Asymptotics of Forward Implied Volatility

Patrick Roome

Department of Mathematics, Imperial College London

Joint work with Antoine Jacquier (Imperial College). Preprint available on ArXiv or SSRN.

Advances in Mathematics of Finance Conference
Warsaw, June 2013

Recap on Spot Implied Volatility

- Consider an asset price process $\left(\mathrm{e}^{X_t}\right)_{t\geq 0}$ with $X_0 = 0$, paying no dividend and assume that interest rates are zero.
- In the Black-Scholes-Merton (BSM) model the no-arbitrage price of a call option at time zero is given by: $C_{\mathrm{BS}}(\tau, k, \sigma) := \mathbb{E}\left(\mathrm{e}^{X_\tau} - \mathrm{e}^k\right)_+ = \mathcal{N}\left(d_+\right) - \mathrm{e}^k \mathcal{N}\left(d_-\right)$, with $d_\pm := -\frac{k}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}$.
- For any given market $C(\tau, k)$ or model price $C(\tau, k) = \mathbb{E}\left(\mathrm{e}^{X_\tau} - \mathrm{e}^k\right)_+$ of a call option at strike e^k and maturity τ we define the spot implied volatility $\sigma_\tau(k)$ as the unique solution to the equation $C(\tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_\tau(k))$.
- Spot implied volatility is the quoting mechanism used in option markets and provides a useful metric to compare options with different strikes and maturities.

Of particular importance is an understanding of the asymptotics and behaviour of the spot implied volatility smile $\sigma_\tau(k)$ in different models. (eg. for calibration purposes)

Of particular importance is an understanding of the asymptotics and behaviour of the spot implied volatility smile $\sigma_\tau(k)$ in different models. (eg. for calibration purposes)

This is a well studied problem using a diverse range of mathematical techniques:

- Berestycki-Busca-Florent (2004): small-maturity asymptotics using PDE methods for continuous time diffusions.
- Henry-Labordère (2009): small-maturity asymptotics using differential geometry.
- Forde et al.(2012), Jacquier et al.(2012): small and large-maturity asymptotics using large deviations and saddlepoint methods.
- Lee (2003), Benaim-Friz (2009): wing asymptotics ($k \to \pm\infty$).
- Fouque et al.(2000, 2011): perturbation techniques in order to study slow and fast mean-reverting stochastic volatility models.
- Deuschel et al.(2012), Osajima (2007), Takahashi (2009) : small noise expansions using Laplace method on Wiener space and Watanabe expansions.

Forward Implied Volatility

- For any $t, \tau > 0$ and $k \in \mathbb{R}$, we define a forward-start option with forward-start date t, maturity τ and strike e^k as a European option with payoff $\left(\mathrm{e}^{X_{t+\tau}}/\mathrm{e}^{X_t} - \mathrm{e}^k\right)^+$.
- In the BSM model its value is simply worth $C_{\mathrm{BS}}(\tau, k, \sigma)$.
- For a given market or model price $C(t, \tau, k)$ of the option at strike e^k, forward-start date t and maturity τ we define the forward implied volatility smile $\sigma_{t,\tau}(k)$ as the unique solution to $C(t, \tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_{t,\tau}(k))$.
- The forward smile is a market defined quantity and naturally extends the notion of the spot implied volatility smile. i.e. when $t = 0$ we recover the spot implied volatility smile.

There are two primary reasons motivating our research into forward smile asymptotics:

There are two primary reasons motivating our research into forward smile asymptotics:

Calibration:

- Forward-start options serve as natural hedging instruments for many exotic securities and it is therefore important for a model to be able to calibrate to liquid forward smiles.

There are two primary reasons motivating our research into forward smile asymptotics:

Calibration:

- Forward-start options serve as natural hedging instruments for many exotic securities and it is therefore important for a model to be able to calibrate to liquid forward smiles.

Model Risk:

- Many models can calibrate to implied volatility smiles (static information) with the same degree of precision and produce radically different prices and risk sensitivities for exotic securities. This can usually be traced back to a complex dependence on the model generated dynamics of implied volatility smiles.
- One metric that can be used to understand the dynamics of implied volatility smiles (Bergomi(2004) calls it a 'global measure' of the dynamics of implied volatilities) is to use the forward smile defined above.
- This allows us to analytically compare models and assess the realism of model generated forward smiles.

