Convex, Discrete and Integral Geometry

05.06.2017 - 09.06.2017

Banach Center, Będlewo, Poland

http://bcc.impan.pl/17Convex

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Research Training Group Methods for discrete structures, Berlin

Main Speakers

Semyon Alesker (Tel Aviv, Israel)

Matthias Beck (San Francisco, U.S.A.)

Andrea Colesanti (Florence, Italy)

Andreas Bernig (Frankfurt, Germany)

Joseph H. G. Fu (Georgia, U.S.A.)

Apostolos Giannopoulos (Athens, Greece)

Michael Joswig (Berlin, Germany)

Monika Ludwig (Vienna, Austria)

Dmitry Ryabogin (Kent, U.S.A.)

Francisco Santos (Cantabria, Spain)

Alina Stancu (Montreal, Canada)

Elisabeth Werner (Cleveland, U.S.A.)

Günter M. Ziegler (Berlin, Germany)

Chuanming Zong (Peking, China)

Convex, Discrete and Integral Geometry

Banach Center, 04.06.-10.06.2017

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
	From 7:30 Breakfast	**From 7:30** Breakfast	**From 7:30** Breakfast	**From 7:30** Breakfast	**From 7:30** Breakfast	**From 7:30** Breakfast
	9:00-9:50 Monika Ludwig	**9:00-9:50** Günter Ziegler	**9:00-9:50** Semyon Alesker	**9:00-9:50** Joseph Fu	**9:00-9:50** Apostolos Giannopoulos	**9:00** Shuttle Bus to Poznań Railway Station
	10:00-10:50 Chuanming Zong	**9:55-10:25** Thomas Jahn	**9:55-10:25** Gil Solanes	**10:00-10:50** Michael Joswig	**10:00-10:50** Dmitry Ryabogin	
	10:50-11:20 Coffee Break	**10:25-10:55** Sören Berg	**10:25-10:55** Thomas Wannerer	**10:50-11:20** Coffee Break	**10:50-11:20** Coffee Break	
	11:20-11:50 René Brandenberg	**10:55-11:25** Coffee Break	**10:55-11:20** Coffe Break	**11:20-11:50** Jin Li	**11:20-11:50** Vladyslav Yaskin	
	11:50-12:20 Katharina Jochemko	**11:25-12:15** Andreas Bernig	**11:20-12:10** Matthias Beck	**11:50-12:20** Ignacio Villanueva	**11:50-12:20** Hannes Pollehn	
	12:20-12:50 Masataka Shibata	**12:20-12:50** Dmitry Faifman	**12:15-12:45** Romanos Malikiosis	**12:20-12:50** Carsten Schütt	**12:20-12:50** Bernardo González	
	13:00-15:30 Lunch	**13:00-15:30** Lunch	**12:45-13:15** Liping Yuan	**13:00-15:30** Lunch	**13:00-15:30** Lunch	
	15:30-16:20 Alina Stancu	**15:30-16:20** Francisco Santos	**13:15** Lunch + Excursion	**15:30-16:20** Elisabeth Werner	**15:30-16:00** Jun O'Hara	
	16:30-17:00 Gabriele Bianchi	**16:25-16:55** Matthias Schymura		**16:30-17:00** Florian Besau	**16:00-16:30** Shigehiro Sakata	
	17:00-17:20 Coffee Break	**16:55-17:10** Coffee Break		**17:00-17:20** Coffee Break	**16:30-17:00** Susanna Dann	
	17:20-17:50 Jesús Yepes	**17:10-18:00** Andrea Colesanti		**17:20-17:50** Galyna Livshyts	**17:00-17:15** Coffee Break	
18:00 Shuttle bus from Poznań Railway Station	**17:50-18:20** Daniel Temesvari	**18:05-18:35** Fabian Mußnig		**17:50-18:20** Vitor Balestro	**17:15-17:45** Antonio Cañete	
	18:20-18:50 Tomasz Kobos	**18:35-19:05** Nico Lombardi		**18:20-18:50** Agnieszka Bogdewicz	**17:45-18:15** Michał Zwierzyński	
19:30pm Dinner	**19:00** Dinner	**19:15** Grill	**19:30** Dinner	**19:00** Social Dinner	**18:15-18:45** Tudor Zamfirescu	
					19:00 Dinner	

Timetable

Monday, 05.06.2017

09:00 - 09:50: Monika Ludwig: Tensor valuations in lattice polytopes.
10:00 - 10:50: Chuanming Zong: Tetrahedral packings, color graphs and optimization.
10:50 - 11:20: *Coffee Break*
11:20 - 11:50: René Brandenberg: Why we should allow non-symmetric gauge bodies when studying radii functionals.
11:50 - 12:20: Katharina Jochemko: h^*-polynomials of zonotopes.
12:20 - 12:50: Masataka Shibata: Symmetric Mahler's conjecture for the volume product in the three dimensional case.
13:00 - 15:30: *Lunch*
15:30 - 16:20: Alina Stancu: Some more results on the L_p Minkowski problem.
16:30 - 17:00: Gabriele Bianchi: The L_p Minkowski problem for $-n < p < 1$.
17:00 - 17:20: *Coffee Break*
17:20 - 17:50: Jesús Yepes Nicolás: On Brunn-Minkowski's inequalities under projections assumptions.
17:50 - 18:20: Daniel Temesvari: Moments of the maximal number of empty simplices of a random point set.
18:20 - 18:50: Tomasz Kobos: Grünbaum distance of two planar convex bodies.
19:00: *Dinner*

Tuesday, 06.06.2017

09:00 - 09:50: Günter M. Ziegler: Semialgebraic sets of f-vectors.
09:55 - 10:25: Thomas Jahn: Hunting for reduced polytopes.
10:25 - 10:55: Sören L. Berg: Discrete slicing problems with low dimensional subspaces.
10:55 - 11:25: *Coffee Break*
11:25 - 12:15: Andreas Bernig: Convolution of valuations.
12:20 - 12:50: Dmitry Faifman: Indefinite Crofton formulas and the centro-affine surface area.
13:00 - 15:30: *Lunch*
15:30 - 16:20: Francisco Santos: Towards a classification of empty lattice 4-simplices.
16:25 - 16:55: Matthias Schymura: On the covering radius of lattice zonotopes and its relation to view-obstructions and the Lonely Runner conjecture.
16:55 - 17:10: *Coffee Break*
17:10 - 18:00: Andrea Colesanti: Functionals subject to concavity or volume conditions.
18:05 - 18:35: Fabian Mußnig: Valuations on Log-concave functions.
18:35 - 19:05: Nico Lombardi: Valuations on the space of quasi-concave functions.
19:15: *Grill*

Wednesday, 07.06.2017

09:00 - 09:50: Semyon Alesker: Some conjectures on intrinsic volumes on Riemannian and Alexandrov spaces.

09:55 - 10:25: Gil Solanes: Integral geometry of the quaternionic plane.

10:25 - 10:55: Thomas Wannerer: Integral geometry in exceptional spheres.

10:55 - 11:20: *Coffee Break*

11:20 - 12:10: Matthias Beck: Polyhedral Number Theory.

12:15 - 12:45: Romanos D. Malikiosis: Polyhedral Gauss sums.

12:45 - 13:15: Liping Yuan: Selfishness of convex bodies.

13:15 - 14:30: *Lunch*

14:30: *Excursion to Rogalin's Palace*

19:30: *Dinner*

Thursday, 08.06.2017

09:00 - 09:50: Joseph H. G. Fu: Riemannian curvature measures.

10:00 - 10:50: Michael Joswig: A tropical isoperimetric inequality.

10:50 - 11:20: *Coffee Break*

11:20 - 11:50: Jin Li: Laplace transforms and function valued valuations.

11:50 - 12:20: Ignacio Villanueva: Radial continuous valuations on star bodies.

12:20 - 12:50: Carsten Schütt: On the geometry of projective tensor products.

