European options sensitivity with respect to the correlation for multidimensional Heston models

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Plan

Plan

From B&S to Heston

- The monotony for the B&S model
- Building the multi-asset Heston model
- Generalization problems

For short maturities

- Theoretical results
- Simulation results

For small values of η_i, ρ_i and $\sqrt{1-\rho_i^2}$

- Theoretical results
- Simulation results
- Studying prices

Final discussion

2D B&S model

Only ρ is unknown

$$dS^1_t = S^1_t \sigma_1 dW^1_t, \quad S^1_0 = x^1_0,$$
$$dS^2_t = S^2_t \sigma_2 (\rho dW^1_t + \sqrt{1-\rho^2} dW^2_t), \quad S^2_0 = x^2_0.$$

Using Itô calculus and B&S PDE : $F(t,s) = E_{t,s}(f(S_T))$

$$E(F(T,\overline{S}_T)) - F(0,S_0) = \Delta\rho E\left\{\int_0^T \sigma_1\sigma_2 \frac{\partial^2 F}{\partial x_1 \partial x_2}(t,\overline{S}_t)\overline{S}^1_t\overline{S}^2_t dt\right\}, \quad (1)$$

with $\Delta\rho = \overline{\rho} - \rho$.

The payoff $f(s) = (a_1 s_1 + a_2 s_2 \pm K)_+$

$$\begin{aligned}\frac{\partial^2 F}{\partial x_1 \partial x_2}(t,x) &= a_1 a_2 E_{t,x}\left(\frac{S^1_T}{x_1}\frac{S^2_T}{x_2}\varepsilon(a_1 S^1_T + a_2 S^2_T \pm K)\right)\\ &= -\frac{a_2}{x_1}\int_{\mathbb{R}^2_+} 1_{a_1 x_1 v_1 + a_2 x_2 v_2 \geq \pm K} \partial_{v_1}\left[v_1 v_2 g(v_1,v_2)\right] dv_1 dv_2\\ &= -\frac{a_1}{x_2}\int_{\mathbb{R}^2_+} 1_{a_1 x_1 v_1 + a_2 x_2 v_2 \geq \pm K} \partial_{v_2}\left[v_1 v_2 g(v_1,v_2)\right] dv_1 dv_2.\end{aligned}$$

Uniqueness of ρ : **implied correlation**.

Only ρ is unknown

$$S_T^1 = x_1 \exp\left(\int_0^T \sqrt{\nu_s^1} dW_s^1 - \frac{1}{2}\int_0^T \nu_s^1 ds\right). \tag{2}$$

$$S_T^2 = x_2 \exp\left(\int_0^T \sqrt{\nu_s^2}\left(\rho dW_s^1 + \sqrt{1-\rho^2} dW_s^2\right) - \frac{1}{2}\int_0^T \nu_s^2 ds\right). \tag{3}$$

$$\begin{aligned} \nu_T^1 &= y_1 + \kappa_1 \textstyle\int_0^T (\theta_1 - \nu_s^1) ds + \eta_1 \int_0^T \sqrt{\nu_s^1} dB_s^1, \\ B_s^1 &= \rho_1 W_s^1 + \sqrt{1-\rho_1^2}\widetilde{W}_s^1. \end{aligned} \tag{4}$$

$$\begin{aligned} \nu_T^2 &= y_2 + \kappa_2 \textstyle\int_0^T (\theta_2 - \nu_s^2) ds + \eta_2 \int_0^T \sqrt{\nu_s^2} dB_s^2, \\ B_s^2 &= \rho_2 \left(\rho W_s^1 + \sqrt{1-\rho^2} W_s^2\right) + \sqrt{1-\rho_2^2}\widetilde{W}_s^2. \end{aligned} \tag{5}$$