Despite the significant research on implied volatility asymptotics, there are virtually no results on the asymptotics of the forward smile:

- Glasserman and Wu (2011) introduced different notions of forward volatilities to assess their predictive values in determining future option prices and future implied volatility,
- Keller-Ressel (2011) studies a very specific type of asymptotic (when the forward-start date becomes large)
- Empirical results have been carried out by practitioners in Bergomi(2004), Bühler(2002) and Gatheral(2006).

General Results

We let (X_ε) be a stochastic process and first develop general option price asymptotics on (X_ε). We will then show how to specialise these general results to forward-start option asymptotics in various regimes. Denote the re-normalised moment generating function (mgf) by $\Lambda_\varepsilon(u) := \varepsilon \log \mathbb{E}\left[\exp\left(\frac{uX_\varepsilon}{\varepsilon}\right)\right]$, for all $u \in \mathcal{D}_\varepsilon \subseteq \mathbb{R}$.

We require the following critical assumptions (call them **Assumption OA**) on the behaviour of our mgf:

- **Expansion property:** For each $u \in \mathcal{D}_0$ we have $\Lambda_\varepsilon(u) = \sum_{i=0}^{2} \Lambda_i(u)\varepsilon^i + \mathcal{O}(\varepsilon^3)$ as ε tends to zero where we define $\mathcal{D}_0 := \lim_{\varepsilon \searrow 0} \mathcal{D}_\varepsilon$.
- **Differentiability:** For small enough ε the map $\Lambda_\varepsilon : \mathcal{D}_0^o \mapsto \mathbb{R}$ is infinitely differentiable where $\mathcal{D}_0^o$ denotes the interior of $\mathcal{D}_0$ in $\mathbb{R}$. This can be relaxed by a $\mathcal{C}^4(\mathcal{D}_0^o)$ condition.
- **Non-degenerate interior:** $0 \in \mathcal{D}_0^o$.
- Λ_0 is **strictly convex** and **essentially smooth** on $\mathcal{D}_0^o$.

Recall: A convex function $h : \mathbb{R} \supset \mathcal{D}_h \to (-\infty, \infty]$ is essentially smooth if $\lim_{n\to\infty} |h'(u_n)| = \infty$ for every sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_h^o$ that converges to a boundary point of $\mathcal{D}_h^o$.

General Option Price Asymptotics

Theorem (Jacquier-R, 2012)

Let (X_ε) satisfy Assumption OA and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying

$$f(\varepsilon)\varepsilon = c + \mathcal{O}(\varepsilon),$$

with constant $c \in \mathcal{D}_0^o \cap \mathbb{R}_+$ as ε tends to zero. Then the following expansion holds for all $k \in \mathbb{R} \backslash \{\Lambda_0'(0), \Lambda_0'(c)\}$ as $\varepsilon \searrow 0$:

$$\begin{aligned} A_c(k,\varepsilon) = \mathbb{E}\left[\left(\mathrm{e}^{X_\varepsilon f(\varepsilon)} - \mathrm{e}^{kf(\varepsilon)}\right)^+\right] \mathbb{1}_{\{k>\Lambda_0'(c)\}} + \mathbb{E}\left[\left(\mathrm{e}^{kf(\varepsilon)} - \mathrm{e}^{X_\varepsilon f(\varepsilon)}\right)^+\right] \mathbb{1}_{\{k<\Lambda_0'(0)\}} \\ - \mathbb{E}\left[\mathrm{e}^{X_\varepsilon f(\varepsilon)} \wedge \mathrm{e}^{kf(\varepsilon)}\right] \mathbb{1}_{\{\Lambda_0'(0)<k<\Lambda_0'(c)\}}, \end{aligned}$$

where $A_c(k,\varepsilon) := \mathrm{e}^{-\Lambda^*(k)/\varepsilon + kf(\varepsilon)} \alpha_0(k,\varepsilon,c)\left(1 + \alpha_1(k,c)\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\right)$ and $\Lambda^* : \mathbb{R} \to \mathbb{R}_+$ is the Fenchel-Legendre transform of Λ_0:

$$\Lambda^*(k) := \sup_{u \in \mathcal{D}_0} \{uk - \Lambda_0(u)\}, \qquad \text{for all } k \in \mathbb{R}.$$