13:00 - 15:30: *Lunch*

15:30 - 16:20: Elisabeth Werner: Recent results on approximation of convex bodies by polytopes.

16:30 - 17:00: Florian Besau: Weighted floating bodies.

17:00 - 17:20: *Coffee Break*

17:20 - 17:50: Galyna Livshyts: On the randomized Log-Brunn-Minkowski inequality.

17:50 - 18:20: Vitor Balestro: Curvature types in normed planes.

18:20 - 18:50: Agnieszka Bogdewicz: On spherical projection of a convex body and the related quotient space.

19:00: *Social dinner*

Friday, 09.06.2017

09:00 - 09:50: Apostolos Giannopoulos: Inequalities about sections and projections of convex bodies.

10:00 - 10:50: Dmitry Ryabogin: On a local version of the fifth Busemann-Petty problem.

10:50 - 11:20: *Coffee Break*

11:20 - 11:50: Vladyslav Yaskin: On polynomially integrable convex bodies.

11:50 - 12:20: Hannes Pollehn: Necessary subspace concentrarion conditions for the even dual Minkowski problem.

12:20 - 12:50: Bernardo González Merino: On two theorems of Minkowski in the Geometry of Numbers.

13:00 - 15:30: *Lunch*

15:30 - 16:00: Jun O'hara: Characterization of unit balls by Riesz energy.

16:00 - 16:30: Shigehiro Sakata: Characterization of regular triangles in terms of critical points of Riesz potentials.

16:30 - 17:00: Susanna Dann: Maximum area circumscribed polygons.

17:00 - 17:15: *Coffee Break*

17:15 - 17:45: Antonio Cañete: Bisections minimizing the maximum relative diameter.

17:45 - 18:15: Michał Zwierzyński: Isoperimetric-type inequalities and equalities for planar ovals.

18:15 - 18:45: Tudor Zamfirescu: Discs held in cages.

19:00: *Dinner*

Main Talks

Some Conjectures on Intrinsic Volumes on Riemannian and Alexandrov Spaces

Semyon Alesker

Tel Aviv University, Israel

Hadwiger's theorem says that linear combinations of intrinsic volumes on convex sets are the only isometry invariant continuous valuations. On the other hand H. Weyl has extended intrinsic volumes beyond convexity, to Riemannian manifolds. We try to understand the continuity properties of this extension under the Gromov-Hausdorff convergence (literally, there is no such continuity in general). First, we describe a new conjectural compactification of the set of all closed Riemannian manifolds with given upper bounds on dimension and diameter and lower bound on sectional curvature. Points of this compactification are pairs: an Alexandrov space and a constructible (in the Perelman-Petrunin sense) function on it up to isometries. Second, conjecturally all intrinsic volumes extend by continuity to this compactification. No preliminary knowledge of Alexandrov spaces will be assumed, though it will be useful.

Polyhedral Number Theory

Matthias Beck

San Francisco State University, U.S.A.

Several problems in number theory concern integer solutions to a linear system of inequalities and equations; the most prominent example involve integer partitions and permutation statistics. We view these enumeration problems geometrically, i.e., as listing/counting integer points in polyhedra. It turns out that from this viewpoint, one can both give "short" proofs of known number-theoretic results (and extensions of them) and ask (and sometimes answer) interesting questions about certain classes of polyhedra. We will survey several results, going in both directions.

Convolution of Valuations

Andreas Bernig

Goethe–Universität Frankfurt, Germany

Valuations play an important role in our understanding of kinematic formulas. There is a tight connection between algebraic structures on valuations such as product, convolution and Alesker-Fourier transform, and integral-geometric formulas, like intersectional and additive kinematic formulas and local kinematic formulas. The convolution was first introduced in the case of smooth translation invariant valuations in a joint work with Joseph Fu. In more recent works, it was extended to generalized translation invariant valuations (joint with Dmitry Faifman); to compactly supported generalized valuations on Lie groups (joint with Semyon Alesker) and to dual area measures. In my talk I will explain how these constructions are related to each other and to integral-geometric formulas.

Functionals Subject to Concavity or Volume Conditions

Andrea Colesanti

Università degli Studi di Firenze, Italy

I will review some results, proved in collaborations with J. Abardia, D. Hug, and E. Saorín-Gómez, which characterize functionals (mainly valuations) defined on the class of convex bodies, subject to invariance conditions, and to other type of constraints, like inequalities of Brunn-Minkowski type, or upper and lower bounds in terms of the volume.

Riemannian Curvature Measures

Joseph H. G. Fu

University of Georgia, Athens, U.S.A.

We introduce (again) a valuation-theoretic enhancement of a classical construction from Riemannian geometry. The best known classical instance arises in Chern's intrinsic proof of the Gauss-Bonnet theorem, in the form of a universal primitive, living in the sphere bundle SM, for the Chern-Gauss-Bonnet curvature form of a smooth oriented Riemannian manifold M^{n+1}. It is expressed as a sum of terms built from the Cartan apparatus of curvature and connection forms. Chern's construction amounts to a particular expression for the Euler characteristic χ as a smooth valuation on M.

We study the vector space of all valuations expressible in this fashion, or, more precisely, of the space of objects that assign such valuations to any given concrete Riemannian manifold. Our main result endows this space with a natural structure as a module over the polynomial algebra $\mathbb{R}[t]$ in one variable. Via the fundamental theorem of algebraic integral geometry, this module constitutes an essential piece of the array of kinematic formulas in any Riemannian isotropic space. As an illustration of this principle we compute in full detail an important map arising in hermitian integral geometry.

This represents joint work with Thomas Wannerer.

Inequalities about Sections and Projections of Convex Bodies

Apostolos Giannopoulos

University of Athens, Greece

We discuss lower dimensional versions of the slicing problem and of the Busemann-Petty problem, both in the classical setting and in the generalized setting of arbitrary measures in place of volume. We introduce an alternative approach which is based on the generalized Blaschke-Petkantschin formula, on asymptotic estimates for the dual affine quermassintegrals and on some new Loomis-Whitney type inequalities in the spirit of the uniform cover inequality of Bollobas and Thomason.

A Tropical Isoperimetric Inequality

Michael Joswig

Technische Universität Berlin, Germany

We state and prove a tropical analog of the (discrete version of the) classical isoperimetric inequality. The planar case is elementary, but the higher-dimensional generalization leads to an interesting class of ordinary convex polytopes. This study is motivated by deep open complexity questions concerning linear optimization and its tropical analogs. This connection will be sketched briefly in the talk.

Joint work with Xavier Allamigeon, Pascal Benchimol, Jules Depersin and Stéphane Gaubert.

Tensor Valuations on Lattice Polytopes

Monika Ludwig

Technische Universität Wien, Austria

Lattice polytopes are convex hulls of finitely many points with integer coordinates in $\mathbb{R}^n$. A function Z from a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ with values in an abelian group is a valuation if

$$Z(P) + Z(Q) = Z(P \cup Q) + Z(P \cap Q)$$

whenever $P, Q, P\cup Q, P\cap Q \in \mathcal{F}$ and $Z(\emptyset) = 0$. The classification of real-valued invariant valuations on lattice polytopes by Betke & Kneser is classical (and will be recalled). It establishes a characterization of the coefficients of the Ehrhart polynomial.

Building on this, classification results are established for tensor valuations on lattice polytopes. The most important tensor valuations are the discrete moment tensor of rank r,

$$L^r(P) = \frac{1}{r!} \sum_{x \in P \cap \mathbb{Z}^n} x^r,$$

where x^r denotes the r-fold symmetric tensor product of the integer point $x \in \mathbb{R}^n$, and its coefficients in the Ehrhart tensor polynomial, called Ehrhart tensors. However, there are additional examples for tensors of rank nine with the same covariance properties.

Based on joint work with Károly J. Böröczky and Laura Silverstein.

On a Local Version of the Fifth Busemann-Petty Problem

Dmitry Ryabogin

Kent State University, U.S.A.

