Stochastic Modelling of Counterparty Credit Risk, CVA and DVA – with a Glimpse at the New Regulatory Framework Basel III

Frank Oertel

University of Southampton
School of Mathematics

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In particular, this presentation is by no means linked to any present and future wording regarding global regulation of CCR including EMIR and CRR (CRD IV).

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2. Total bilateral valuation adjustment

3. Unilateral CVA and Basel III

Counterparty credit risk - Outline I

Suppose there are two parties who are trading a portfolio of OTC derivative contracts such as, e. g., a portfolio of CDSs. (Bilateral) counterparty credit risk (CCR) is the risk that *at least one* of those two parties in that derivative transaction will default prior to the expiration of the contract and will be unable to make all contractual payments to its counterpart.

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Future cashflow exchanges are not known with certainty today. The main feature that distinguishes CCR from the risk of a standard loan is the uncertainty of the exposure at any future date. Hence, regarding the modelling of the exposure a simulation of future cashflow exchanges is necessary.

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- Modelling of the "exposure", i. e., the expected loss to a party when their counterparty defaults before maturity of the financial contract;
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- Calculation of a value adjustment on top of the "CCR free" market value of a transaction, implying a "pricing of CCR";
- Basel III capital charge for the "value adjustment volatility risk";
- Impact of central clearing through a CCP.

CCR - The framework I

Consider two arbitrary parties who trade with each other according to an underlying financial contract - “party 0” and “party 2”, say. Let $k \in \{0, 2\}$ and $T > 0$ the final maturity of this financial contract.

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- Let $X = (X(t))_{0 \leq t \leq T}$ denote a stochastic process describing a cash flow between party 0 and party 2 (or a random sequence of prices). If, at time t, X_t is seen from the point of view of party k, we denote its value equivalently as $X_t(k)$ or $X_t(k, 2-k)$ or $X_k(t)$ - depending on its eligibility.

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- We will make use of the important notation $Y_t(k \mid l)$ to describe a random cash flow amount from the point of view of party k at time t contingent on the default of party l, where $l \in \{0, 2\}$.

CCR - The framework II

- "Contingent cash flows" between two trading parties can be embedded into the model of a generalised weighted and directed Erdős-Rényi random graph ("network"), consisting of counterparties as nodes and current exposures as edges, leading to matrix-valued stochastic processes of type $(\omega, t) \mapsto \mathbf{A}(\omega, t)$, where $\mathbf{A}(\omega, t)_{k,l} := Y_t(k \mid l)(\omega)$ describes a contingent cash flow between the nodes k and l, given that l will default until T (with probability 1).

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- Clearing through a central counterparty (CCP) changes this graph to a (possibly disconnected) tree.
- Notice that the permutation $s : \{0, 2\} \longrightarrow \{0, 2\}, k \mapsto 2 - k$ is bijective. It satisfies $s \circ s = s$. (If s should permute the numbers 1 and 2 instead, then put $s(k) := 3 - k$.)

Modelling of default times

Let τ_0 denote the default time of the investor and τ_2 the default time of the counterparty. Suppose that the underlying financial market model is arbitrage-free. Let $(\Omega, \mathcal{G}, \mathbf{G}, \mathbb{Q})$ be a filtered probability space (satisfying the usual conditions) such that for all $t \geq 0$ the σ-algebra $\mathcal{G}_t$ contains both, the market information up to time t and the information whether the default of the investor or its counterpart has occurred or not up to time t. $\mathbb{Q}$ is a (not necessarily unique) “spot martingale measure”.

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All possible bilateral CCR scenarios

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Moreover, $N \in \mathcal{G}_{\tau^*}$, $A_k^- \in \mathcal{G}_{\tau^*}$ *and hence also* $A_{sim} = \left(N \,\dot\cup\, A_0^- \,\dot\cup\, A_2^-\right)^c \in \mathcal{G}_{\tau^*}$.

Mark-to-Market value (MtM)

The trading book of a bank is based on the principle of "fair value accounting" (FAS 157 (US), respectively IAS 39 (EU)). It refers to accounting for the "fair value" of an asset or liability based on the current market price. Positions are "marked to market" on daily basis (via calibration of models to market data).

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Deals are "fair" at the commencement of the financial transaction. I. e., their net present value at $t = 0$ equals zero: $M(0) := NPV(0) := 0$. However, as time goes by, the financial transaction is "marked to market", implying that at $0 < t \leq T$ $M(t) \equiv NPV(t) \neq 0$.

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Seen from $t = 0$, $M(t) : \Omega \to \mathbb{R}$ is a real-valued random variable (which can have a negative value if the trade is "out-of-the-money").