Specialising to Forward-Start Options

- In this general framework we derived asymptotics as ε tends to zero for option prices with payoff given by $\left(\mathrm{e}^{X_\varepsilon f(\varepsilon)} - \mathrm{e}^{kf(\varepsilon)}\right)^+$.
- We can now specialise to forward-start options and define the forward price process $X_\tau^{(t)} := X_{t+\tau} - X_t$ for any $t \geq 0, \tau > 0$.
- If we define the process $(X_\varepsilon) := (X_{\varepsilon\tau}^{(\varepsilon t)})$ and set $f(\varepsilon) \equiv 1$ then we obtain asymptotics of forward-start options for small forward-start dates and maturities, which we call **diagonal small-maturity asymptotics**.
- If for a fixed $t \geq 0$ we define the process $(X_\varepsilon) := (\varepsilon X_{\tau/\varepsilon}^{(t)})$ and set $f(\varepsilon) = 1/\varepsilon$ we obtain **large-maturity asymptotics** of forward-start options. Note in this case that the strike now depends on ε.
- In general we can consider any regime and re-scaling of the forward price process that is financially meaningful and meets the assumptions above.
- These forward-start option asymptotics can then be converted into corresponding forward smile asymptotics in various regimes.

General Forward Smile Asymptotics I

Corollary (Diagonal Small-Maturity Forward Smile Asymptotic)

Suppose that $\left(X_{\varepsilon\tau}^{(\varepsilon t)}\right)_{\varepsilon>0}$ satisfies Assumption OA and $\Lambda_0'(0) = 0$. The following expansion then holds for all $k \in \mathbb{R}$ as ε tends to zero:

$$\sigma^2_{\varepsilon t, \varepsilon\tau}(k) = v_0(k,t,\tau) + v_1(k,t,\tau)\varepsilon + v_2(k,t,\tau)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right),$$

where $v_0(\cdot,t,\tau)$, $v_1(\cdot,t,\tau)$ and $v_2(\cdot,t,\tau)$ are continuous functions on $\mathbb{R}$.

Recall our assumption: $\Lambda_\varepsilon(u) = \sum_{i=0}^{2} \Lambda_i(u)\varepsilon^i + \mathcal{O}(\varepsilon^3)$, as ε tends to zero.

- The functions v_0, v_1, and v_2 depend on derivatives of the functions Λ_0, Λ_1 and Λ_2 evaluated at a strike dependent point $u^*(k)$ given as the solution to the equation $\Lambda_0'(u^*(k)) = k$.
- The strict convexity and essential smoothness assumption always ensures that there exists a unique solution to this equation for all $k \in \mathbb{R}$.
- In this way we have explicitly **related the expansion of the forward mgf to the expansion of the forward smile.**

General Forward Smile Asymptotics II

Corollary (Large-Maturity Forward Smile Asymptotic)

Suppose that $\left(\tau^{-1}X_\tau^{(t)}\right)_{\tau>0}$ satisfies Assumption OA with $\varepsilon = \tau^{-1}$ and $\Lambda_0(1) = 0$ with $1 \in \mathcal{D}_0^o$. The following expansion then holds for all $k \in \mathbb{R}$ as τ tends to infinity:

$$\sigma_{t,\tau}^2(k\tau) = v_0^\infty(k,t) + \frac{v_1^\infty(k,t)}{\tau} + \frac{v_2^\infty(k,t)}{\tau^2} + \mathcal{O}\left(\frac{1}{\tau^3}\right),$$

where $v_0^\infty(\cdot,t)$, $v_1^\infty(\cdot,t)$ and $v_2^\infty(\cdot,t)$ are continuous functions on $\mathbb{R}$.