Let K be an origin-symmetric convex body in $\mathbb{R}^n$, $n \geq 3$, satisfying the following condition: there exists a constant c such that for all directions ξ in $\mathbb{R}^n$,

$$h_K(\xi)\,\mathrm{vol}_{n-1}(K \cap \xi^\perp) = c.$$

(Here $\xi^\perp$ stands for a subspace of $\mathbb{R}^n$ of co-dimension 1 orthogonal to a given direction ξ, and $h_K(\xi)$ is the support function of K in this direction). The fifth Busemann-Petty problem asks if K must be an ellipsoid. We give an affirmative answer to this question for origin-symmetric convex bodies that are sufficiently close to an Euclidean ball in the Banach-Mazur distance.

This is a joint work with María Ángeles Alfonseca, Fedor Nazarov and Vlad Yaskin.

Towards a Classification of Empty Lattice 4-Simplices

Francisco Santos

University of Cantabria, Spain

A *lattice d-simplex* is the convex hull of $d+1$ affinely independent integer points in $\mathbb{R}^d$. It is called *empty* if it contains no integer point apart of the $d+1$ vertices. The *determinant* of the lattice d-simplex $T = \mathrm{conv}(p_0, \dots, p_d)$ is the absolute value of $\begin{vmatrix} 1 & \dots & 1 \\ p_0 & \dots & p_d \end{vmatrix}$. It equals $d!$ times the Euclidean volume of T.

Empty simplices are the fundamental building blocks in the theory of lattice polytopes, in the sense that every lattice polytope P (i.e., polytope with integer vertices) can be triangulated into empty simplices. (Consider, for example, a Delaunay triangulation of the set of integer points in P). In particular, it is very useful to have classifications or, at least, structural results, concerning the list of all empty simplices in a given dimension.

In dimension two this is trivial as a consequence of Pick's Theorem: Every empty triangle is a *unimodular triangle*, that is, it has volume $1/2$ (or, equivalently, determinant 1). In particular, they are all *unimodularly equivalent.* That is, if T_1 and T_2 are two empty triangles then there is an affine map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with integer coefficients and determinant 1 such that $T_2 = f(T_1)$.

In higher dimension it is still true that all unimodular simplices are empty, and that they are equivalent, but they are no longer the only empty simplices. Still, in dimension three there is a quite simple classification of empty simplices due to White:

Theorem 1 (White 1964 [6]). *Every empty tetrahedron of determinant q is unimodularly equivalent to*

$$T(p,q) := \mathrm{conv}\{(0,0,0), (1,0,0), (0,0,1), (p,q,1)\},$$

for some $p \in \mathbb{Z}$ with $\gcd(p,q) = 1$. Moreover, $T(p,q)$ is $\mathbb{Z}$-equivalent to $T(p',q)$ if and only if $p' = \pm p^{\pm 1} \pmod{q}$.

A key for the proof of this classification is the fact that all empty 3-simplices have *lattice width* equal to one. That is, for any empty 3-simplex T there is an affine integer functional $f : \mathbb{R}^3 \to \mathbb{R}$ with $T \in f^{-1}([0, 1])$. (Put differently, T lies between two consecutive parallel lattice planes). In the statement of Theorem 1, empty 3-simplices are given coordinates so that their width is one with respect to the functional $f(x, y, z) = z$.

In dimension 4 a full classification of empty simplices is not known, but the following facts are known:

(1) There are infinitely many empty 4-simplices of width 1 (e.g., cones over empty tetrahedra).

(2) There are infinitely many empty 4-simplices of width 2 [3].

(3) Every empty 4-simplex T is cyclic [1]; that is, $\mathbb{Z}^d/\Lambda(T)$ is a cyclic group, where Λ is the linear lattice generated by (differences of) vertices of T . (Observe that the order of this group equals the determinant of T).

We are interested in the classification of all empty lattice 4-simplices. Athough we still do not have a complete one, we know that with finitely many exceptions, all empty 4-simplices have width at most two. (The lattice width of a lattice polytope is the minimum width of a 1-dimensional affine integer projection of it). More precisely, in this talk I will discuss the following three statements, each of which implies the stated finiteness:

Theorem 2 ([2])**.** *For each n, there are only finitely many lattice 4-polytopes with exactly n lattice points and of width larger than two.*

Theorem 3 ([4])**.** *There are exactly 179 empty lattice 4-simplices of width larger than two. Their (normalized) volumes range from 41 to 179 and they all have width three except for a single example of width four, and volume 101.*

Almost Theorem 4 ([1])**.** *Except for finitely many examples, every empty lattice 4-simplex belongs to one of the 29 families found by Mori, Morrison and Morrison [5]. All simplices in these families have width one or two.*

Theorem 3 was conjectured by Haase and Ziegler [3].

As implicit in our statement, we have found a gap in the proof of Theorem 4 and, in fact, the statement is not true: there are infinite families that do not exactly fit in the Mori, Morrison, Morrison classification. Still, the following weaker version of the theorem is true:

Theorem 5 (corrected version of Theorem 4)**.** *Except for finitely many examples, every empty lattice 4-simplex belongs to one of the 29 families found by Mori, Morrison and Morrison [5], or to a family obtained as a direct sum of one of these with a finite set.*

This is joint work with M. Blanco, C. Haase, J. Hoffman and O. Iglesias-Valiño.

References

[1] M. Barile, D. Bernardi, A. Borisov and J.-M. Kantor. On empty lattice simplices in dimension 4, *Proc. Amer. Math. Soc.* 139(12):4247–4253, 2011.

[2] M. Blanco, C. Haase, J. Hoffman, F. Santos. The finiteness threshold width of lattice polytopes, `arXiv:1607.00798v2`.

[3] C. Haase, G. M. Ziegler. On the maximal width of empty lattice simplices. *Europ. J. Combinatorics* 21 (2000), 111–119.

[4] O. Iglesias-Valiño, F. Santos. Classification of empty lattice 4-simplices, in preparation.

[5] S. Mori, D.R. Morrison and I. Morrison. On four-dimensional terminal quotient singularities. *Math. Comput.* 51 (1988), no. 184, 769–786.

[6] G. K. White. Lattice tetrahedra. *Canadian J. Math.* 16 (1964), 389–396.

Some More Results on the L_p Minkowski Problem

Alina Stancu

Concordia University, Montréal, Canada

I will report on some cases of the L_p Minkowski problem that are not, or less, known in which the methods vary from classical convex geometry techniques to PDEs tools for the smooth problem.

Recent Results on Approximation of Convex Bodies by Polytopes

Elisabeth Werner

Case Western Reserve University, Cleveland, U.S.A.

We discuss several recent results related to approximation of convex bodies by polytopes.

The first recent result, obtained jointly with J. Grote, generalizes a theorem by Ludwig, Schuett and Werner on approximation of a convex body K in the symmetric difference metric by an arbitrarily placed polytope with a fixed number of vertices.

The second recent result is by S. Hoehner, C. Schuett and E. Werner. It gives a lower bound, in the surface deviation, on the approximation of the Euclidean ball by an arbitrary positioned polytope with a fixed number of k-dimensional faces.

Semialgebraic Sets of f-Vectors

Günter M. Ziegler

Freie Universität Berlin, Germany

Steinitz (1906) characterized the sets of all f-vectors of 3-dimensional polytopes as all the integer points in a 2-dimensional rational cone. Grünbaum and co-workers characterized the sets of all pairs (f_i, f_j) that appear for 4-dimensional polytopes and got complete and reasonably simple answers: They found in all cases that this is the set of all integer points between fairly obvious upper and lower bounds, with finitely-many exceptions. On the other hand, a characterization of all f-vectors of 4-polytopes remains elusive, while the characterization of all f-vectors of simplicial d-polytopes exists (the "g-theorem") but is complicated.