Money conservation property I

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Let $0 \leq t \leq T$ and $k \in \{0, 2\}$. A cash flow $X = (X_t)_{0 \leq t \leq T}$ between two trading parties satisfies the Money Conservation Property (MCP) at t iff there exists $k \in \{0, 2\}$ such that

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$$\big(X_t(k, 2-k) + X_t(2-k, k)\big)\big|A = 0 \text{ for all } A \in \{A_0^-, A_2^-, A_{\mathsf{sim}}, N\} .$$

Money conservation property II

In particular, (at t) such “MCP processes” X satisfy the following important property in relation to CCR:

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(ii) $\Lambda_t(k,2-k) := M_k(t) - LGD_{2-k}(M_k(t))^+$.

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$$\Pi_k^{(s,t]} := D(0,s)^{-1} \int_{(s,t]} D(0,u) d\Phi_k(u) ,$$

where Φ_k (viewed from party k) denotes a non-vulnerable cumulative dividend process of the portfolio over the time horizon $[s,t]$ and $D(0,\cdot)$ a continuous $\mathbf{G}$-adapted discount factor process (which both are assumed to be of finite variation).

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Representation of the MtM M value - FTAP II

Assume throughout that our financial market model does not allow arbitrage and that each CCR free contingent claim between party k and party $2-k$ in the portfolio (or netting set) is attainable therein.

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Assume throughout that our financial market model does not allow arbitrage and that each CCR free contingent claim between party k and party $2-k$ in the portfolio (or netting set) is attainable therein. Thus, seen from party k's point of view the CCR free mark-to-market process $M_k = (M_k(t))_{0 \leq t \leq T} \equiv (M_t(k))_{0 \leq t < T}$ is then given by

$$M_k(t) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} \middle| \mathcal{G}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\int_{(t,T]} D(t,u)d\Phi_k(u) \middle| \mathcal{G}_t\right] = -M_{2-k}(t)\,,$$

where $\mathbb{Q}$ is a "spot martingale measure" (due to the risk-neutral valuation formula). In the following we fix $\mathbb{Q}$ and occasionally omit its extra description in the notation of (conditional) expectation operators.

The main CCR building blocks

$$\Pi_k^{(t,u]} = -\Pi_{2-k}^{(t,u]}$$

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Random NPV (or MtM) of $\Pi_k^{(t,u]}$ – represented as conditional expectation w.r.t. $\mathbb{Q}$, given the “information” $\mathcal{G}_t$

The main CCR building blocks

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$0 < \mathrm{LGD}_k := 1 - R_k \leq 1$	k's (random) Loss Given Default
$D(t, u) := D(0, u)/D(0, t)$	discount factor at time t for time $u > t$ (can be random)

Valuation of defaultable claims I

Defaultable claims can be valued by interpreting them as portfolios of claims between non-defaultable counterparties including the riskless claim and mutual default protection contracts. Let $k \in \{0, 2\}$. Seen e. g. from the point of view of party k the latter says:

Party k sells to party $2 - k$ default protection on party $2 - k$ contingent to an amount specified by an ISDA close-out rule.

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Party k sells to party $2 - k$ default protection on party $2 - k$ contingent to an amount specified by an ISDA close-out rule.

Let $t \in [0, T]$ such that $]\!]t, \tau^*]\!] \neq \emptyset$. Let

- $M_t(k)$ be the mark-to-market value to party k in case both, party k and party $2 - k$ do not default until T;
- $CVA_t(k \mid 2 - k)$ be the value of default protection that party k sells to party $2 - k$ contingent on the default of party $2 - k$.

Valuation of defaultable claims II

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$$2-k \text{ rejects and requires } M_t(k)+CVA_t(2-k \mid k) \text{ instead} \; .$$

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$$-V_t(k) = -\Big(M_t(k) - \big(CVA_t(k \,|\, 2-k) - CVA_t(2-k \,|\, k)\big)\Big) \overset{(!)}{=} V_t(2-k)\,;$$

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Hence, the deal between k and $2-k$ is agreed!

CCR - Closing out practice I

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Since the market position of k is unchanged after replacing the contract, the loss is determined by the contract's "replacement value" at (or shortly after) τ_k. In general, the partial reimbursement to party $2-k$ involves a payment of $(100\,\mathrm{LGD}_k)\%$ of the "replacement value" at (or shortly after) τ_k.

CCR - Closing out practice II

This process is known as “close-out”. The ISDA Master Agreement defines the term “close-out amount” *to be the amount of the losses or costs of the surviving party* $2-k$ *that would incur by replacement or provision of an economic equivalent*. To this end, ISDA introduced so called “close-out rules”.

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This leads us to the introduction of a useful definition which generalises the ISDA close-out approach (as we will see very soon).

