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A General Approach for Stochastic Correlation using Hyperbolic Functions

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Outline

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2. Stochastic Model for Correlation
   - Building-up the Model
   - Stochastically correlated Brownian Motions
   - Model calibration
3. Example: Pricing Quanto
4. Conclusions

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4. Conclusions

The correlated Brownian Motions

Using the notation

$$dW_t^1 dW_t^2 = \rho dt,$$

the two BMs W_t^1 and W_t^2 are correlated with $\rho \in [-1, 1]$.

Linear Correlation

Correlation $\rho(X_1, X_2)$ between random variables X_1 and X_2 reads

$$\rho(X_1, X_2) := \frac{\text{cov}(X_1, X_2)}{\sqrt{\text{var}(X_1)\text{var}(X_2)}}$$

Example: Heston Model *[Heston 1993]*:

$$\begin{aligned} dS_t &= \mu S dt + \sqrt{\nu(t)} S dW_t^1 \\ d\nu(t) &= \kappa(\theta - \nu(t))dt + \sigma\sqrt{\nu(t)} dW_t^2 \end{aligned}$$

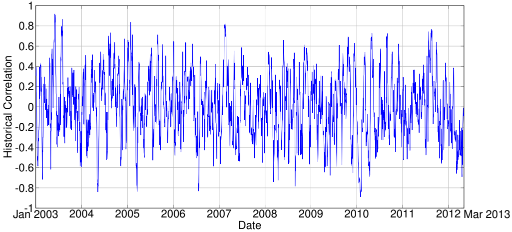
μ, κ, θ and σ are positive constants

The estimator of correlation

The correlation $\rho_T(X_1, X_2)$ for the time region T with observed values $\hat{X}_1(t)$ and $\hat{X}_2(t), t \in T$ can be estimated as

$$\rho_T(X_1, X_2) \approx \hat{\rho}_T = \frac{\sum_{t \in T}(\hat{X}_1(t) - \frac{1}{n_T}\sum_{t \in T}\hat{X}_1(t))(\hat{X}_2(t) - \frac{1}{n_T}\sum_{t \in T}\hat{X}_2(t))}{\sqrt{\sum_{t \in T}(\hat{X}_1(t) - \frac{1}{n_T}\sum_{t \in T}\hat{X}_1(t))^2 \sum_{t \in T}(\hat{X}_2(t) - \frac{1}{n_T}\sum_{t \in T}\hat{X}_2(t))^2}}$$

where n_T is the number of pairs $(\hat{X}_1(t), \hat{X}_2(t))$ with $t \in T$.

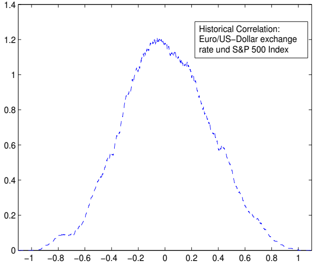


Historical Correlation between Euro/US-Dollar exchange Rate and S&P 500 *[yahoo.com]*

- Not constant over times
- As stochastic process?

Empirical Density Function

Bandwidth: $\frac{1}{20}$



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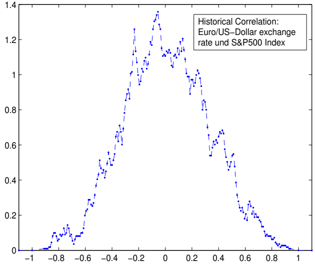


Empirical Density function

The correlation should

- only take values on $[-1, 1]$
- vary around a mean value
- the probability mass tends to zero in the boundary values

Outline

Model

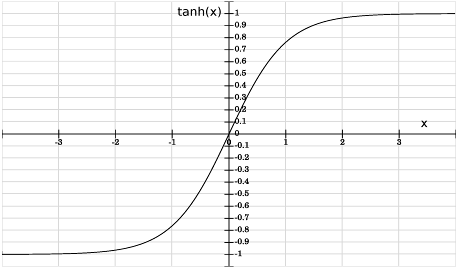
Based on the SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad t \geq 0,\ X_0 = x_0$$

we assume a **stochastic correlation**

$$\rho_t = \tanh(X_t)$$

where $\rho_0 = \tanh(x_0) \in (-1, 1)$.