Two essential assumptions

$$\textbf{(A1)} \quad 4\kappa_i\theta_i > \eta_i^2,$$

and

$$\textbf{(A2)} \quad |\rho| < 1, \quad |\rho_1| < 1, \quad |\rho_2| < 1.$$

Proposition

If the assumptions **(A1)** & **(A2)** are fulfilled and f is convex then

$$\frac{E(f(\overline{S}_T)) - E(f(S_T))}{\Delta\rho} = E\left\{\int_0^T \overline{S}_t^1 \overline{S}_t^2 \sqrt{\overline{\nu}_t^1 \overline{\nu}_t^2} E_{t,\overline{S}_t,\overline{\nu}_t}\left[\partial^2_{s_1,s_2} f(S_T) \frac{S_T^1}{S_t^1} \frac{S_T^2}{S_t^2} \alpha^1_{t,T} \alpha^2_{t,T}\right] dt\right\} \quad (6)$$

with $\Delta\rho = \overline{\rho} - \rho$

and
$$\begin{cases} \alpha^1_{t,T} = 1 + \eta_1\rho_1 \left(\int_t^T \frac{\dot{\nu}_s^1}{2\sqrt{\nu_s^1}} dW_s^1 - \frac{1}{2}\int_t^T \dot{\nu}_s^1 ds\right), \\ \alpha^2_{t,T} = 1 + \eta_2\rho_2 \left(\int_t^T \frac{\dot{\nu}_s^2}{2\sqrt{\nu_s^2}} \left(\rho dW_s^1 + \sqrt{1-\rho^2} dW_s^2\right) - \frac{1}{2}\int_t^T \dot{\nu}_s^2 ds\right) \end{cases}$$

where $\{\dot{\nu}_s^i\}_{t \le s \le T}^{i=1,2}$ are the flow derivatives.

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The flow regularity :

$y \mapsto \nu_s(y) = y + \kappa \int_t^s (\theta - \nu_u(y))du + \eta \int_t^s \sqrt{\nu_u(y)} dB_u$

Some preliminary computations

Let $\tau(y) = \inf\{s > 0 : \nu_s(y) = 0\}$, for $s < \tau(y)$

$$\dot{\nu}_s = \frac{\sqrt{\nu_s}}{\sqrt{y}} \exp\left(-\frac{\kappa s}{2} - \frac{1}{2}\left(\kappa\theta - \frac{\eta^2}{4}\right)\int_0^s \frac{dr}{\nu_r}\right). \tag{7}$$

Proposition (based on Vostrikova 09)

When **(A1)**: $4\kappa\theta > \eta^2$ is fulfilled, there exists a modification $\widetilde{\nu}$ of ν such that $(0,\infty) \ni y \mapsto \widetilde{\nu}_s$ is $\mathcal{C}^1$ in probability sense. Moreover, $\partial_y \widetilde{\nu}$ coincides with $\dot{\nu} := \partial_y \nu$ on $[0, \tau(y)[$ and the former derivative vanishes on $[\tau(y), \infty[$.

ν^2 regularity according to ρ, where :

$$\nu_T^2 = y_2 + \kappa_2 \int_0^T (\theta_2 - \nu_s^2) ds + \eta_2 \int_0^T \sqrt{\nu_s^2} dB_s^2,$$

$$B_s^2 = \rho_2 \left(\rho W_s^1 + \sqrt{1-\rho^2} W_s^2 \right) + \sqrt{1-\rho_2^2} \widetilde{W}_s^2.$$

Theorem ν^2 is continuous with respect to $\rho \in (-1,1)$ and if **(A0)**: $2\kappa_2\theta_2 \geq \eta_2^2$ is fulfilled, ν^2 is differentiable and its derivative satisfies the SDE

$$\begin{aligned} \partial_\rho \nu_s^2 &= -\kappa_2 \int_0^s \partial_\rho \nu_u^2 du + \eta_2 \int_0^s \frac{\partial_\rho \nu_u^2}{2\sqrt{\nu_u^2}} dB_u^2 \\ &+ \eta_2 \rho_2 \int_0^s \sqrt{\nu_u^2} \left(dW_u^1 - \frac{\rho}{\sqrt{1-\rho^2}} dW_u^2 \right) \end{aligned} , \quad \partial_\rho \nu_0^2 = 0 \qquad (8)$$