In our work we start with a definition of "simple answers": We call a set of integer points "semi-algebraic" if it is the set of all integer points in a semi-algebraic set defined over the integers. We then prove that, in particular, the set of pairs (f_i, f_j) for 4-dimensional polytopes is semi-algebraic, with one exception: The set of pairs (f_1, f_2) is NOT semi-algebraic. Similarly, we show that the set of f-vectors of simplicial d-polytopes is semi-algebraic for $d < 6$, but NOT for $d = 6$. This also implies that the set of f-vectors of all 6-polytopes is not semi-algebraic.

Work in progress, joint with Hannah Schäfer Sjöberg.

Tetrahedral Packings, Color Graphs and Optimization

Chuanming Zong

Tianjin University / Peking University, China

2,300 years ago, Aristotle claimed that identical regular tetrahedra can fill the whole space without gap. Unfortunately, this statement is wrong. In 1900, Hilbert listed sphere packings and tetrahedron packings as the third part of his 18th problem. In this talk, we will report some progresses on tetrahedral packings, and present a new approach based on color graphs and optimization.

Contributed Talks

Curvature Types in Normed Planes

Vitor Balestro

Campus Nova Friburgo, Brazil

Heuristically, *curvature* can be regarded as "amount of rotation". Having this in mind, we can extend the usual Euclidean curvature to a smooth and strictly convex normed plane simply by considering, in such a plane, the natural tools to measure area and length (namely, a fixed area element and the *Minkowski length*). In this process, we define two pairs of *dual* curvatures, for which all the questions posed and answered for the Euclidean curvature also make sense.

Let $(X, \|\cdot\|)$ be a smooth and strictly convex normed plane with unit ball B and unit circle S. Let $l(\gamma)$ denote the usual *Minkowski length* of a curve γ in X, and let $[\cdot,\cdot]$ denote a fixed nondegenerate symplectic bilinear form (which yields an area measure). The first natural way to define a curvature concept in a normed plane is to measure the variation of the area swept in the unit ball by the unit tangent field. Formally, we let $\varphi : [0, 2\lambda(S)] \to X$ be a parametrization of the unit circle by twice the area of the sector from $\varphi(0)$ to $\varphi(u)$, and let s be an arc-length parameter in the curve γ. Then, we let $u(s) : [0, l(\gamma)] \to [0, 2\lambda(S)]$ be the function such that $\gamma'(s) = \varphi(u(s))$, and we define the *Minkowski curvature* of γ at $\gamma(s)$ to be

$$k_m(s) := u'(s).$$

Analogously, if we consider the variation of the arc-length, instead of area, determined in the unit circle by the unit tangent field to γ we obtain the *arc-length curvature.* These curvature concepts are *dual*, in the sense that the Minkowski curvature is the arc-length curvature in the *anti-norm.*

These curvature types, however, do not behave well when we want to define an *osculating circle* (i.e., a 2nd order contact attached to the curve at some point). For this sake, we define the *circular curvature* of a curve γ by considering the variation of the arc-length of the unit circle determined when we regard the unit tangent field γ as tangents to S. In other words, if $\varphi(t) : [0, l(S)] \to X$ is an arc-length parametrization of the unit circle, then we let $t(s) : [0, l(\gamma)] \to [0, l(S)]$ be the function such that $\gamma'(s) = \frac{d\varphi}{dt}(t(s))$. Then we define the *circular curvature* to be

$$k_c(s) := t'(s).$$

In the Euclidean plane, one can obtain the usual curvature by considering the *Frenet frame.* We can do the same here, but extending the usual orthogonality concept to *Birkhoff orthogonality* (which we will denote by $\dashv_B$), and regarding two curvatures instead of one. Let $n_\gamma(s)$ denote, for each $s \in [0, l(\gamma)]$, the unique vector such that $\gamma'(s) \dashv_B n_\gamma(s)$ and $[\gamma'(s), n_\gamma(s)] = 1$. The field $n_\gamma(s)$ is unit in the anti-norm, and then we can measure the variation of the area swept by it in the unit *anti-circle*, with respect to the usual arc-length of γ. By doing so, we obtain the *normal curvature* (which we denote by k_n).

For the Minkowski curvature and the normal curvature the following *Frenet formulas* hold:

$$\gamma''(s) = k_m(s) n_\gamma(s); \text{ and}$$
$$n'_\gamma(s) = -k_n(s)\gamma'(s).$$

Also, the normal curvature is dual to the circular curvature, in the sense as before: one gets the other when we switch the norm by the anti-norm. Notice that these curvature types coincide if and only if the plane is *Radon* (like also for Minkowski curvature and arc-length curvature, clearly).

Of course, defining these curvature types gives rise to several questions. We can mention, e.g., the classification of the curves of constant curvature, the invariance under isometries of the plane, and the eventual validity of the four vertex theorem. One can prove that portions of circles and anti-circles are the (unique) curves with constant circular and normal curvatures, respectively, that all the curvature types are invariant under isometries of the plane and that the four vertex theorem is valid for all the curvature types.

It is worth mentioning that, as in the Euclidean case, the sum of the *curvature radii* (i.e., the inverse of the circular curvature) at opposite points of a *curve of constant width* is constant. This observation allows us to prove that, as also in the Euclidean subcase, if any affine diameter of a constant width curve divides it into two portions of equal Minkowski length, then this curve is a Minkowski circle.

Finally, since we have the notion of osculating circle, we can define evolutes, involutes and parallels of a curve. This can be made by using the techniques of Singularity Theory, by considering squared distance functions. A little surprising, the behavior of these concepts in the general case remains analogous to the Euclidean subcase. This suggests that an inner product is not strictly necessary in developing this theory.

This is a joint work with Horst Martini and Emad Shonoda.

Discrete Slicing Problems with Low Dimensional Subspaces

Sören Lennart Berg

Technische Universität Berlin, Germany

In recent years, popularity of *discrete slicing problems*, where structure and properties of lattice points of a given (symmetric) convex body are studied with regard to affine and linear subspaces, steadily increased. In this talk we briefly summarize some of the recent results, and focus on low dimensional subspaces. Moreover, we will see that this assumption can imply interesting arithmetic properties, such as *sum-freeness*, of the underlying set of lattice points.

This is joint work with Martin Henk.

Weighted Floating Bodies

Florian Besau

Goethe–Universität Frankfurt, Germany

The volume of the classical floating body of a convex body naturally gives rise to the affine surface area of the body. In joint work together with M. Ludwig and E. Werner, we were able to show, that one may replace both notions, volume and the floating body, with weighted ones, to obtain a generalization of the classical affine surface area – the weighted floating area. This weighted floating area still shares many important properties with the classical affine surface area and is also closely tied to the non-uniform random approximation of a convex body by polytopes.

For a polytope the affine surface area, as well as the weighted floating area, vanish and the rate of convergence in the random approximation increases. Instead of the affine surface area, a combinatorial invariant – the number of flags – is now determining the asymptotic behavior. Again, the volume of the classical floating body of a polytope gives rise to this invariant. Together with C. Schütt and E. Werner, we are currently working on an investigation into the behavior of the weighted floating body of a polytope.

The L_p Minkowski Problem for $-n < p < 1$

Gabriele Bianchi

Università degli Studi di Firenze, Italy

I will present some results obtained with Károly J. Böröczky and Andrea Colesanti on the L_p Minkowski problem for $-n < p < 1$. In particular I will concentrate on the smoothness of the solution.

On Spherical Projection of a Convex Body and the Related Quotient Space

Agnieszka Bogdewicz

Politechnika Warszawska, Poland

Consider $A \in \mathcal{K}_0^n$. Let $\mathcal{F}(A) := \{A(u) \,|\, u \in S^{n-1}\}$ be the set of all *faces* of A. We say that $S_0 \subset S^{n-1}$ is *spherically convex* whenever *cone* $S_0 := \bigcup_{x \in S_0} \Delta(0, x)$ is a convex subset of $\mathbb{R}^n$. Let $\mathcal{K}(S^{n-1})$ be the family of closed, spherically convex subsets of S^{n-1}.