Tangens Hyperbolicus

Stochastic Correlation Process

Applying *Itô's Lemma*

$$d\rho_t = d\tanh(X_t) = \frac{\partial \tanh(X_t)}{\partial t}dt + \frac{\partial \tanh(X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 \tanh(X_t)}{\partial x^2}(dX_t)^2$$

we obtain

$$d\rho_t = (1-\rho_t^2)\left((\tilde{a} - \rho_t\tilde{b}^2)dt + \tilde{b}dW_t\right), \quad t \geq 0$$

where $\rho_0 \in (-1,1), \tilde{a} = a(t, \text{artanh}(\rho_t))$ and $\tilde{b} = b(t, \text{artanh}(\rho_t))$

The generated correlation

- only takes value on $(-1,1)$ agreed with market observation
- can vary around a mean value
- approaches zero in the boundary values

Example

We choose the *Ornstein-Uhlenbeck process*

$$dX_t = \kappa(\mu - X_t)dt + \sigma dW_t \quad \kappa, \sigma > 0 \text{ and } X_0, \mu \in \mathbb{R}$$

Applying *Itô's Lemma* with $\rho_t = \tanh(X_t)$

$$d\rho_t = \frac{\partial \tanh(X_t)}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 \tanh(X_t)}{\partial x^2} \sigma^2 dt$$

gives **stochastic correlation process** as

$$d\rho_t = (1 - \rho_t^2)\left(\kappa(\mu - \text{artanh}(\rho_t)) - \rho_t \sigma^2\right) dt + (1 - \rho_t^2)\sigma dW_t,$$

where $t \geq 0, \rho_0 \in (-1, 1), \kappa, \sigma > 0$ and $\mu \in \mathbb{R}$.

For $t \to \infty$ the probability density function $f(\tilde{\rho})$ can be derived as

$$f(\tilde{\rho}) = \frac{1}{1 - \tilde{\rho}^2} \cdot \frac{\sqrt{\kappa}}{\sigma\sqrt{\pi}} \cdot \mathrm{e}^{-\frac{\kappa(\text{artanh}(\tilde{\rho}) - \mu)^2}{\sigma^2}}$$

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Stochastically correlated BMs

Based on two independent Brownian motions W_t^2 and W_t^3 we define

$$W_t^1 = \int_0^t \rho_s dW_s^2 + \int_0^t \sqrt{1-\rho_s^2} dW_s^3$$

which satisfies $W_0^1 = 0, \mathbb{E}\left[(W_t^1)^2\right] = t$ and $\mathbb{E}[W_t^1|\mathcal{F}_s] = W_s^1, t \geq s$.

By the stochastic correlation process ρ_t correlated this two Brownian motions W_t^1 and W_t^2 holds

$$\mathbb{E}\left[W_t^1 \cdot W_t^2\right] = \mathbb{E}\left[\int_0^t \rho_s ds\right]$$

which agrees for constant correlation ρ with

$$dW_t^1 \cdot dW_t^2 = \rho dt.$$

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The Fokker-Planck equation

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Recall the *stochastic correlation process*

$$d\rho_t = \underbrace{(1-\rho_t^2)(\tilde{a}-\rho_t\tilde{b}^2)}_{:=\hat{a}(t,\rho_t)}dt + \underbrace{(1-\rho_t^2)\tilde{b}}_{:=\hat{b}(t,\rho_t)}dW_t, \quad t \geq 0$$

where $\rho_0 \in (-1,1)$. Assuming it possesses a transition density $p(t,\tilde{\rho}|\rho_0)$ which satisfies the *Fokker-Planck equation*

