# Gambling in Contests with Regret

Han Feng David Hobson

University of Warwick

11th June 2013

# Introduction

- Seel & Strack[1] introduced a gambling contest
- Each player
  - Privately observes a drifting Brownian motion (starts above zero, absorbed at zero)
  - Chooses when to stop it
- The player with the highest stopped value wins
- Objective: to maximise the probability of winning

[1] Christian Seel & Philipp Strack 2012. Gambling in contests. Forthcoming in Journal of Economic Theory.

# Motivation

- Stylised model for competition between fund managers
  - Best performing manager gets a prize
- Simple contest
  - Rich and subtle solutions

# The Model

- $n$ players with labels $i \in I = \{1, 2, \ldots, n\}$
- Player $i$ privately observes a BM $X^i = (X^i_t)_{t \in \mathbb{R}^+}$
  - $X^i$ is absorbed at zero.
  - $X^i_0 = x_0 > 0$ is a contant.
- Processes $X^i$ are independent.

# Stategies

- $\mathcal{F}_t^i = \sigma(\{X_s^i : s < t\})$ and $\mathbb{F}^i = (\mathcal{F}_t^i)_{t \geq 0}$
- Strategies of player $i$: $\mathbb{F}^i$-stopping times $\tau^i$
  - Require $\tau^i \leq H_0^i = \inf\{t \geq 0 : X_t^i = 0\}$.
- Notice: player $i$ can observe neither $X^j$ nor $\tau^j$ for any $j \neq i$.

# Payoffs

- Player $i$ wins 1 if $X^i_{\tau^i} > X^j_{\tau^j}$ $\forall j \neq i$.
- Ties are broken evenly.
- Payoff:

$$\frac{1}{k}\mathbf{1}_{\{X^i_{\tau^i} = \max_{j \in I} X^j_{\tau^j}\}},$$

  where $k = \left|\left\{i \in I : X^i_{\tau^i} = \max_{j \in I} X^j_{\tau^j}\right\}\right|$.
- Insight: payoff only deponds upon $\tau^i$ via $X^i_{\tau^i}$.

# Two Stages

- Two stages:
  - Find an optimal target distribution $F^i$
  - Verify that $\exists \tau^i$ such that $X^i_{\tau^i} \sim F^i$
- First stage: to find Nash equilibria
- Second stage: the Skorokhod embedding problem
  - Any distribution on $\mathbb{R}^+$ with mean $x_0$ can be embedded with a finite stopping time $\tau$.

# Nash Equilibrium

## Nash equilibrium

$(F^i)_{i \in I}$ is a Nash equilibrium if, for each $i \in I$, if the other agents use stopping rules $\tau^j$ such that $X^j_{\tau^j} \sim F^j$, then the optimal target distribution for agent $i$ is $F^i$, and she may use any stopping rule $\tau^i$ such that $X^i_{\tau^i} \sim F^i$.

# Symmetry & Atom-free

A Nash equilibrium is

- Symmetric if $F^i$ does not depend on $i$
- Atom-free if each $F^i$ is atom-free

Theorem *1 [Seel & Strack 2012]*

*Any Nash equilibrium has the property that it is symmetric and atom-free.*

# Solution to Original Contest

**Theorem** *2 [Seel & Strack 2012]*

*There exists a symmetric, atom-free Nash equilibrium for the problem for which $X^i_{\tau^i}$ has law $F(x)$, where for $x \geq 0$*

$$F(x) = \min\left\{ \sqrt[n-1]{\frac{x}{nx_0}}, 1 \right\}.$$

Observations:

- Randomised strategies $\Rightarrow$ the stopped level is stochastic.
- Set of stopped levels is bounded above by $nx_0$.

# Our Aim

- Different proof based on a Lagrangian approach
- Our aim:
  - to consider more general processes
  - to add regret

# Contests with Regret

- An extension: adding a penalty
  - Agent is penalised if her strategy is suboptimal.
- Payoff:

$$\mathbf{1}_{\{X^i_{\tau^i} = \max_{j \in I} X^j_{\tau^j}\}} - K \mathbf{1}_{\{X^i_{\tau^i} < \max_{j \neq i} X^j_{\tau^j} < M^i_{\tau^i}\}},$$

  where $K \geq 0$ is a constant and

$$M^i_{\tau^i} = \max\{X^i_t; 0 \leq t \leq \tau^i\}.$$

- Nash equilibrium: Symmetric and atom-free

# Problem

Given that $X^j_{\tau^j} \sim F \ \forall j \neq i$, agent $i$ aims to choose a feasible measure $\nu(x, m)$ for $(X^i_{\tau^i}, M^i_{\tau^i})$ to maximise

$$\begin{aligned}
&\mathbb{E}\left[F(X^i_{\tau^i})^{n-1}\right] - K\mathbb{E}\left[F(M^i_{\tau^i})^{n-1} - F(X^i_{\tau^i})^{n-1}\right] \\
&= (1+K)\mathbb{E}\left[F(X^i_{\tau^i})^{n-1}\right] - K\mathbb{E}\left[F(M^i_{\tau^i})^{n-1}\right] \\
&= \int_0^\infty \int_0^\infty \left[(1+K)F(x)^{n-1} - KF(m)^{n-1}\right] \nu(dx, dm).
\end{aligned}$$