The SDE (8) can be solved by a variation of constants method, to obtain

$$\partial_\rho \nu_s^2 = \dot{\nu}_s^2 \left(\eta_2 \rho_2 \int_0^s \frac{\sqrt{\nu_u^2}}{\dot{\nu}_u^2} \left(dW_u^1 - \frac{\rho}{\sqrt{1-\rho^2}} dW_u^2 \right) \right). \qquad (9)$$

The payoff $f(s) = (a_1 s_1 - a_2 s_2)_+$, $(a_1, a_2) \in (\mathbb{R}_+^*)^2$

Assumption (A3)

$$\exists C \in \mathbb{R}^* \quad \textit{such that} \quad \ln\left(\frac{a_2 x_2}{a_1 x_1}\right) = C\sqrt{T} + o(\sqrt{T})$$

with $x_1 = S_0^1$ and $x_2 = S_0^2$.

Theorem Under the assumptions **(A0)**, **(A2)** and **(A3)**. For short maturities, the price derivative with respect to ρ can be asymptotically approximated by

$$\frac{\partial}{\partial \rho} E\left((a_1 S_T^1 - a_2 S_T^2)_+\right) \underset{T \sim 0}{=} -a_2 x_2 \sqrt{\frac{T \nu_0^1 \nu_0^2}{2\pi\lambda}} \exp\left(-\frac{1}{2}\left[\frac{C}{\sqrt{\lambda}}\right]^2\right) + o(\sqrt{T})$$

with $\lambda = \nu_0^1 + \nu_0^2 - 2\rho\sqrt{\nu_0^1 \nu_0^2}$ and the constant C comes from **(A3)**.

Under the assumptions (A1), (A2) and (A3)

$(a_1 s_1 - a_2 s_2)_+$, $(a_1, a_2) \in (\mathbb{R}_+^*)^2$

The derivative of the price according to ρ, for short maturities, is given by

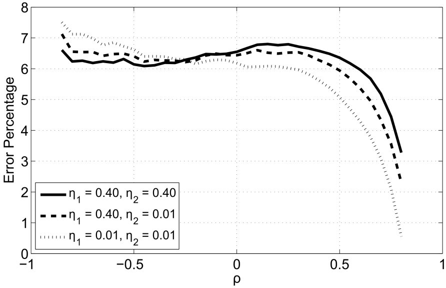
$$-a_2 x_2 \sqrt{\frac{T\nu_0^1 \nu_0^2}{2\pi\lambda}} \exp\left(-\frac{1}{2}\left[\frac{\ln\left(\frac{a_2 x_2}{a_1 x_1}\right)}{\sqrt{T\lambda}} + \frac{\sqrt{T\lambda}}{2} \right]^2 \right), \quad \lambda = \nu_0^1 + \nu_0^2 - 2\rho\sqrt{\nu_0^1 \nu_0^2}. \quad (10)$$

$K \neq 0$, $(a_1 s_1 - a_2 s_2 + K)_+$, $(a_1, a_2) \in (\mathbb{R}_+^*)^2$

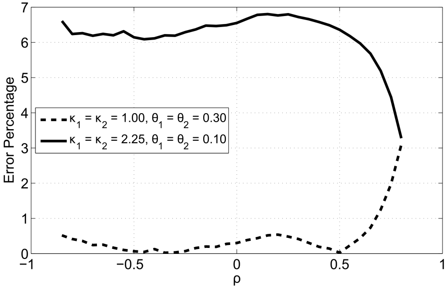
Denoting $\widetilde{x}_1 = x_1 + \frac{K}{a_1}$ and $\widetilde{x}_2 = x_2 - \frac{K}{a_2}$, we approximate the derivative according to ρ by

$$\begin{aligned} &\frac{-a_2 \widetilde{x}_2}{2} \sqrt{\frac{T\nu_0^1 \nu_0^2}{2\pi\lambda}} \exp\left(-\frac{1}{2}\left[\frac{\ln(a_2 \widetilde{x}_2 / a_1 x_1)}{\sqrt{T\lambda}} + \frac{\sqrt{T\lambda}}{2} \right]^2 \right) \\ &+ \frac{-a_2 x_2}{2} \sqrt{\frac{T\nu_0^1 \nu_0^2}{2\pi\lambda}} \exp\left(-\frac{1}{2}\left[\frac{\ln(a_2 x_2 / a_1 \widetilde{x}_1)}{\sqrt{T\lambda}} + \frac{\sqrt{T\lambda}}{2} \right]^2 \right). \end{aligned} \quad (11)$$