$$\frac{\partial}{\partial t}p(t,\tilde{\rho}) + \frac{\partial}{\partial\tilde{\rho}}(\hat{a}(t,\tilde{\rho})p(t,\tilde{\rho})) - \frac{1}{2}\frac{\partial^2}{\partial\tilde{\rho}^2}(\hat{b}(t,\tilde{\rho})^2p(t,\tilde{\rho})) = 0$$

with the conditions

$$\int_{-1}^{1} p(t,\tilde{\rho})d\tilde{\rho} = 1 \text{ and } \int_{-1}^{1}\tilde{\rho}\cdot p(t,\tilde{\rho})d\tilde{\rho} \to_{t\to\infty} mean\ value$$

and the stationary density can be computed as

$$p(\tilde{\rho}) := \lim_{t\to\infty} p(t,\tilde{\rho}|\rho_0).$$

Example

Correlation process based on the Ornstein-Uhlenbeck process

$$d\rho_t = \underbrace{(1-\rho_t^2)(\kappa(\mu - \mathrm{artanh}(\rho_t)) - \rho_t\sigma^2)}_{:=\hat{a}(t,\rho_t)} dt + \underbrace{(1-\rho_t^2)\sigma}_{:=\hat{b}(t,\rho_t)} dW_t$$

where $t \geq 0$ and $\rho_0 \in (-1, 1)$.

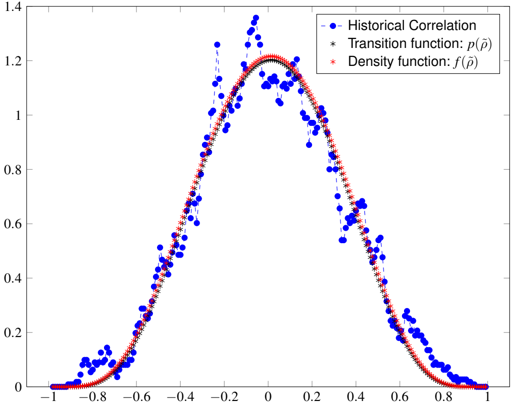
The stationary density can be computed as

$$p(\tilde{\rho}) = \frac{c}{\tilde{\rho}^2 - 1} \cdot \mathrm{e}^{-\frac{\kappa \mathrm{artanh}(\tilde{\rho})}{\sigma^2}(\mathrm{artanh}(\tilde{\rho}) - 2\mu)}$$

with c such that $\int_{-1}^{1} p(\tilde{\rho}) = 1$.

Compare to function $f(\tilde{\rho})$

$$f(\tilde{\rho}) = \frac{1}{1-\tilde{\rho}^2} \cdot \frac{\sqrt{\kappa}}{\sigma\sqrt{\pi}} \cdot \mathrm{e}^{-\frac{\kappa(\mathrm{artanh}(\tilde{\rho}) - \mu)^2}{\sigma^2}}$$



The estimated parameters for:

$p(\tilde{\rho})$: $\kappa = 25.66, \mu = 0.01, \sigma = 2.31$ and $c = -1.2$ $(MSE = 0.0011)$

$f(\tilde{\rho})$: $\kappa = 23.33, \mu = 0.01$ and $\sigma = 2.24$ $(MSE = 0.0014)$

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Quanto (Quantity Adjusting Option)

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A call on the S&P500 with payoff in Euro

$$\underbrace{exchangeRate}_{:=R_t}{}_0 \cdot (\underbrace{S\&P500}_{:=S_t}{}_T - Strike)^+$$

is modeled by

$$\begin{cases} dS_t = \mu_S S_t dt + \sigma_S S_t dW_t^S \\ dR_t = \mu_R R_t dt + \sigma_R R_t dW_t^R \end{cases}$$

where W_t^S and W_t^R are correlated with

$$d\rho_t = (1-\rho_t^2)(\kappa(\mu - \text{artanh}(\rho_t)) - \rho_t \sigma^2)dt + (1-\rho_t^2)\sigma dW_t$$