CCR - Closing out practice III

Definition (Generalised close-out cash flow)

Let $k \in \{0, 2\}$ and $f, g : [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. Let H_k be the stochastic process (seen from the viewpoint of party k), defined as

$$
\begin{aligned}
H_k(t) \quad := \quad & f\left(\mathrm{LGD}_{2-k}, (M_{2-k}(t))^-, (M_{2-k}(t))^+\right)\mathbb{1}_{A^-_{2-k}} \\
- \quad & f\left(\mathrm{LGD}_k, (M_k(t))^-, (M_k(t))^+\right)\mathbb{1}_{A^-_k} \\
+ \quad & g\left(\mathrm{LGD}_{2-k}, (M_{2-k}(t))^-, (M_{2-k}(t))^+\right)\mathbb{1}_{A_{\mathrm{sim}}} \\
- \quad & g\left(\mathrm{LGD}_k, (M_k(t))^-, (M_k(t))^+\right)\mathbb{1}_{A_{\mathrm{sim}}}
\end{aligned}
$$

where $0 \leq t \leq T$. If $H_k(T) = 0$, then H_k is called a (symmetric) generalised close-out cash flow.

CCR - Closing out practice IV

Notice that - by construction - a generalised close-out cashflow already satisfies the MCP (due to its symmetry!) and $H_k \mathbb{1}_N \equiv 0$.

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Let $\tau : \Omega \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ be an arbitrary random "time" (e. g., a stopping time) and $X : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}$ a (real-valued) stochastic process. On $\{\tau < \infty\}$ the random variable $X_\tau : \Omega \longrightarrow \mathbb{R}$ is defined through

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega) = X(\tau(\omega), \omega)\,.$$

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$$X_\tau(\omega) := X_{\tau(\omega)}(\omega) = X(\tau(\omega), \omega) \,.$$

Suppose that $\Omega \setminus N \neq \emptyset$. Choose $\omega \in \Omega \setminus N$. Given the - simplifying - assumption that there is no strictly positive margin period of risk, a non-zero close-out has to be settled at $\tau^*(\omega) = \min\{\tau_0(\omega), \tau_2(\omega), T\}$. For simplicity, let us also assume that no collateral is exchanged between party k and party $2 - k$ (until τ^*).

Bipartite ISDA CCR free close-out I

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In the following we fix a very important building block which will appear repeatedly. It plays a fundamental role in ISDA's CCR free close-out rule. Let $k \in \{0, 2\}$ and $0 \leq t \leq T$. We put

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$$\Lambda_k(t) := M_k(t) - \mathrm{LGD}_{2-k}(M_k(t))^+ \overset{(!)}{=} f^*(\mathrm{LGD}_{2-k}, (M_{2-k}(t))^-, (M_{2-k}(t))^+)$$

where $f^*(l, m_-, m_+) := m_-(1-l) - m_+$ for all $(l, m_-, m_+) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+$

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Bipartite ISDA CCR free close-out II

Let $k \in \{0, 2\}$. Next, we will give a precise representation of the generalised close-out cash flow random variable $H_k(\tau^*)$ (seen from the viewpoint of party k), if both parties “symmetrically” apply the same ISDA close-out rule.

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To this end, recall that for all $k \in \{0, 2\}$ we always have

$$H_k(\tau^*) \stackrel{\checkmark}{=} H_k(\tau^*)1\!\!1_{A_0-} + H_k(\tau^*)1\!\!1_{A_2-} + H_k(\tau^*)1\!\!1_{A_{\text{sim}}} ,$$

since no non-zero close-out is required if both parties do not default strictly before T. In fact, by construction, we have $H_k(\tau^*)1\!\!1_N = H_k(T)1\!\!1_N = 0 \stackrel{\checkmark}{=} H_{2-k}(\tau^*)1\!\!1_N$ for all $k \in \{0, 2\}$.

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Fix $k \in \{0, 2\}$. Firstly, suppose $A^{-}_{2-k} \neq \emptyset$. Let $\omega \in A^{-}_{2-k}$. Seen from the viewpoint of party k the (symmetrically) bipartite ISDA CCR free close-out rule (in a given netting set) is reflected in the following table:

	$M_k(\tau_{2-k})(\omega) > 0$	$M_k(\tau_{2-k})(\omega) \leq 0$
Party k receives from party $2-k$	$R_{2-k}(\omega) \cdot M_k(\tau_{2-k})(\omega)$	0
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Let $0 \leq t \leq T$. Let $H_k(t) := H_t(k, 2-k)$ denote the random amount of the close-out cash flow at t seen from the viewpoint of party k. Since in any case $A^{-}_{2-k} \subseteq \{\tau_{2-k} = \tau^*\}$ (and $\mathbb{1}_\emptyset = 0$) the above table shows that in fact $H_k(\tau^*)\big|A^{-}_{2-k} = \Lambda_k(\tau^*)\big|A^{-}_{2-k}$, which is equivalent to:

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$$1\!\!1_{A^-_{2-k}} H_k(\tau^*) = 1\!\!1_{A^-_{2-k}} \left(R_{2-k}(M_k(\tau^*))^+ - (-M_k(\tau^*))^+\right)$$