We define *spherical projection* $\pi_A : \mathcal{F}(A) \to \mathcal{K}(S^{n-1})$ and the equivalence relation $\equiv_\pi$ in $\mathcal{K}_0^n$. We consider properties of the related quotient space and of orbits of $\equiv_\pi$.

This is joint work with Maria Moszyńska.

Why We Should Allow Non-Symmetric Gauge Bodies When Studying Radii Functionals

René Brandenberg

Technische Universität München, Germany

In this talk we summarize some recent developments in the theory of radii of convex bodies, which were mainly possible because of generalizing the definitions of the radii functionals, allowing non-symmetric gauge bodies.

Bisections Minimizing the Maximum Relative Diameter

Antonio Cañete

Universidad de Sevilla, Spain

Given a centrally symmetric planar convex body C, a *bisection* of C is a division of C into two connected subsets $\{C_1, C_2\}$ determined by a simple curve with endpoints in ∂C. For each bisection P of C, the *maximum relative diameter* is defined by

$$d_M(P) = \max\{D(C_1), D(C_2)\},$$

where $D(C_i)$ denotes the Euclidean diameter of C_i, $i = 1, 2$.

In this talk we will focus on the bisections of C which minimize the maximum relative diameter functional d_M, showing that they are not unique, and establishing some sufficient conditions for a bisection to be minimizing. Moreover, we shall study when the so-called *standard bisection* (determined by two symmetric inradius segments) is minimizing. We will also see the relation of these questions with classical Borsuk's problem in $\mathbb{R}^2$.

This is part of a joint work with Uwe Schnell (University of Applied Sciences Zittau/Görlitz) and Salvador Segura (University of Alicante).

Maximum Area Circumscribed Polygons

Susanna Dann

Technische Universität Wien, Austria

A convex polygon Q is *circumscribed* about a convex polygon P if every vertex of P lies on at least one side of Q. We present an algorithm for finding a maximum area convex polygon circumscribed about any given convex n-gon in $O(n^3)$ time. As an application, we disprove a conjecture of Farris. Moreover, for the special case of regular n-gons we find an explicit solution.

This is joint work with Markus Aussenhofer (University of Wien), Zsolt Lángi (Budapest University of Technology) and Géza Tóth (Alfréd Rényi Institute of Mathematics, Budapest).

Indefinite Crofton Formulas and the Centro-Affine Surface Area

Dmitry Faifman

Universtity of Toronto, Canada

The Euclidean intrinsic volumes are the most important examples of valuations, which inspired many central notions in convex, differential and integral geometry. In recent joint works with Alesker and Bernig, the intrinsic volumes corresponding to a general quadratic form were constructed and studied. In this talk, we will discuss how to write Crofton-Kubota formulas for those intrinsic volumes. As an application, we will produce explicit Crofton formulas for a valuation of a different kind - the centro-affine surface area. A central ingredient in the above is the explicit evaluation of a new Selberg-type integral.

On Two Theorems of Minkowski in the Geometry of Numbers

Bernardo González Merino

Technische Universität München, Germany

In 1896 Minkowski proved his 'First Fundamental Theorem', which states that every 0-symmetric convex body whose only interior lattice point is the origin 0, has volume bounded from above by 2^n.

Replacing the volume by the number of lattice points in the set (known as the *lattice point enumerator*) he showed an analogous 'discrete' result to his First Fundamental Theorem, proving that 3^n is the right upper bound in this case. He also showed that if the set is moreover strictly convex, the latter further reduces to $2^{n+1} - 1$.

In this talk we will show how to extend these two 'discrete' theorems of Minkowski, by allowing any number of interior lattice points in the convex set, following in some sense the ideas that Blichfeldt and Van der Corput developed in order to extend the First Fundamental Theorem of Minkowski.

This is a joint work with G. Averkov, I. Paschke, M. Schymura, and S. Weltge.

Hunting for Reduced Polytopes

Thomas Jahn

Technische Universität Chemnitz, Germany

A convex body is said to be *reduced* if it has no proper compact convex subset sharing the same minimum width. Reduced bodies appear naturally as extremal bodies in Steinhagen's inequality or Pál's problem, and are conceptually related to diametrically complete and constant width bodies. In spite of the strikingly simple definition, there are still gaps in understanding reducedness. One of these gaps is the study of reduced polytopes in d-dimensional Euclidean space, $d \geq 3$, concerning first and foremost their mere existence.

This is joint work with Bernardo González Merino, Alexandr Polyanskii and Gerd Wachsmuth.

h^*-Polynomials of Zonotopes

Katharina Jochemko

Royal Institute of Technology (KTH) Stockholm, Sweden

The Ehrhart polynomial counts the number of lattice points in integer dilates of a lattice polytope. A central question in Ehrhart theory is to characterize all possible Ehrhart polynomials. An important tool is the h^*-polynomial of a lattice polytope, which encodes the Ehrhart polynomial in a certain binomial basis. One open question coming from commutative algebra is whether the h^*-polynomial of an integrally closed lattice polytope is always unimodal. Schepers and Van Langenhoven (2011) proved this for lattice parallelepipeds.

Using the interplay of geometry and combinatorics, we generalize their result to zonotopes by interpreting their h^*-polynomials in terms of certain refined descent statistics on permutations. From that we obtain that the h^*-polynomial of a zonotope is unimodal with peak in the middle and, moreover, that it has only real roots. Moreover, we are able to give a complete description of the convex hull of all h^*-polynomials of zonotopes in a given dimension: it is a simplicial cone spanned by refined Eulerian polynomials.

This is joint work with Matthias Beck and Emily McCullough (both San Francisco State University).

Grünbaum Distance of Two Planar Convex Bodies

Tomasz Kobos

Jagiellonian University, Poland

For two (not necessarily symmetric) convex bodies $K, L \subset \mathbb{R}^n$ the classical *Banach-Mazur distance* is defined as

$$d_{BM}(K,L) = \inf_{T \in GL(n), u,v \in \mathbb{R}^n} \{r > 0 : K + u \subset T(L+v) \subset r(K+u)\}.$$

One of variations of this notion is so called *Grünbaum distance*, which for two convex bodies $K, L \subset \mathbb{R}^n$ is defined as

$$d_G(K,L) = \inf_{T \in GL(n), u,v \in \mathbb{R}^n} \{|r| : r \in \mathbb{R},\ K + u \subset T(L+v) \subset r(K+u)\}.$$

In simple words, we fit an affine copy of one body between the other body and its positive **or** negative homothetic copy. Clearly $d_G(K,L) \leq d_{BM}(K,L)$ for every pair of convex bodies (K,L).

Maximal Banach-Mazur distance between symmetric convex bodies in $\mathbb{R}^n$ is asymptotically of order n, but the exact value is known only for $n = 2$ and it is equal to $\frac{3}{2}$ (see [1], [5]). Maximal Banach-Mazur distance between arbitrary convex bodies in $\mathbb{R}^n$ is not determined even in the asymptotic setting. However, maximal Grünbaum distance between two convex bodies in $\mathbb{R}^n$ have been determined and it is equal to n. This was a conjecture of Grünbaum (see [3]) and have been settled by Gordon, Litvak, Meyer and Pajor (see [2]), who used an innovative tool of John's decomposition in the general case. Moreover if S is a simplex and K is symmetric then one can easily prove that $d_G(K,S) = n$. Jiménez and Naszódi have therefore conjectured that the equality $d_G(K,L) = n$ implies that K or L is a simplex (see [4]). They have proved their conjecture in the case where K or L is strictly convex or smooth.

The aim of the talk is to present a sketch of the proof that if $K, L \subset \mathbb{R}^2$ are convex bodies such that $d_G(K, L) = 2$ then K or L is a triangle. Conjecture of Jiménez and Naszódi is therefore true for $n = 2$, without any other additional conditions on K or L.