# Constraints on optimal $\nu$

Constraints on optimal $\nu$:

- $\nu$ is a probability measure on $[0,\infty) \times [0,\infty)$ that has no mass on $\{(x,m) : m < x \text{ or } m < x_0\}$.
- $\mathbb{E}[X_\tau] = x_0 \Rightarrow \int_0^\infty \int_0^\infty x\nu(dx,dm) = x_0$.
- $(X_{t\wedge\tau})_{t\geq 0}$ is a *u.i.* martingale & Doob's (sub)martingale

  inequality $\Rightarrow \mathbb{E}[X_\tau - z; M_\tau \geq z] = 0, \forall z \geq x_0$.

  $$\Rightarrow \int_{x=0}^\infty \int_{m=z}^\infty (x-z)\nu(dx,dm) = 0, \forall z \geq x_0.$$

# Optimisation Problem

- Let $\mathcal{E}(x_0)$ be the set of measures $\nu$ on $[0,\infty) \times [0,\infty)$ that has no mass on $\{(x,m) : m < x \text{ or } m < x_0\}$.
- Given $F(x)$, the agent solves

$$\max_{\nu \in \mathcal{E}(x_0)} \left\{ \int_0^\infty \int_0^\infty \left[ (1+K)F(x)^{n-1} - KF(m)^{n-1} \right] \nu(dx, dm) \right\}$$

subject to $\int_0^\infty \int_0^\infty x\nu(dx,dm) = x_0$, $\int_0^\infty \int_0^\infty \nu(dx,dm) = 1$

and $\int_{x=0}^\infty \int_{m=z}^\infty (x-z)\nu(dx,dm) = 0 \ \forall z \geq x_0$.

# Lagrangain Approach

- Lagrangian:

$$\mathcal{L}_F(\nu; \lambda, \gamma, \eta) = \int_0^\infty \int_0^\infty L(x, m)\nu(dx, dm) + \lambda x_0 + \gamma,$$

  where

$$L(x, m) = (1 + K)\psi(x) - K\psi(m) - \lambda x - \gamma - \int_{x_0}^{m} \eta(z)(x - z)dz$$

  and $\psi(x) = F(x)^{n-1}$.

- Expect: $L(x, m) = 0 \Leftrightarrow \nu(dx, dm) > 0$.

# When $\nu(dx, dm) > 0$

- Recall: payoff is $(1 + K)F(X_\tau)^{n-1} - KF(M_\tau)^{n-1}$
- To maximise the payoff,
  - for any feasible $X_\tau$, find the joint law of $(X_\tau, M_\tau)$ for which $M_\tau$ is as small as possible in distribution $\Leftarrow$ Perkins[2] and Hobson and Pedersen[3]
  - maximise over feasible laws of $X_\tau$.

[2] E. Perkins 1986. The Cereteli-Davis solution to the $H^1$-embedding problem and an optimal embedding in Brownian motion.

[3] D. G. Hobson and J. L. Pedersen 2002. The Minimum Maximum of a Continuous Martingale with Given Initial and Terminal Laws

### Smallest $M_\tau$

Given $X_\tau$, the joint law of $(X_\tau, M_\tau)$ for which $M_\tau$ is minimised is such that mass is placed only on the set

$$A = \{(x, x); x \geq x_0\} \cup \{(x, \Phi(x)); x < x_0\}$$

where $\Phi : (0, x_0) \mapsto (x_0, \infty)$ is a decreasing function (and if $X_\tau$ is atom-free, a strictly decreasing function).

# Distribution of $M_\tau$

- The conditional distribution of $M_\tau$ given $X_\tau$ is

$$M_\tau = \begin{cases} X_\tau & , \text{ if } X_\tau \geq x_0, \\ \Phi(X_\tau) & , \text{ if } 0 \leq X_\tau < x_0. \end{cases} \quad (1)$$

- Expected payoff:

$$\begin{aligned} &\mathbb{E}\left[(1+K)F(X_\tau)^{n-1} - KF(M_\tau)^{n-1}\right] \\ &= \begin{cases} \mathbb{E}\left[F(X_\tau)^{n-1}\right], & \text{if } X_\tau \geq x_0, \\ \mathbb{E}\left[(1+K)F(X_\tau)^{n-1} - KF(\Phi(X_\tau))^{n-1}\right], & \text{if } 0 \leq X_\tau < x_0. \end{cases} \end{aligned}$$

### Smallest $M_\tau$

Given $X_\tau$, the joint law of $(X_\tau, M_\tau)$ for which $M_\tau$ is minimised is such that mass is placed only on the set

$$A = \{(x,x); x \geq x_0\} \cup \{(x,\Phi(x)); x < x_0\}$$

where $\Phi : (0, x_0) \mapsto (x_0, \infty)$ is a decreasing function (and if $X_\tau$ is atom-free, a strictly decreasing function).