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Fix $k \in \{0,2\}$. Firstly, suppose $A_{2-k}^{-} \neq \emptyset$. Let $\omega \in A_{2-k}^{-}$. Seen from the viewpoint of party k the (symmetrically) bipartite ISDA CCR free close-out rule (in a given netting set) is reflected in the following table:

	$M_k(\tau_{2-k})(\omega) > 0$	$M_k(\tau_{2-k})(\omega) \leq 0$
Party k receives from party $2-k$	$R_{2-k}(\omega) \cdot M_k(\tau_{2-k})(\omega)$	0
Party k pays to party $2-k$	0	$-M_k(\tau_{2-k})(\omega)$

Let $0 \leq t \leq T$. Let $H_k(t) := H_t(k, 2-k)$ denote the random amount of the close-out cash flow at t seen from the viewpoint of party k. Since in any case $A_{2-k}^{-} \subseteq \{\tau_{2-k} = \tau^*\}$ (and $1\!\!1_{\emptyset} = 0$) the above table shows that in fact $H_k(\tau^*)\big|A_{2-k}^{-} = \Lambda_k(\tau^*)\big|A_{2-k}^{-}$, which is equivalent to:

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Bipartite ISDA CCR free close-out II

Fix $k \in \{0, 2\}$ and suppose now that $A_k^- \neq \emptyset$. Let $\omega \in A_k^-$. Seen from the viewpoint of party $2-k$ the (symmetrically) bipartite ISDA CCR free close-out rule (in a given netting set) is reflected in the following table:

	$M_{2-k}(\tau_k)(\omega) > 0$	$M_{2-k}(\tau_k)(\omega) \leq 0$
Party $2-k$ receives from party k	$R_k(\omega) \cdot M_{2-k}(\tau_k)(\omega)$	0
Party $2-k$ pays to party k	0	$-M_{2-k}(\tau_k)(\omega)$

Hence,

$$\mathbb{1}_{A_k^-} H_{2-k}(\tau^*) = \mathbb{1}_{A_k^-} \left(R_k(M_{2-k}(\tau^*))^+ - (-M_{2-k}(\tau^*))^+ \right) = \mathbb{1}_{A_k^-} \Lambda_{2-k}(\tau^*) \,.$$

Bipartite ISDA CCR free close-out III

Observe that $H_{2-k}(\tau^*)$ is seen from the viewpoint of party $2-k$. Hence, a strict implication of our basic accounting property and the MCP again implies the following important

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Fix $k \in \{0, 2\}$. Firstly let us assume that there are no simultaneous defaults $($i. e., $A_{sim} = \emptyset)$. Then the ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $F_k(\tau^*)$, where

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$$F_k(\tau^*) \quad := \quad \mathbb{1}_{A^-_{2-k}} \Lambda_k(\tau^*) - \mathbb{1}_{A^-_k} \Lambda_{2-k}(\tau^*)$$

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$$F_k(\tau^*) := \mathbb{1}_{A^-_{2-k}} \Lambda_k(\tau^*) - \mathbb{1}_{A^-_k} \Lambda_{2-k}(\tau^*)$$

$$= \mathbb{1}_{A^-_k} \mathit{LGD}_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A^-_{2-k}} \mathit{LGD}_{2-k}(M_k(\tau^*))^+$$

$$- \mathbb{1}_{A^-_k} M_{2-k}(\tau^*) + \mathbb{1}_{A^-_{2-k}} M_k(\tau^*)$$

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F_k(\tau^*) \quad &:= \quad \mathbb{1}_{A_{2-k}^-}\Lambda_k(\tau^*) - \mathbb{1}_{A_k^-}\Lambda_{2-k}(\tau^*) \\
&= \quad \mathbb{1}_{A_k^-} LGD_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^-} LGD_{2-k}(M_k(\tau^*))^+ \\
&\qquad - \mathbb{1}_{A_k^-} M_{2-k}(\tau^*) + \mathbb{1}_{A_{2-k}^-} M_k(\tau^*) \\
&\overset{(MCP)}{=} \quad \mathbb{1}_{A_k^-} LGD_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^-} LGD_{2-k}(M_k(\tau^*))^+ \\
&\qquad + (\mathbb{1}_{A_0^- \cup A_2^-}) M_k(\tau^*).
\end{aligned}
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Bipartite ISDA CCR free close-out IV

Fix $k \in \{0, 2\}$. The case of simultaneous defaults – seen from the point of view of party k - can be treated in the same way. Hence, by another application of the MCP we arrive at the following ISDA CCR free close-out cash flow random variable at τ^* (coinciding with the fourth term in formula (5) of Gregory's paper [6]):

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$$G_k(\tau^*) \quad := \quad \mathbb{1}_{A_{\text{sim}}} \Lambda_k(\tau^*) - \mathbb{1}_{A_{\text{sim}}} \Lambda_{2-k}(\tau^*)$$

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&\quad + \mathbb{1}_{A_{\text{sim}}} 2M_k(\tau^*).
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Bipartite ISDA CCR free close-out V

Observation

Let $k \in \{0, 2\}$. The total ISDA CCR free close-out cash flow random variable at τ^* seen from the point of view of party k is given by $H_k^*(\tau^*) := F_k(\tau^*) + G_k(\tau^*)$.