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Laplace Transforms and Function Valued Valuations

Jin Li

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Let $p \geq 1$. It was shown before that $\mathrm{GL}(n)$ compatible valuations taking values on the space of p homogeneous and continuous functions on $\mathbb{R}^n$ are basically L_p Minkowski valuations. We found that the Laplace transform on convex bodies is another function valued valuation which is continuous, positively $\mathrm{GL}(n)$ covariant and logarithmic translation covariant. Conversely, these properties turn out to be sufficient to characterize this transform. The classic Laplace transform is also characterized after extension.

This is joint work with Dan Ma.

On the Randomized Log-Brunn-Minkowski Inequality

Galyna Livshyts

Georgia Institute of Technology, U.S.A.

Given a scalar $\lambda \in [0, 1]$, and a pair of convex bodies K and L containing the origin in their interior, with support functions h_K and h_L, respectively, their geometric average is defined as follows:

$$K^\lambda L^{1-\lambda} := \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K^\lambda(u) h_L^{1-\lambda}(u) \ \forall u \in \mathbb{S}^{n-1}\}.$$

Böröczky, Lutwak, Yang, Zhang conjectured that

$$|K^\lambda L^{1-\lambda}| \geq |K|^\lambda |L|^{1-\lambda},$$

for every pair of symmetric convex bodies K and L. Saroglou showed that proving this conjecture true is equivalent to proving, in every dimension, that

$$\gamma(K^\lambda L^{1-\lambda}) \geq \gamma(K)^\lambda \gamma(L)^{1-\lambda}$$

for some probability log-concave measure γ and when K and L are images of the unit cube under linear operators. This suggests a more general matrix form to the Log-Brunn-Minkowski inequality. We verify this matrix inequality on average for random matrices, and prove concentration bounds, in the case when the restriction of the measure γ on a parallelepiped is positively log-concave.

This is a joint work with Paata Ivanishvili, Christos Saroglou and Ionel Popescu.

Valuations on the Space of Quasi-Concave Functions

Nico Lombardi

Università degli Studi di Firenze, Italy

We present some results concerning valuations defined on quasi-concave functions. We recall that a function $f : \mathbb{R}^N \to \mathbb{R}^+$ is said to be *quasi-concave* if the set

$$L_t(f) = \{x \in \mathbb{R}^N \mid f(x) \geq t\}$$

is a convex body or the empty set, for all $t > 0$. Let $\mathcal{C}^N$ be the set of quasi-concave functions.

We define a valuation on $\mathcal{C}^N$ as a functional $\mu : \mathcal{C}^N \to \mathbb{R}$ such that

$$\mu(f) + \mu(g) = \mu(f \vee g) + \mu(f \wedge g),$$

where $\vee$ and $\wedge$ are the maximum and minimum operators, for all f, g and $f \vee g$ quasi-concave functions.

There are many connections between quasi-concave functions and convex bodies which inspire the study of valuations on $\mathcal{C}^N$.

We present some characterization results for valuations on $\mathcal{C}^N$. First of all, with the additional hypothesis of continuity (w.r.t. a suitable topology) and rigid motion invariance, we present a characterization theorem as Hadwiger Theorem. Moreover we consider the homogeneous case and we show a decomposition theorem as McMullen Theorem.

This is joint work with Andrea Colesanti and Lukas Parapatits.

Polyhedral Gauss sums

Romanos Diogenes Malikiosis

Technische Universität Berlin, Germany

We define certain natural finite sums of n'th roots of unity, called $G_P(n)$, that are attached to each convex integer polytope P, and which generalize the classical 1-dimensional Gauss sum $G(n)$ defined over $\mathbb{Z}/n\mathbb{Z}$, to higher dimensional abelian groups and integer polytopes. We consider the finite Weyl group $\mathcal{W}$, generated by the reflections with respect to the coordinate hyperplanes, as well as all permutations of the coordinates; further, we let $\mathcal{G}$ be the group generated by $\mathcal{W}$ as well as all integer translations in $\mathbb{Z}^d$. We prove that if P multi-tiles $\mathbb{R}^d$ under the action of $\mathcal{G}$, then we have the closed form $G_P(n) = \mathrm{vol}(P)G(n)^d$. Conversely, we also prove that if P is a lattice tetrahedron in $\mathbb{R}^3$, of volume $1/6$, such that $G_P(n) = \mathrm{vol}(P)G(n)^d$, for $n \in \{1, 2, 3, 4\}$, then there is an element g in $\mathcal{G}$ such that $g(P)$ is the fundamental tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$.

Valuations on Log-concave Functions

Fabian Mussnig

Technische Universität Wien, Austria

A function Z defined on the subset $\mathcal{S}$ of a lattice $(\mathcal{L}, \vee, \wedge)$ and taking values in an abelian semigroup is called a *valuation* if

$$\mathrm{Z}(f \vee g) + \mathrm{Z}(f \wedge g) = \mathrm{Z}(f) + \mathrm{Z}(g),$$

whenever $f, g, f \vee g, f \wedge g \in \mathcal{S}$.

In the classical theory, valuations on the set of convex bodies (non-empty, compact, convex sets), $\mathcal{K}^n$, in $\mathbb{R}^n$ are studied, where $\vee$ and $\wedge$ denote union and intersection, respectively. The celebrated Hadwiger classification theorem gives a complete classification of continuous, rotation and translation invariant valuations on $\mathcal{K}^n$ and provides a characterization of intrinsic volumes. Among them, the Euler characteristic and volume form the basis of all continuous and $\mathrm{SL}(n)$ invariant valuations on the space of convex bodies that contain the origin. Moreover, valuations with values in $\mathcal{K}^n$ together with Minkowski addition, also called Minkowski valuations, have attracted interest. Furthermore, since several important geometric operators like the Steiner point and the moment vector are not translation invariant, translation covariant valuations were studied.

More recently, valuations were defined on function spaces. For a space $\mathcal{S}$ of real-valued functions $f \vee g$ denotes the pointwise maximum of f and g while $f \wedge g$ denotes their pointwise minimum. Motivated by previous results on convex bodies, a classification of continuous, homogeneous, $\mathrm{SL}(n)$ and translation covariant Minkowski valuations on the space of log-concave functions is established. Thereby, the recently introduced level set body, its reflection and the moment vector of a log-concave function are characterized. In the proof of this result, a classification of continuous, homogeneous, $\mathrm{SL}(n)$ and translation invariant real-valued valuations is established, where suitable analogs of the Euler characteristic and volume for log-concave functions are characterized.

Characterization of Unit Balls by Riesz Energy

Jun O'Hara

Chiba University, Japan

We show that unit balls can be characterized by regularized Riesz energy, which is generalization of $\int_{M\times M} |x-y|^z dxdy$ (M is a compact body in the Euclidean space) via analytic continuation ($z \in \mathbb{C}$) or equivalently by Hadamard regularization.

Necessary Subspace Concentrarion Conditions for the Even Dual Minkowski Problem

Hannes Pollehn

Technische Universität Berlin, Germany

Recently Huang, Lutwak, Yang and Zhang introduced a broad class of geometric measures related to convex bodies. Among these are the dual curvature measures which are the counterparts to the classical curvature measures of convex bodies in the dual Brunn-Minkowski theory. Here we discuss the associated dual Minkowski problem and show necessary subspace concentration conditions for dual curvature measures $\widetilde{C}_q(K, \cdot)$ of an n-dimensional symmetric convex body in the cases $q \leq n$ and $q \geq n+1$.

This is joint work with Károly Böröczky Jr. and Martin Henk.