Varying η, κ and θ, $T = 0.2$



(a)

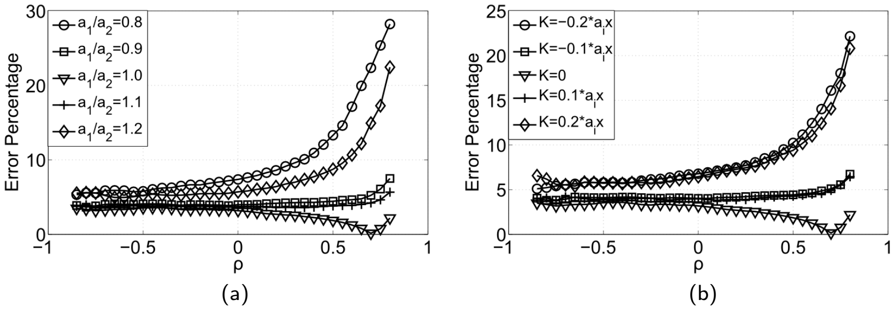


(b)

(a) $\kappa_1 = \kappa_2 = 2.25$, $\theta_1 = \theta_2 = 0.1$, $\nu_0^1 = \nu_0^2 = 0.5$, $a_1 x_1 = a_2 x_2 = 100$ *and* $K = 0$,

(b) $\eta_1 = \eta_2 = 0.4$, $\nu_0^1 = \nu_0^2 = 0.5$, $a_1 x_1 = a_2 x_2 = 100$ *and* $K = 0$.

Maturity $T = 0.1$, $f(s) = (a_1 s_1 + a_2 s_2 \pm K)_+$



(a) $\kappa_1 = \kappa_2 = 3.0$, $\theta_1 = \theta_2 = 0.1$, $\eta_1 = \eta_2 = 0.1$, $\nu_0^1 = \nu_0^2 = 0.5$, $x_1 = x_2 = 100$ *and* $K = 0$,

(b) $\kappa_1 = \kappa_2 = 3.0$, $\theta_1 = \theta_2 = 0.1$, $\eta_1 = \eta_2 = 0.1$, $\nu_0^1 = \nu_0^2 = 0.5$ *and* $a_1 x_1 = a_2 x_2 = 100$.

Good approximation when

- $T \leq 0.1$ and the payoff is less than 20% ITM or OTM, the approximation (11) can be accepted when $\theta_i/\nu_0^i > 1/4$ and $\kappa_i < 3$.
- $T \leq 0.2$ and the payoff is less than 20% ITM or OTM, the approximation (11) can be accepted when $\theta_i/\nu_0^i > 1/4$ and $\kappa_i < 1.5$.

Plan

Theorem Under the assumptions **(A1)** and **(A2)** and $f(s) = (a_1 s_1 + a_2 s_2 \pm K)_+$ with $a_1, a_2 \in (\mathbb{R}^*)^2$ then $E(f(S_T))$ is differentiable according to ρ with

$$\partial_\rho E(f(S_T)) = E\left\{\int_0^T S_t^1 S_t^2 \sqrt{\nu_t^1 \nu_t^2} E_{t,S_t,\nu_t}\left[\partial^2_{s_1,s_2} f(S_T)\dot{S}_T^1 \dot{S}_T^2 \alpha^1_{t,T}\alpha^2_{t,T}\right]dt\right\} \quad (12)$$

and

$$\alpha^1_{t,T} = 1 + \eta_1\rho_1\left(\int_t^T \frac{\dot{\nu}_s^1}{2\sqrt{\nu_s^1}}dW_s^1 - \frac{1}{2}\int_t^T \dot{\nu}_s^1 ds\right),$$

$$\alpha^2_{t,T} = 1 + \eta_2\rho_2\left(\int_t^T \frac{\dot{\nu}_s^2}{2\sqrt{\nu_s^2}}\left(\rho dW_s^1 + \sqrt{1-\rho^2}dW_s^2\right) - \frac{1}{2}\int_t^T \dot{\nu}_s^2 ds\right),$$