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$$\begin{aligned} H_k^*(\tau^*) &= F_k(\tau^*) + G_k(\tau^*) \\ &= \mathbb{1}_{A_k^-} \mathrm{LGD}_k(M_{2-k}(\tau^*))^+ - \mathbb{1}_{A_{2-k}^-} \mathrm{LGD}_{2-k}(M_k(\tau^*))^+ \\ &\quad + (\mathbb{1}_{A_0^- \cup A_2^-}) M_k(\tau^*) \\ &\quad + \mathbb{1}_{A_{\mathrm{sim}}} \left(2M_k(\tau^*) - \mathrm{LGD}_{2-k}(M_k(\tau^*))^+ + \mathrm{LGD}_k(M_{2-k}(\tau^*))^+\right) \end{aligned}$$

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&\quad + M_k(\tau^*) + \mathbb{1}_{A_{\text{sim}}} \left((M_k(\tau^*))^+ - \boxed{(M_{2-k}(\tau^*))^+} \right).
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Bipartite ISDA CCR free close-out VI

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Let $k \in \{0, 2\}$ and $t \in [0, T]$. Seen from the viewpoint of party k, the random variable $H_k^*(\tau^*)$ is given as

$$\begin{aligned} H_k^*(\tau^*) &= M_k(\tau^*) - B_{\tau^*}(k, 2-k) \\ &= M_k(\tau^*) - \left(X_{2-k}(\tau^*) - X_k(\tau^*)\right), \end{aligned}$$

where $B_{\tau^*}(k, 2-k) = X_{2-k}(\tau^*) - X_k(\tau^*) \overset{\checkmark}{=} -B_{\tau^*}(2-k, k)$ *and*

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Actually, we have seen more. Namely,

Bipartite ISDA CCR free close-out VII

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Let $k \in \{0,2\}$ and $t \in [0,T]$. Seen from the viewpoint of party k, the random variable $H_k^*(t)$ is given as

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Bipartite ISDA CCR free close-out VIII

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The ISDA CCR free restrictions simply say that both parties, party 0 and party 2 close-out their positions according to the lines of the bipartite ISDA CCR free close-out rule by taking into account precisely all - possible - default scenarios.

Bipartite ISDA CCR free close-out IX

Definition

Let $k \in \{0, 2\}, t \in [0, T]$ and

$$f^*(l, m_-, m_+) := m_-(1 - l) - m_+ \, ,$$

where $(l, m_-, m_+) \in [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+$.

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$$\begin{aligned} H_k^*(\omega, t) \quad &:= \quad \mathbb{1}_{A_{2-k}^- \dot\cup A_{\text{sim}}}(\omega) f^*\left(\text{LGD}_{2-k}, (M_{2-k}(\omega, t))^-, (M_{2-k}(\omega, t))^+\right) \\ &- \quad \mathbb{1}_{A_k^- \dot\cup A_{\text{sim}}}(\omega) f^*\left(\text{LGD}_k, (M_k(\omega, t))^-, (M_k(\omega, t))^+\right) \end{aligned}$$

is called a symmetrically bipartite ISDA CCR free close-out cash flow (seen from the viewpoint of party k).

Bipartite ISDA CCR free close-out IX

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Hence, H^* is a special case of a generalised close-out cash flow in our sense.

Vulnerable cash flows I

Fix $k \in \{0, 2\}$ and $t \in [0, T]$. Let $t \leq \tau^*(\omega)$. Based on our previous representation of $H_k^*(\tau^*)(\omega)$ discounting to t immediately implies

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Theorem and Definition

Let $t \in [0, T]$. On $\{t \leq \tau^*\}$ the t-discounted bipartite ISDA CCR free close-out cash flow amount seen from the point of view of party k at t is given by $D(t, \tau^*)H_k^*(\tau^*)$.

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on $\{t \leq \tau^*\}$, where (as before) $B_{\tau^*}(k,2-k) = X_{2-k}(\tau^*) - X_k(\tau^*)$ and

$$X_k(\tau^*) = \left(\mathbb{1}_{A_k^- \cup A_{sim}} LGD_k - \mathbb{1}_{A_{sim}}\right)(M_{2-k}(\tau^*))^+ \,.$$

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Fix $k \in \{0, 2\}$. Let $0 \leq t \leq u \leq T$. Recall that $\Pi_k^{(t,u]}$ defines the random CCR free cumulative cash flow from the claim in $(t, u]$, discounted to time t (seen from k's point of view).