- If our asset price process $\left(\mathrm{e}^{X_t}\right)_{t\geq 0}$ is a true martingale (remember interest rates are zero) then $\Lambda_0(1) = 0$.
- Similar remarks as on the previous slide apply to the large-maturity asymptotic.
- Note that by setting $t = 0$ we obtain spot implied volatility asymptotics for free in both corollaries.

Application: Heston

In Heston the (log) stock price process is the unique strong solution to the following SDEs:

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, && X_0 = 0,\\
\mathrm{d}V_t &= \kappa\left(\theta - V_t\right)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t, && V_0 = v > 0,\\
\mathrm{d}\left\langle W, Z\right\rangle_t &= \rho\mathrm{d}t,
\end{aligned}
$$

with $\kappa > 0$, $\xi > 0$, $\theta > 0$ and $|\rho| < 1$.

We first use our asymptotic results to try and understand **how the Heston forward smile is different to the Heston spot implied volatility smile.**

Heston Diagonal Small-Maturity

In the diagonal small-maturity case the Heston model satisfies the assumptions of the corollary. We use our asymptotic result to establish that the Heston forward convexity as ε tends to zero is given by

$$\partial_k^2 \sigma_{\varepsilon t, \varepsilon \tau}(0) = \partial_k^2 \sigma_{0, \varepsilon \tau}(0) + \xi^2 t / (4 \tau v^{3/2}) + \mathcal{O}(\varepsilon).$$

- At zeroth order in ε the wings of the forward smile increase to arbitrarily high levels with decreasing maturity.
- Quoting Bühler(2002) from an empirical investigation: "Heston implied forward volatility: Short skew becomes U-shaped - this is inconsistent with observations." This is exactly what we see with our asymptotic.
- We can go one step further and conjecture: For fixed $t > 0$ the Heston forward smile blows up to infinity (except ATM) as the maturity tends to zero. This has been proved in our latest paper.
- In fact this is why we considered diagonal small-maturity asymptotics and not the small-maturity asymptotic of $\sigma_{t,\tau}$ for fixed $t > 0$. In Heston (among other models) Assumption OA is not met in this small-maturity case but this degenerate behaviour does not occur in the diagonal small-maturity regime.

Heston Forward Smile Explosion

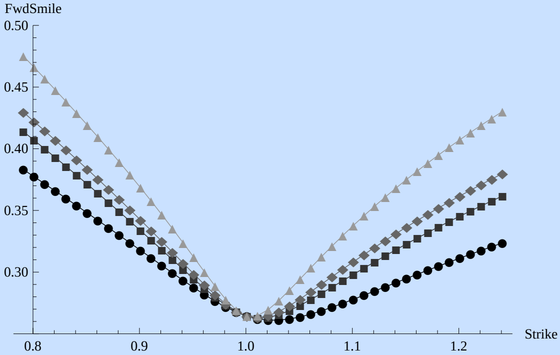


Figure: We plot forward smiles with forward-start date $t = 1/2$ and maturities $\tau = 1/6, 1/12, 1/16, 1/32$ given by circles, squares, diamonds and triangles respectively using the Heston parameters $v = 0.07, \theta = 0.07, \kappa = 1, \rho = -0.6, \xi = 0.5$ and the asymptotic.

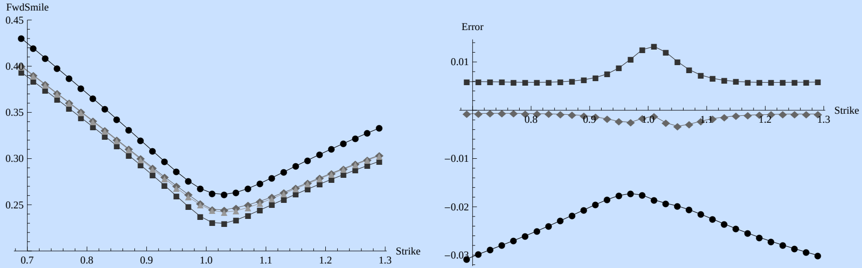
Heston Large-Maturity

In the Heston large-maturity forward smile case it is natural to conjecture that the limiting forward smile will be the same as the limiting large-maturity spot smile since as the maturity increases the forward smile may become more and more independent of the forward-start date t.

