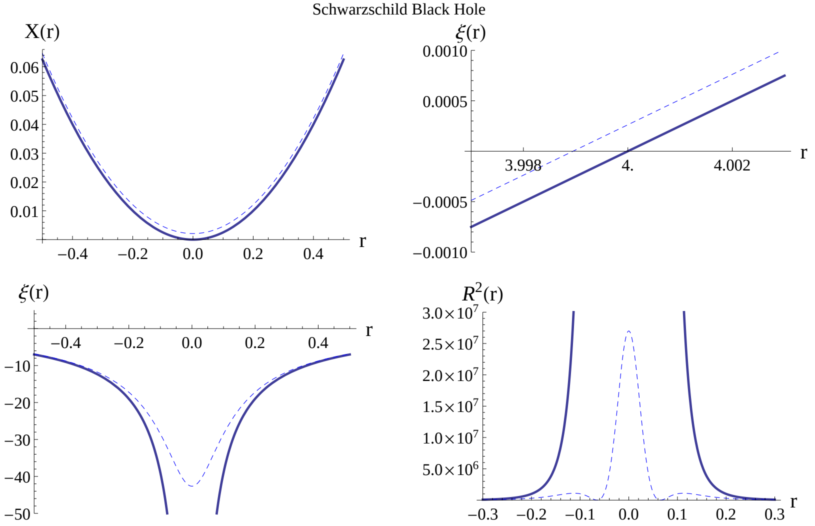
$$T_c = \frac{X_h^b \sqrt{B\,(b+1)} X_c^{\frac{a}{2}}}{4\pi \sqrt{\left(X_c^{b+1} - X_h^{b+1} \right)}}$$

		T_c	F_c	S	E_c	C_c
1)	$a > b+1 \wedge b > -1$	∞	∞	$S < 0$	0	0^+
2)	$a = b+1 \wedge b > -1$	$T_c > 0$	$F_c > 0$	$S < 0$	0	0^+
3)	$-1 < a < b+1 \wedge b > -1$	0	0	$S < 0$	0	0^+
4)	$a = -1 \wedge b > -1$	0	0	$-\infty$	0	$C_c > 0$
5)	$-b-3 < a < -1 \wedge b > -1$	0	0	$-\infty$	0	∞
6)	$a = -b-3 \wedge b > -1$	0	$F_c > 0$	$S > 0$	0	∞
7)	$a < -b-3 \wedge b > -1$	0	∞	∞	∞	$-\infty$
	$b <= -1$	Excluded by $\mathrm{Im}(F_c) \neq 0$ and $\mathrm{Im}(E_c) \neq 0$				

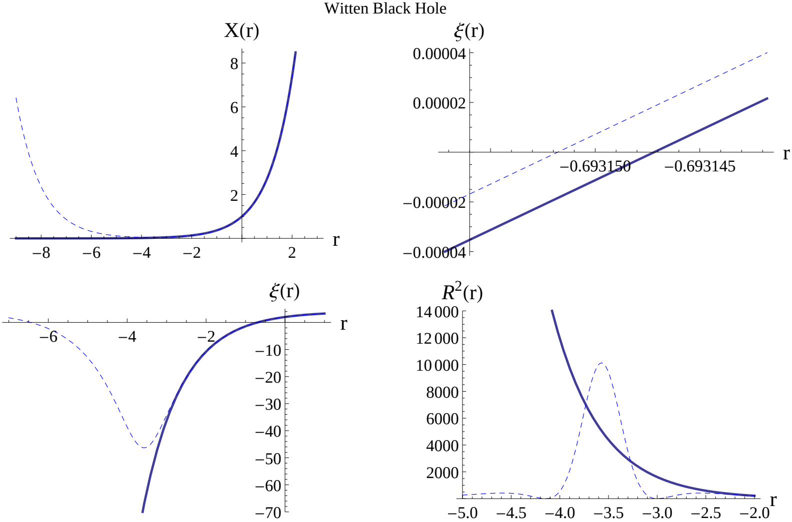
Results: the ab-family



Schwarzschild Black Hole



Witten Black Hole



Resolving the initial cosmological singularity

Europhys.Lett. 101 (2013) 10001 - arXiv:1203.4936
SZ with M. Fanuel

Classical action

$$S = \alpha \int dt \, \frac{1}{2} N(t) a^3 \left[-\frac{1}{N^2(t)} \left(\frac{\dot{a}}{a} \right)^2 - \frac{\Lambda}{3} + \frac{k}{a^2} \right]$$

Classical Hamiltonian

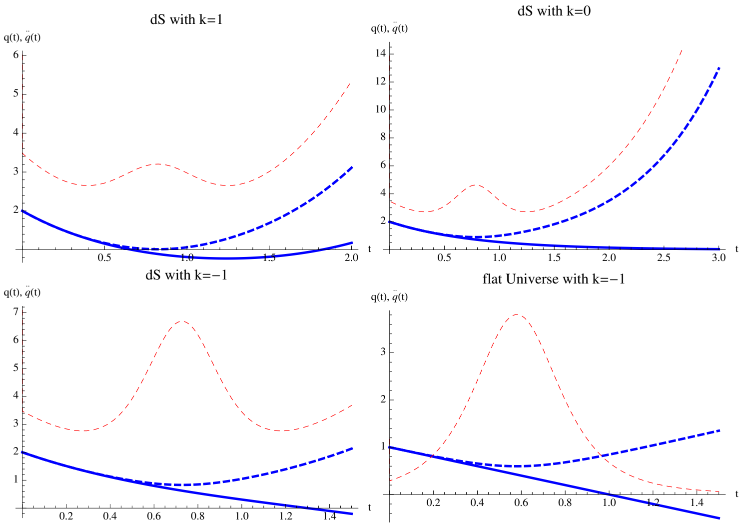
$$\mathcal{H}(p,q) = -\frac{p(t)^2}{2q(t)} - \frac{1}{2}\kappa q(t) + \frac{1}{6}\Lambda q(t)^3$$

Enhanced Hamiltonian

$$h(p,q) = -\frac{\delta(-1)p(t)^2}{2q(t)} - \frac{\gamma(3)}{2q(t)^3} + \frac{1}{6}\delta(3)\Lambda q(t)^3 - \frac{1}{2}\kappa q(t)$$

where $\delta = \delta(\beta)$ and $\gamma = \gamma(\hbar, \beta)$

Solving the singularity



- Affine quantisation:
  powerful tools for consistent implementations of $q > 0$ type conditions
- Affine coherent states and the weak correspondence principle
  non-perturbative quantum dynamics in a semiclassical formulation
- Singularities are sistematically removed for BH and FLRW cosmology
- Thermodynamical consistency with classical solutions

Perpectives:

1. Full treatment of 2d GDTs
2. Implications for horizons
3. Significance of the β parameter

Thank you!