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Fix $k \in \{0,2\}$. Let $0 \leq t \leq u \leq T$. Recall that $\Pi_k^{(t,u]}$ defines the random CCR free cumulative cash flow from the claim in $(t,u]$, discounted to time t (seen from k's point of view). Let us similarly denote by $\widehat{\Pi}_k^{(t,T]}$ the random cumulative cash flow from the claim in $(t,T]$, discounted to time t (seen from k's point of view), yet accounting for CCR now.

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By construction $\widehat{\Pi}_k^{(t,T]}$ should include both first-to-default scenarios *and* the scenario of a simultaneous default of both parties. To derive the structure of $\widehat{\Pi}_k^{(t,T]}$, we again assume that the MCP holds, as well as our basic accounting principle, and that each party applies the bipartite ISDA CCR free close-out rule.

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Hence, on $\{t \leq \tau^*\}$ we put

$$\widehat{\Pi}_k^{(t,T]} := \Pi_k^{(t,\tau^*]} + D(t, \tau^*) H_k^*(\tau^*)$$

Vulnerable cash flows III

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Recall that we also assume that the No-Arbitrage Principle is satisfied. Hence, under inclusion of all those listed assumptions, we obtain the following

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Let $k \in \{0,2\}$ *and* $t \in [0,T]$*. On* $\{t \leq \tau^*\}$*, the random variable* $\widehat{\Pi}_k^{(t,T]}$ *can be written as*

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In particular, we have $\widehat{\Pi}_k^{(t,T]} \overset{(MCP)}{=} -\widehat{\Pi}_{2-k}^{(t,T]}$.

Hence, a consequent application of the conditional expectation $\mathbb{E}^{\mathbb{Q}}\big[\,\cdot\,\big|\,\mathcal{G}_{\tau^*}\big]$ to $\widehat{\Pi}_k^{(t,T]}$, together with some stochastic analysis lead to the following crucial

Vulnerable cash flows IV

Theorem (Market prices of total bipartite CCR)

Let $k \in \{0,2\}$ and $t \in [0,T]$. Assume that each party applies the bipartite ISDA CCR free close-out rule. If both, the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \leq \tau^*\}$, we have

$$\mathbb{E}^{\mathbb{Q}}\big[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\big] \quad =$$

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$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\right] = M_t(k)$$

Vulnerable cash flows IV

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$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\right] &= M_t(k) \\ &+ \mathbb{E}^{\mathbb{Q}}\left[D(t,\tau^*)\left(\mathbb{1}_{A_k^- \cup A_{sim}} LGD_k - \mathbb{1}_{A_{sim}}\right)(M_{2-k}(\tau^*))^+ \Big| \mathcal{G}_t\right] \end{aligned}$$

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$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\right] &= M_t(k) \\
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&- \mathbb{E}^{\mathbb{Q}}\left[D(t,\tau^*)\left(\mathbb{1}_{A_{2-k}^- \cup A_{sim}} LGD_{2-k} - \mathbb{1}_{A_{sim}}\right)(M_k(\tau^*))^+ \Big| \mathcal{G}_t\right].
\end{aligned}$$

Vulnerable cash flows V

Latter result contains an important and well-known special case. Namely the first appearance of “FTDCVA” and “FTDDVA”:

Vulnerable cash flows V

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Corollary (Brigo-Capponi (2009))

Let $k \in \{0,2\}$, $t \in [0,T]$ and assume that each party applies the bipartite ISDA CCR free close-out rule. Further assume that both, the No-Arbitrage Principle and the MCP are satisfied and that the basic accounting rule holds. If there are no simultaneous defaults (i. e., if $A_{sim} = \emptyset \quad \mathbb{Q}$-a. s.) then on $\{t \leq \tau^*\}$, we have

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\right] &\overset{(!)}{=} M_t(k) \\
&+ \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_k^-} LGD_k D(t,\tau^*)\left(M_{2-k}(\tau^*)\right)^+ |\mathcal{G}_t\right] \\
&- \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{A_{2-k}^-} LGD_{2-k} D(t,\tau^*)\left(M_k(\tau^*)\right)^+ |\mathcal{G}_t\right].
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TCVA, TDVA and TBVA I

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Let $k = 0$ or $k = 2$, $t \in [0, T]$ and $t \leq \tau^*$. Put

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$$\mathsf{TBVA}_t(k, 2-k) := \mathsf{TCVA}_t(k|2-k) - \mathsf{TCVA}_t(2-k|k),$$

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$$\text{TCVA}_t(k|2-k) := \mathbb{E}^{\mathbb{Q}}\left[\left(\mathbb{1}_{A^-_{2-k} \cup A_{\text{sim}}} \text{LGD}_{2-k} - \mathbb{1}_{A_{\text{sim}}}\right) D(t, \tau^*)(M_k(\tau^*))^+ \middle| \mathcal{G}_t\right],$$

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TCVA, TDVA and TBVA I

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TCVA, TDVA and TBVA I

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TCVA, TDVA and TBVA II

Definition (ctd.)

(i) The real-valued $\mathcal{G}_t$-measurable random variable TCVA$_t(k \mid 2-k)$ is called Total Credit Valuation Adjustment at t, seen from the viewpoint of party k.