- In the Heston large-maturity case the name of the game is finding conditions on the parameters of the model such that the essential smoothness assumption is verified.
- This is most easily stated as a condition on the Heston correlation: If $\rho_- \leq \rho \leq \min(\rho_+, \kappa/\xi)$ then the large-maturity asymptotic result holds for Heston.
- Here $\rho_\pm$ are functions of the forward-start date and Heston model parameters. Also $-1 \leq \rho_- < 0$ and $0 < \min(\rho_+, \kappa/\xi) \leq 1$.
- Further the limiting forward smile under these conditions is the same as the limiting spot smile.
- So our intuition is only partly correct: The limiting large-maturity forward smile will be the same as the limiting large-maturity spot smile for correlations "close to zero"!

Heston Numerics

- We compare here the true Heston forward smile and the asymptotics developed in the paper.
- We calculate forward-start option prices using an inverse Fourier transform representation in Lee(2004) and a global adaptive Gauss-Kronrod quadrature scheme.
- We then compute the forward smile $\sigma_{t,\tau}$ and compare it to the zeroth, first and second order asymptotics.
- We do this for the Heston diagonal small-maturity asymptotic and the Heston large-maturity asymptotic.
- Results are in line with expectations and the higher the order of the asymptotic the closer we match the true forward smile.

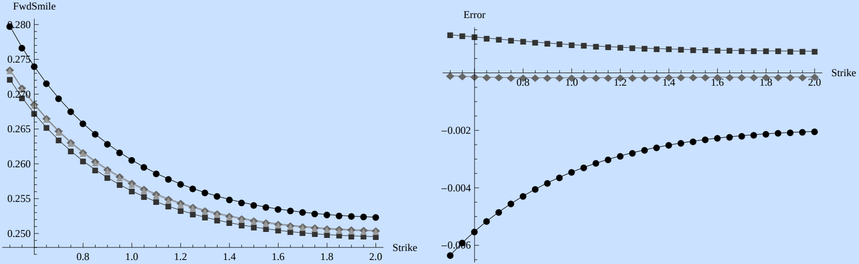


(a) Heston diagonal small-maturity vs Fourier inversion.

(b) Errors

Figure: In (a) circles, squares and diamonds represent the zeroth, first and second order asymptotics respectively and triangles represent the true forward smile using Fourier inversion. In (b) we plot the differences between the true forward smile and the asymptotic. We use $t = 1/2$ and $\tau = 1/12$ and the Heston parameters $v = 0.07$, $\theta = 0.07$, $\kappa = 1$, $\xi = 0.34$, $\rho = -0.8$.

Recall that in the large-maturity case we require $\rho_- \leq \rho \leq \min(\rho_+, \kappa/\xi)$. For the parameter choice in the figure below we have $\rho_- = -0.65$ and the condition is satisfied.



(a) Heston Large-Maturity vs Fourier Inversion.

(b) Errors

Figure: In (a) circles, squares and diamonds represent the zeroth, first and second order asymptotics respectively and triangles represent the true forward smile using Fourier inversion. In (b) we plot the differences between the true forward smile and the asymptotic. We use $t = 1$ and $\tau = 5$ and the Heston parameters $v = 0.07$, $\theta = 0.07$, $\kappa = 1.5$, $\xi = 0.34$, $\rho = -0.25$.

Conclusions

- We developed an expansion formula for option prices in a general framework.
- We showed how this formula could then be specialised to forward-start options in different financial regimes of interest. (eg. large and small-maturities)
- We then converted these options prices into expansions for the forward smile.

We then applied our results to the Heston model and made the following observations:

- For small-maturities the wings of the forward smile increase to arbitrarily high levels even for very negative correlations. This is very different to the small-maturity spot smile.
- If one believes that the small-maturity forward smile should be downward sloping (like the spot smile) Heston will have structural issues incorporating this effect.
- For large maturities and correlation "close to zero" the limiting forward smile is the same as the limiting spot smile.

We study Heston and other models in more detail in our paper.