Thus, if :

c1) $\{\eta_i\rho_i\}_{i=1,2} = 0$ or

c2) $\eta_1\rho_1 = 0$, $\eta_2\sqrt{1-\rho_2^2} = 0$ and $2\kappa_2 - \eta_2\rho_2 > 0$ or

c3) $\eta_2\rho_2 = 0$, $\eta_1\sqrt{1-\rho_1^2} = 0$ and $2\kappa_1 - \eta_1\rho_1 > 0$,

then the price is monotonous with respect to ρ : Increases if $a_1 a_2 > 0$, decreases if $a_1 a_2 < 0$.

The considered payoff

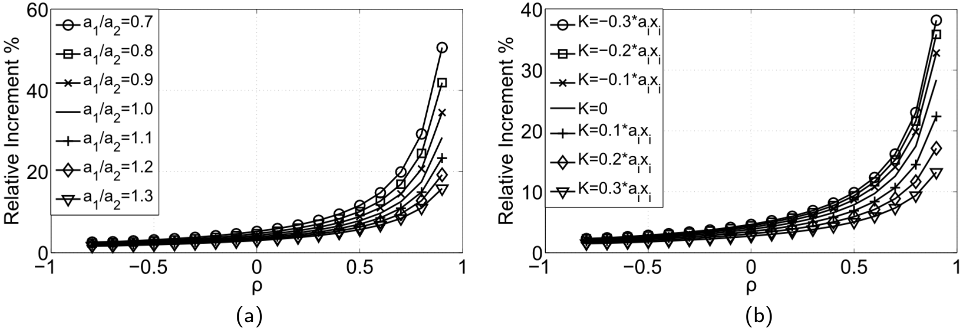
$$f(s) = (a_1 s_1 + a_2 s_2 \pm K)_+$$

Relative Increment %

$$100 * \frac{F(\rho_i) - F(\rho_{i+1})}{F(\rho_i)}, \tag{13}$$

$F(\rho_i)$ is the price obtained by Monte Carlo.

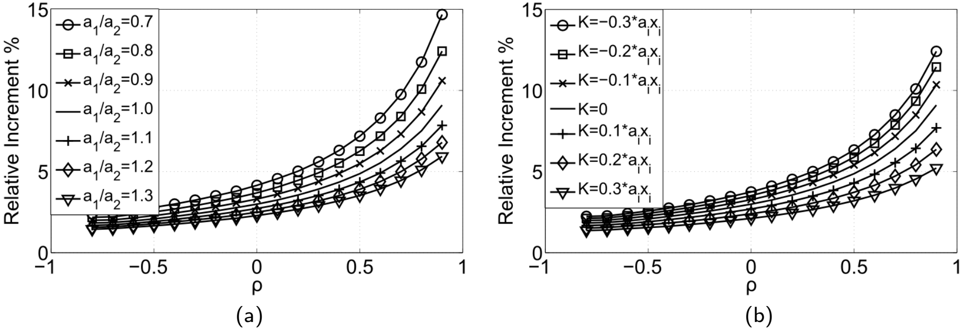
Maturity $T = 5$, η very small



(a) $\eta_1 = \eta_2 = 0.1,\ \kappa_1 = \kappa_2 = 3.0,\ \theta_1 = \theta_2 = 0.2,\ \nu_0^1 = \nu_0^2 = 0.4,\ x_1 = x_2 = 100$ *and* $K = 0$,

(b) $\eta_1 = \eta_2 = 0.1,\ \kappa_1 = \kappa_2 = 3.0,\ \theta_1 = \theta_2 = 0.2,\ \nu_0^1 = \nu_0^2 = 0.4$ *and* $a_1 x_1 = a_2 x_2 = 100$