TCVA, TDVA and TBVA II

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The word “Total” should reflect the total coverage of all four possible cases: $\Omega = N \cup A_0^- \cup A_2^- \cup A_{\text{sim}}$.

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$$\text{“}\,\text{TDVA}_t(k, 2-k \big| k) := \text{TCVA}_t(2-k, k \big| k)\,\text{''}$$

(where by definition $X_t(a, b|m)$ is seen from the viewpoint of the trading party a, given arbitrary trading parties a, b and $m \in \{a, b\}$)?

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TCVA, TDVA and TBVA IV

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Observation

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TCVA, TDVA and TBVA IV

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So, what would (ii) say if in addition both parties did default simultaneously? Does then (ii) just state that the required partial reimbursement $R_k(M_{2-k}(\tau^*))^+$ of party $2-k$ by party k could be ignored by party k then?

Market price of total bilateral CCR I

Recall that we also have assumed that the discount factor process $D(0, \cdot)$ is $\mathbf{G}$-adapted.

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Theorem

Let $k \in \{0,2\}$ and $t \in [0,T]$. Assume that each party applies the bipartite ISDA CCR free close-out rule. If both, the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \leq \tau^*\}$, we have

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\right] = M_t(k) - \mathit{TBVA}_t(k),$$

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where the stopped process $D(0,\cdot\wedge\tau^*)\,\mathit{TBVA}^{\tau^*}(k,2-k)$ is a $\mathbf{G}$-martingale under $\mathbb{Q}$ (seen from the viewpoint of party k), satisfying $\mathit{TBVA}^{\tau^*}(k,2-k) = -\mathit{TBVA}^{\tau^*}(2-k,k)$.

Market price of total bilateral CCR II

Theorem (ctd.)

Moreover,

$$\mathit{TBVA}_{\tau^*}(k, 2-k) = B_{\tau^*}(k, 2-k),$$

where $B_{\cdot}(k, 2-k)$ *is a* $\mathbf{G}$*-adapted stochastic process, satisfying the ISDA CCR free restrictions and* $B_{\cdot}(k, 2-k) = -B_{\cdot}(2-k, k)$*.*

Market price of total bilateral CCR II

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In particular, we have $\mathbb{E}^{\mathbb{Q}}\big[\widehat{\Pi}_k^{(t,T]}|\mathcal{G}_t\big] = -\mathbb{E}^{\mathbb{Q}}\big[\widehat{\Pi}_{2-k}^{(t,T]}|\mathcal{G}_t\big]$, implying that both parties 0 and 2 would then agree on the "$\mathbb{Q}$-market price of total bilateral CCR".

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In fact, we not only have an existence result. The listed properties already lead to the following "uniqueness" result:

Market price of total bilateral CCR III

Theorem

Let $k \in \{0, 2\}$ *and* $t \in [0, T]$*.*

Market price of total bilateral CCR III

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Let $k \in \{0, 2\}$ and $t \in [0, T]$. Consider
$Z_t(k) := M_t(k) - \Delta_t(k, 2-k)$*, where (as usual in this talk)*
$M_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\Pi_k^{(t,T]} | \mathcal{G}_t\right]$ *denotes the CCR free mark-to-market value of the portfolio to party k.*

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- $\Delta_{\tau^*}(0, 2) = -\Delta_{\tau^*}(2, 0)$ *for all* $s \in [0, T]$;

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- $\Delta_{\tau^*}(0, 2) = -\Delta_{\tau^*}(2, 0)$ *for all* $s \in [0, T]$;
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- $\Delta_{\tau^*}(0,2) = -\Delta_{\tau^*}(2,0)$ *for all $s \in [0,T]$;*
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- $D(0,\cdot)$ *is $\mathbf{G}$-adapted,*
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If the No-Arbitrage Principle and the MCP is satisfied, and if the basic accounting rule holds, then on $\{t \leq \tau^*\}$, we have $\Delta_t(k, 2-k) = TBVA_t(k, 2-k)$ and $Z_t(k) = \mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right]$.

Basel III and unilateral CVA

The confusion continues since:

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Special Case (**Basel III** $\rightsquigarrow$ only one party defaults!)

Fix $k \in \{0, 2\}$. Assume that in addition $\tau_k = +\infty$ (i. e., no default of party k). Then $A^-_{2-k} = \{\tau_{2-k} \leq T\}$, $A^-_k = \emptyset$ and $A_{sim} = \emptyset$. Consequently, $TDVA_t(k, 2-k) = 0$,

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_k^{(t,T]} | \mathcal{G}_t\right] = M_t(k) - TCVA_t(k \,|\, 2-k)\,,$$

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and

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_{2-k}^{(t,T]} | \mathcal{G}_t\right] = M_t(2-k) + TDVA_t(2-k, k).$$

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and

$$\mathbb{E}^{\mathbb{Q}}\left[\widehat{\Pi}_{2-k}^{(t,T]} | \mathcal{G}_t\right] = M_t(2-k) + TDVA_t(2-k, k).$$

Hence, if party k *were the investor, and if* $\tau_k = +\infty$ *the unilateral CVA* $UCVA_t(k \,|\, 2-k) := TCVA_t(k \,|\, 2-k)$ *would have to be paid by party* $2-k$ *to the default free party* k *at* t *to cover a potential default of party* $2-k$ *after* t.