Characterization of Regular Triangles in Terms of Critical Points of Riesz Potentials

Shigehiro Sakata

Miyazaki University, Japan

For a convex body K in $\mathbb{R}^n$, Professor Maria Moszyńska looked for a "good" position of the origin for K and investigated a maximizer of the function

$$\Phi_K(x) := \int_{S^{n-1}} \rho_{K-x}(u)^\alpha d\sigma(u), \quad x \in \mathrm{int}K,$$

where $\rho_{K-x}(u) = \max\{\lambda \,|\, \lambda u + x \in K\}$ $(u \in S^{n-1})$ is the radial function of K with respect to x. A maximizer of Φ_K is called a *radial center* of order α of K. It is easy to show that Φ_K is proportional to the *Riesz potential* of order α,

$$V_K^{(\alpha)}(x) := \int_K |x-y|^{\alpha-n} dy.$$

Professor Jun O'Hara showed that the family of radial centers with respect to α includes the centroid (center of mass) and the Chebyshev center of K. Thus, in general, radial centers of K move in α. We will consider the shape of a triangle with radial centers not moving in α.

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On the Geometry of Projective Tensor Products

Carsten Schütt

Universität Kiel, Germany

We study the volume ratio of projective tensor products $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$ where $1 \leq p \leq q \leq r \leq \infty$. As a consequence of the Bourgain-Milman upper estimate on the volume ratio of Banach spaces by the cotype 2 constant, we obtain information on the cotype of $\ell_p^n \widehat{\otimes}_\pi \ell_q^n \widehat{\otimes}_\pi \ell_r^n$. Our results naturally generalize to k-fold projective tensor products $\ell_{p_1}^n \otimes_\pi \cdots \otimes_\pi \ell_{p_k}^n$.

This is joint work with Ohad Giladi, Joscha Prochno, Nicole Tomczak-Jaegermann, and Elisabeth Werner.

On the Covering Radius of Lattice Zonotopes and its Relation to View-Obstructions and the Lonely Runner Conjecture

Matthias Schymura

Freie Universität Berlin, Germany

The covering radius of a convex body K in $\mathbb{R}^n$ is the minimal dilation factor $r > 0$ such that the lattice arrangement $rK + \mathbb{Z}^n$ covers the whole space. This classic concept in the geometry of numbers has been of interest ever since its introduction and connects fruitfully to Number Theory, Integer Programming and Convex Geometry, among others.

The view-obstruction problem of Cusick asks for the minimal edge length in a lattice arrangement of cubes that is necessary to obstruct every non-trivial view through space. We give a reformulation of this question as a study of covering radii of lattice zonotopes. This allows to apply tools from Convex and Discrete Geometry, which we use to estimate the necessary edge length in terms of the rational dimension of the view direction. In terms of zonotopes we need to estimate the covering radii in dependence on the number of generators.

Our investigations are motivated by the popular Lonely Runner Conjecture, a problem in Diophantine Approximation originally posed by Jörg Wills in 1967. In the talk, I will discuss consequences of our findings to this perennial problem.

This is joint work with Romanos-Diogenes Malikiosis (`http://arxiv.org/abs/1609.01939`).

Symmetric Mahler's Conjecture for the Volume Product in the Three Dimensional Case

Masataka Shibata

Tokyo Institute of Technology, Japan

Mahler conjectured the inequality for the volume product:

$$|K||K^{\circ}| \geq \frac{4^n}{n!}$$

for any symmetric convex body $K \subset \mathbb{R}^n$ and its polar K°. In the case $n = 1$ is trivial, and Mahler showed the inequality in the case $n = 2$. For general n, Reisner showed that the inequality holds if K is a zonoid, and Saint Raymond showed that the inequality holds if K is symmetric with respect to each of the coordinate hyperplanes. However, the conjecture is still open even in the case $n = 3$. In this talk, we discuss our recent results about the conjecture in the three dimensional case.

The talk is based on a joint work with Hiroshi Iriyeh.

Integral Geometry of the Quaternionic Plane

Gil Solanes

Serra Húnter Fellow, Universitat Autònoma de Barcelona, Spain

A valuation is a finitely additive functional on the space of convex bodies. Hadwiger's classification theorem states that the space $\mathrm{Val}^{SO(n)}$ of continuous rigid motion invariant valuations is spanned by the so-called intrinsic volumes. This yields a simple proof of the principal kinematic formula of Blaschke and Santaló.

It was shown by Alesker [1] that the space of continuous, translation-invariant and G-invariant valuations Val^G has finite dimension for any group $G \subset O(n)$ acting transitively on the sphere S^{n-1}. The list of such groups, obtained by Montgomery-Samelson and Borel, is the following

$$G = SO(n), U(n), SU(n), Sp(n), Sp(n)U(1), Sp(n)Sp(1),$$

plus the exceptional groups $G = G_2, Spin(7), Spin(9)$. In particular, kinematic formulas exist for each of these groups. The explicit computation of these formulas is a difficult problem that has been recently solved for $G = SO(n), U(n), SU(n), G_2, Spin(7)$ in [4, 2, 3]. This was possible thanks to a new approach developed by Bernig and Fu, based on algebraic structures discovered by Alesker on the space of valuations.

The cases $G = Sp(n)$, $Sp(n)U(1)$, $Sp(n)Sp(1)$, $Spin(9)$ are still open, except for $Sp(2)Sp(1)$, which is the subject of this talk. We will present the classification of $Sp(2)Sp(1)$-invariant valuations on the quaternionic plane [5], as well as the kinematic formulas of this space [6]. This is a first step in the investigation of integral geometry of quaternionic spaces.

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Moments of the Maximal Number of Empty Simplices of a Random Point Set

Daniel Temesvari

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For a finite set X of n points from $\mathbb{R}^M$, the degree of an M-element subset $\{x_1,\dots,x_M\}$ of X is defined as the number of empty simplices, i.e., the number of points $z \in X \setminus \{x_1,\dots,x_M\}$ such that the intersection of the convex hull of $\{x_1,\dots,x_M,z\}$ with the set X contains only the points $x_1,\dots,x_M,z$. Furthermore, the degree of the set X, denoted by $\deg(X)$, is introduced as the maximal degree of any of its m-element subsets.

The purpose of this talk is to show that the moments of the degree of X satisfy $\mathbb{E}[\deg(X)^k] \geq cn^k/\log n$, for some constant $c > 0$, if the elements of the set X are chosen uniformly from a convex body $W \subset \mathbb{R}^n$.

Radial Continuous Valuations on Star Bodies

Ignacio Villanueva

Universidad Complutense de Madrid, Spain

We characterize rotation invariant radial continuous valuations defined on the n-dimensional star bodies by means of an integral in S^{n-1} with respect to the Lebesgue measure. Next we study to which extent this characterization remains valid if we remove the condition of rotation invariance. The talk is based on *Radial continuous rotation invariant valuations on star bodies, Adv. Math. 291 (2016)* and *Radial continuous valuations on star bodies and star sets, arXiv:1611.03345.*

This is joint work with Pedro Tradacete.

Integral Geometry in Exceptional Spheres

Thomas Wannerer

Friedrich-Schiller-Universität Jena, Germany

The kinematic formulas for spheres go back to Blaschke, Chern, and Santaló and are in some sense entirely classical. For spheres of dimension 6 and 7 however, new kinematic formulas have been discovered recently. In this talk we will report on ongoing joint work with Gil Solanes on kinematic formulas in these exceptional spheres. We will put our results within the framework of Alesker's theory of valuations and we will explain how properties of the octonions shape the integral geometry in these special dimensions. Implications for the integral geometry of general isotropic spaces will be touch upon briefly. Finally, we will present a generalization of the Klain-Schneider characterization of simple valuations.

On Polynomially Integrable Convex Bodies

Vladyslav Yaskin

University of Alberta, Canada

Let K be a convex body in $\mathbb{R}^n$. The parallel section function of K in the direction $\xi \in S^{n-1}$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap \{(x,\xi) = t\}), \quad t \in \mathbb{R},$$

where (x, ξ) is the scalar product in $\mathbb{R}^n$.

A convex body K in $\mathbb{R}^n$ is called *polynomially integrable* (of degree N) if

$$A_{K,\xi}(t) = \sum_{k=0}^{N} a_k(\xi)\, t^k$$

for some integer N, all $\xi \in S^{n-1}$ and all t for which the set $K \cap \{x : (x,\xi) = t\}$ is non-empty. Here, a_k are functions on the sphere. We assume that the function a_N is not identically zero.