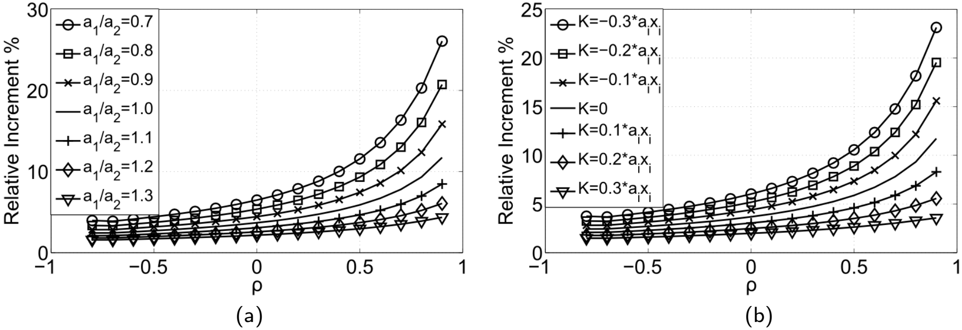
Maturity $T = 5$, η large



(a) $\eta_1 = \eta_2 = 1.5$, $\kappa_1 = \kappa_2 = 3.0$, $\theta_1 = \theta_2 = 0.2$, $\nu_0^1 = \nu_0^2 = 0.4$, $x_1 = x_2 = 100$ *and* $K = 0$,

(b) $\eta_1 = \eta_2 = 1.5$, $\kappa_1 = \kappa_2 = 3.0$, $\theta_1 = \theta_2 = 0.2$, $\nu_0^1 = \nu_0^2 = 0.4$ *and* $a_1 x_1 = a_2 x_2 = 100$.

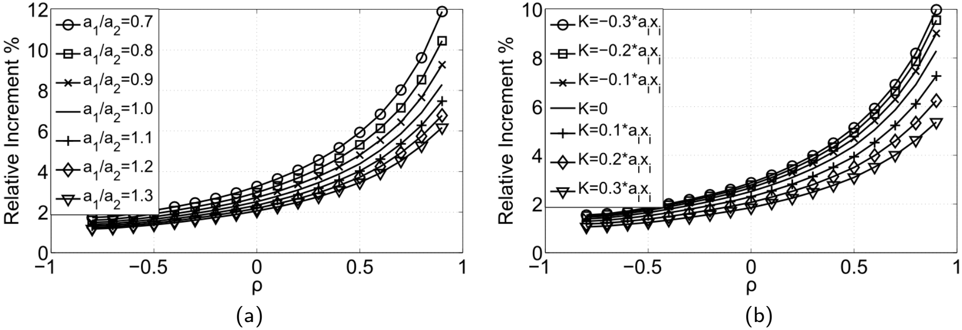
Maturity $T = 1$, η large



(a) $\eta_1 = \eta_2 = 1.5$, $\kappa_1 = \kappa_2 = 3.0$, $\theta_1 = \theta_2 = 0.2$, $\nu_0^1 = \nu_0^2 = 0.4$, $x_1 = x_2 = 100$ *and* $K = 0$,

(b) $\eta_1 = \eta_2 = 1.5$, $\kappa_1 = \kappa_2 = 3.0$, $\theta_1 = \theta_2 = 0.2$, $\nu_0^1 = \nu_0^2 = 0.4$ *and* $a_1 x_1 = a_2 x_2 = 100$.

Maturity $T = 10$, η large



(a) $\eta_1 = \eta_2 = 1.5$, $\kappa_1 = \kappa_2 = 3.0$, $\theta_1 = \theta_2 = 0.2$, $\nu_0^1 = \nu_0^2 = 0.4$, $x_1 = x_2 = 100$ *and* $K = 0$,

(b) $\eta_1 = \eta_2 = 1.5$, $\kappa_1 = \kappa_2 = 3.0$, $\theta_1 = \theta_2 = 0.2$, $\nu_0^1 = \nu_0^2 = 0.4$ *and* $a_1 x_1 = a_2 x_2 = 100$.

Plan

Results to explore

- Asymptotic behavior for large T
- Generalization to other SDEs

Using GPUs

- Studying the asymptotic error within seconds
- Simulating 2^{22} trajectories with $\delta t = 0.01$ and $T = 10$ within minutes

Calibration Dichotomy by Monte Carlo algorithm is implemented in Premia 15

Thank you

Questions ?