Excerpt from Basel III (ACVA, Para 98)

A. Banks with IMM approval and Specific Interest Rate Risk VaR model[34] approval for bonds: Advanced CVA risk capital charge

98. Banks with IMM approval for counterparty credit risk and approval to use the market risk internal models approach for the specific interest-rate risk of bonds must calculate this additional capital charge by modelling the impact of changes in the counterparties' credit spreads on the CVAs of all OTC derivative counterparties, together with eligible CVA hedges according to new paragraphs 102 and 103, using the bank's VaR model for bonds. This VaR model is restricted to changes in the counterparties' credit spreads and does not model the sensitivity of CVA to changes in other market factors, such as changes in the value of the reference asset, commodity, currency or interest rate of a derivative. Regardless of the accounting valuation method a bank uses for determining CVA, the CVA capital charge calculation must be based on the following formula for the CVA of each counterparty:

$$CVA = (LGD_{MKT}) \cdot \sum_{i=1}^{T} Max\left(0; exp\left(-\frac{s_{i-1} \cdot t_{i-1}}{LGD_{MKT}}\right) - exp\left(-\frac{s_i \cdot t_i}{LGD_{MKT}}\right)\right) \cdot \left(\frac{EE_{i-1} \cdot D_{i-1} + EE_i \cdot D_i}{2}\right)$$

Where

- t_i is the time of the i-th revaluation time bucket, starting from t_0=0.
- t_T is the longest contractual maturity across the netting sets with the counterparty.

CVA risk in Basel III: Flaws I

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An analysis of “CVA volatility risk” and its capitalisation should particularly treat the following serious flaws:

(i) CVA risk (and hedges) extend far beyond the risk of credit spread changes. It includes all risk factors that drive the underlying counterparty exposures as well as dependent interactions between counterparty exposures and the credit spreads of the counterparties (and their underyings). By solely focusing on credit spreads, the Basel III UCVA VaR and stressed VaR measures in its advanced approach for determining a CVA risk charge do not reflect the real risks that drive the P&L and earnings of institutes. Moreover, banks typically hedge these non-credit-spread risk factors. The Basel III capital calculation does not include these hedges.

CVA risk in Basel III: Flaws II

(ii) The non-negligible and non-trivial problem of a more realistic inclusion of WWR should be analysed deeply. In particular, the “alpha” multiplier $1.2 \leq \alpha$ should be revisited, and any unrealistic independence assumption should be strongly avoided.

(iii) Credit and market risks in UCVA are not different from the same risks, embedded in many other trading positions such as corporate bonds, CDSs, or equity derivatives. CVA risk can be seen as just another source of market risk. Consequently, it should be managed within the trading book. Basel III requires that the CVA risk charge is calculated on a stand alone basis, separated from the trading book. This seems to be an artificial segregation. A suitable approach would be to include UCVA and all of its hedges into the trading book capital calculation.

CVA risk in Basel III: Flaws III

(iv) Basel III considers unilateral CVA only. More precisely, the regulatory calculation of the ACVA is based on UCVA_0 – as opposed to the calculations of CVA in FAS 157 respectively IAS 39! Latter explicitly include the $(\mathrm{U})\mathrm{DVA}_0$. Hence, there exists a non-trivial mismatch between regulation and accounting! Moreover, as we have seen a thorough and appropriate treatment of a market price of (bilateral) CCR leads to TBVA_0 and not to UCVA_0. Consequently, further research is necessary. There is work in progress such as e.g. the running “Fundamental Review of the Trading Book” or running projects in the RTF subgroup of the BCBS – hopefully leading to necessary improvements of Basel III.

Structure of $\text{TCVA}_t(k \,|\, 2-k)$

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$$\boxed{\text{TCVA}_k(t, T, \text{LGD}_{2-k}, \tau_k, \tau_{2-k}, D(t,\tau^*), M_k(\tau^*))} \quad !$$

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To cover dynamically changing stochastic dependence between all embedded risk factors, a truly dynamic copula model has to be constructed ($\rightsquigarrow$ Bielecki, Crépey, Frey, Jeanblanc and many more).

Further important topics (not discussed here)

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- Risk mitigants such as collateral and margins;
- Re-hypothecation of collateral and funding;
- Margin period of risk;
- TBVA and clearing through a CCP (a CCP could also default) $\rightsquigarrow$ systemic risk?!

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Are there any questions, comments or remarks?