No-Arbitrage Condition requires

- $\frac{1}{R_0}\exp(r_e T)\mathbb{E}[R_T] = \exp(r_d T) \Rightarrow \mu_R = r_d - r_e$
- $\frac{1}{R_0}\frac{1}{S_0}\mathbb{E}[S_T R_T] = \exp(r_d T) \Rightarrow \mu_S = r_d - \mu_R - \sigma_S \sigma_R \frac{1}{T}\int_0^T \rho_t dt$

r_e and r_d denote the risk-free interest rates in Euro/US-Dollar

Conditional Monte Carlo Approach

Price of the Quanto in the Black-Scholes formula with continuous dividend yield

$$C_{\text{Quanto}}(S_0, K, r_d, D(\rho_t), \sigma_S, T) = R_0 \left(S_0 \mathrm{e}^{-D(\rho_t)T} \mathcal{N}(d_1) - K \mathrm{e}^{-r_d T} \mathcal{N}(d_2) \right)$$

with

$$d_1 = \frac{\log(\frac{S_0}{K}) + ((r_d - D(\rho_t)) + \frac{\sigma_S^2}{2})/T}{\sigma_S \sqrt{T}}, \quad d_2 = d_1 - \sigma_S \sqrt{T}$$

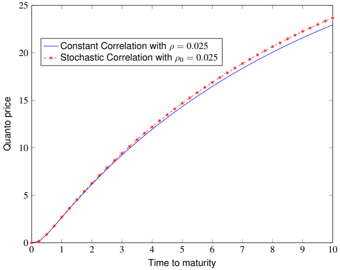
and

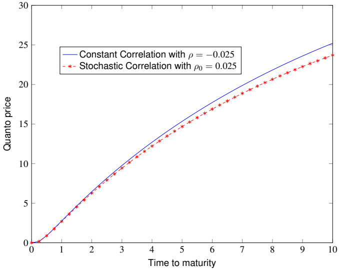
$$D(\rho_t) = r_d - r_e + \sigma_S \sigma_R \frac{1}{T} \int_0^T \rho_t dt.$$

The fair price P_0 is given by

$$P_0 = \mathbb{E}\left[\mathbb{E}[C_{\text{Quanto}}(S_0, K, r_d, D(\rho_t), \sigma_S, T) | \mathcal{F}(\rho_{\{0 \leq s \leq t\}})] \right]$$

Correlation Risk





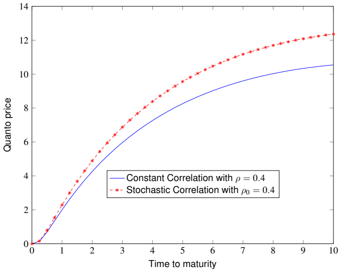
BS parameters:

$$K = 120,\ S_0 = 100,\ R_0 = 1,\ r_d = 0.05,\ r_e = 0.03,\ \sigma_S = 0.2,\ \sigma_R = 0.4$$

Correlation process parameters:

$\kappa = 23.33, \mu = 0.01$ and $\sigma = 2.24$

Correlation Risk



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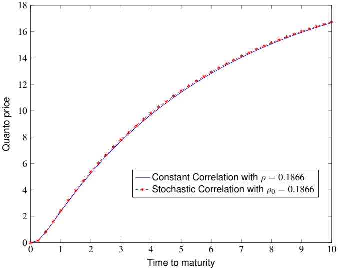
Correlation Risk



BS parameters:

$K = 120,\ S_0 = 100,\ R_0 = 1,\ r_d = 0.05,\ r_e = 0.03,\ \sigma_S = 0.2,\ \sigma_R = 0.4$

Correlation process parameters:

$\kappa = 43.33, \mu = 0.2,$ and $\sigma = 2.24$

$\Rightarrow mean\ value = 0.1866$

Outline

Conclusions

- Correlation modeled as fixed number may lead to correlation risk
- Correlation can be modeled as the Hyperbolic Funtion $tanh$ of a stochastic process
- Effect of considering stochastic correlation on pricing the quanto as Example

Thank you for your attention!

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