We prove that the only smooth convex bodies with this property in odd dimensions are ellipsoids, if $N \geq n-1$. This is in contrast with the case of even dimensions and the case of odd dimensions with $N < n-1$, where such bodies do not exist, as it was recently shown by Agranovsky.

This is joint work with A. Koldobsky and A. Merkurjev.

On Brunn-Minkowski's Inequalities under Projections Assumptions

Jesús Yepes Nicolás

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The well-known Brunn-Minkowski inequality asserts that $\mathrm{vol}((1-\lambda)K+\lambda L)^{1/n}$, for $K, L \in \mathcal{K}^n$ convex bodies, is a concave function in $\lambda \in [0,1]$, where the exponent $1/n$ is moreover the best that one may expect for such an inequality.

Nevertheless, from a classic result by Bonnesen, we may state that

$$\mathrm{vol}((1-\lambda)K+\lambda L) \geq (1-\lambda)\mathrm{vol}(K) + \lambda \mathrm{vol}(L),$$

provided that K and L have a projection onto a hyperplane of the same measure.

The natural hypothesis of a common $(n-k)$-plane projection of the sets does not imply, however, that the $(1/k)$-th powered volume function is concave.

In this talk we will discuss which is the, somehow, best projection type assumption that is needed in order to get concavity for $\mathrm{vol}((1-\lambda)K+\lambda L)^{1/k}$. In the same way, we will show that such a result can be obtained in a more general setting: for compact sets on the one hand and, on the other, for its functional analogue, via the Prékopa-Leindler inequality.

This is joint work with María A. Hernández Cifre (University of Murcia).

Selfishness of Convex Bodies

Liping Yuan

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Let $\mathcal{F}$ be a family of sets in $\mathbb{R}^d$. A set $M \subset \mathbb{R}^d$ is called $\mathcal{F}$-*convex* if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathcal{F}$ such that $x, y \in F$ and $F \subset M$.

We call a family $\mathcal{F}$ of compact sets *complete* if $\mathcal{F}$ contains all compact $\mathcal{F}$-convex sets. A single convex body K will be called *selfish*, if the family of all convex bodies similar to K (resulting from an isometry and a dilation) is complete. We investigate here the selfishness of convex bodies.

This is joint work with Tudor Zamfirescu.

Discs Held in Cages

Tudor Zamfirescu

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A *cage* is defined as the 1-skeleton of a convex polyhedron. More than half a century ago, Coxeter, Besicovitch, Aberth and Valette investigated 3-dimensional balls held by cages, and determined the minimal total length of those cages. In this talk we consider discs, i.e. 2-dimensional balls, held by cages, and try to determine the various positions the discs may have when being "almost fixed" by the cage. We shall determine the exact number of such positions for tetrahedral cages and the maximal number for pentahedral cages.

Isoperimetric-type Inequalities and Equalities for Planar Ovals

Michał Zwierzyński

Politechnika Warszawska, Poland

During this talk we will introduce definitions of affine λ-equidistants of planar curves, including the Wigner caustic, and of the Constant Width Measure Set. We will show the geometric properties of these sets and also we will show that oriented areas of these sets give the improvement in the classical isoperimetric inequality for planar closed convex smooth curves (ovals).

The Wigner caustic can be viewed as the measure of centrally symmetric property of an oval and the Constant Width Measure Set can be viewed as the measure of constant width property of an oval.

The oriented area of the Wigner caustic gives the exact relation between the length and the area of the region bounded by an oval when the oval is an oval of constant width. The same holds for the Constant Width Measure Set and centrally symetric ovals.

It turns out that the linear combination of the oriented areas of the Wigner caustic and the Constant Width Measure Set gives the isoperimetric equality for ovals.

References

[1] M. Zwierzyński: "The improved isoperimetric inequality and the Wigner caustic of planar ovals", J. Math. Anal. Appl. (2016), `http://dx.doi.org/10.1016/j.jmaa.2016.05.016`.

[2] M. Zwierzyński: "The Constant Width Measure Set, The Spherical Measure Set and isoperimetric equalities for planar ovals", `arXiv:1605.02930`.

[3] M. Zwierzyński: "Isoperimetric equalities for rosettes", `arXiv:1605.08304`.

Posters Contributions

On a Discrete Brunn-Minkowski Inequality

David Iglesias López

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Relating the volume with the Minkowski (vectorial) addition of compact (not necessarily convex) sets, one is led to the famous Brunn-Minkowski inequality. One form of it states that $V_n(K+L)^{1/n} \geq V_n(K)^{1/n} + V_n(L)^{1/n}$ for any compact sets $K, L \subset \mathbb{R}^n$. The Brunn-Minkowski inequality is one of the most powerful results in Convex Geometry and beyond: its equivalent analytic version (Prékopa-Leindler inequality) and the fact that the compactness assumption can be 'weakened' to consider just Lebesgue measurable sets, have allowed it to move in much wider fields.

In 2001, Gardner & Gronchi moved it to the discrete setting and proved a discrete version of the Brunn-Minkowski inequality for finite subsets of the integer lattice $\mathbb{Z}^n$: if $A, B \subset \mathbb{Z}^n$ are finite with $\dim B = n$, then $|A+B| \geq |D^B_{|A|} + D^B_{|B|}|$; here $|\cdot|$ denotes the cardinal function, and $D^B_{|A|}$, $D^B_{|B|}$ are, roughly speaking, like an intersection of some simplices with $\mathbb{Z}^n$, having the same cardinality as A and B, respectively.

In this poster we aim to present a new type of discrete Brunn-Minkowski inequality. More precisely, we will show that if $A, B \subset \mathbb{Z}^n$ are finite, then

$$|\overline{A} + B|^{1/n} \geq |A|^{1/n} + |B|^{1/n},$$

where $\overline{A}$ is an *extension* of A which is obtained by adding some new integer points in a particular way. We also prove that the inequality is sharp, providing examples for the equality case, and show that the number of additional points is somehow controlled, and depends on the structure of A.

This is joint work with María A. Hernández Cifre and Jesús Yepes Nicolás.

Classification of Empty Lattice 4-Simplices

Oscar Iglesias-Valiño

Universidad de Cantabria, Spain

A lattice d-simplex is the convex hull of $d+1$ affinely independent integer points in $\mathbb{R}^d$. It is called empty if it contains no lattice point apart of its $d+1$ vertices. The classification of empty 3-simplices is known since 1964 (White), based on the fact that they all have width one. But for dimension 4 no complete classification is known.

Haase and Ziegler (2000) computed all empty 4-simplices up to determinant 1000 and based on their results conjectured that after determinant 179 all empty 4-simplices have width one or two. We prove this conjecture as follows:

- We show that no empty 4-simplex of width three or more can have determinant greater than 5058, by combining the recent classification of hollow 3-polytopes (Averkov, Krümpelmann and Weltge, to appear) with general methods from the geometry of numbers.

- We continue the computations of Haase and Ziegler up to determinant 7600, and find that no new 4-simplices of width larger than two arise.

In particular, we give the whole list of empty 4-simplices of width larger than two, which is as computed by Haase and Ziegler: There is a single empty 4-simplex of width four (of determinant 101), and 178 empty 4-simplices of width three, with determinants ranging from 41 to 179.

Valuations on Lipschitz Functions

Daniele Pagnini

Università degli Studi di Firenze, Italy

We prove a characterization theorem for continuous, dot product-invariant and rotation-invariant valuations defined on the space $\mathrm{Lip}(S^1)$ of Lipschitz continuous functions on the bidimensional sphere: we show that every such valuation admits a precise integral representation.

Moreover, we prove that a continuous valuation on $\mathrm{Lip}(S^2)$ is uniquely determined by the values it attains on the set of convex functions defined on S^2.

Participants List

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