Let $\phi$ be inverse to $\Phi$.

- Expect: $\nu(dx, dm) > 0 \Leftrightarrow$ either $x = m$ or $x = \phi(m)$.

# Conjectures

- Recall that $L(x, m) = 0 \Leftrightarrow \nu(dx, dm) > 0$. Thus

$$L(m, m) = 0; \qquad L(\phi(m), m) = 0. \tag{2}$$

- Since $L(x, m) \leq 0$ for any $0 \leq x \leq m$, $\phi(m) \leq m$ and $L(\phi(m), m) = 0$, we expect

$$\frac{\partial L}{\partial x}(\phi(m), m) = 0. \tag{3}$$

# Candidate Solution

Candidate solution comes from

$$\begin{cases} \psi(m) - \lambda m - \gamma - \int_{x_0}^{m} \eta(z)(x-z)dz = 0, \\ (1+K)\psi(\phi(m)) - K\psi(m) - \lambda\phi(m) - \gamma - \int_{x_0}^{m} \eta(z)(x-z)dz = 0, \\ (1+K)\psi'(\phi(m)) - \lambda - \int_{x_0}^{m} \eta(z)dz = 0. \end{cases} \tag{4}$$

# Simplification

- (4) can be rewritten as

$$\begin{cases} \phi'(m)\psi'(m) = (1+K)\theta'(m), \\ K\psi'(m) = (y-\phi(m))\psi''(m), \\ \frac{m-\phi(m)}{n-1}\theta'(m) = \left(\psi(m)^{\frac{1}{n-1}} - 1\right)\theta(m)^{\frac{n-2}{n-1}} - \theta(m). \end{cases} \tag{5}$$

- Boundary conditions: $\phi(x_0) = x_0$, $\psi(r) = 1$, $\psi'(r-) = \frac{K+1}{r}$, $\psi''(r-) = \frac{K(K+1)}{r^2}$ and $\theta(x_0) = \psi(x_0)$.
- Remarks:
  - $r = \sup\{x \geq 0 : F(x) < 1\}$; $\phi : [x_0, r] \mapsto [0, x_0]$.
  - $\theta(x) = F(\phi(x))^{n-1}$ for $x_0 \leq x \leq r$; $\psi(x) = F(x)^{n-1}$ for $x_0 \leq x \leq r$.

# Solution

## Lemma 1

Let $J(u)$ solve the ordinary differential equation

$$J'(u) = \frac{J(u) + 1 - (1-u)^{1/(n-1)}}{(K+1)\left[1 - u - J(u)^{n-1}\right]} \tag{6}$$

subject to $J(0) = 0$ and $u \geq 0$. Let

$$u^* = \sup\left\{u : J(u) < (1-u)^{1/(n-1)}\right\}.$$

i) Define

$$H(z) = \frac{K}{(K+1)\left[z - J(1-z)^{n-1}\right]}$$

on $[z^*, 1]$, where $z^* = 1 - u^*$. Then $z^* > 0$, $H$ is positive on $(z^*, 1)$ and $\int_{z^*}^{1} \exp\left(\int_{w}^{1} H(v)dv\right) dw < (K+1)$.

# Solution

ii) Define

$$r = \frac{x_0(K+1)}{(K+1) - \int_{z^*}^{1} \exp\left(\int_w^1 H(v)dv\right) dw}$$

and

$$\Psi(z) = \frac{r}{K+1}\left[(K+1) - \int_z^1 \exp\left(\int_w^1 H(v)dv\right) dw\right]$$

on $[z^*, 1]$. Let $\psi = \Psi^{-1}$ be the inverse function of $\Psi$. Then $x_0 < r < \infty$ and $\psi : [x_0, r] \mapsto [0, 1]$ is a strictly increasing and strictly convex function that satisfies $\psi(r) = 1$, $\psi'(r-) = \frac{K+1}{r}$ and $\psi''(r-) = \frac{K(K+1)}{r^2}$.

# Solution

iii) Define

$$\phi(m) = m - \frac{K\psi'(m)}{\psi''(m)}.$$

Then $\phi : [x_0, r] \mapsto [0, x_0]$ is a strictly decreasing function with $\phi(x_0) = x_0$.

iv) Define

$$\theta(m) = \psi(x_0) + \frac{1}{K+1}\int_{x_0}^{m} \phi'(z)\psi'(z)dz$$

Then $\theta : [x_0, r] \mapsto [0, 1]$ is a strictly decreasing function with $\theta(x_0) = \psi(x_0)$.

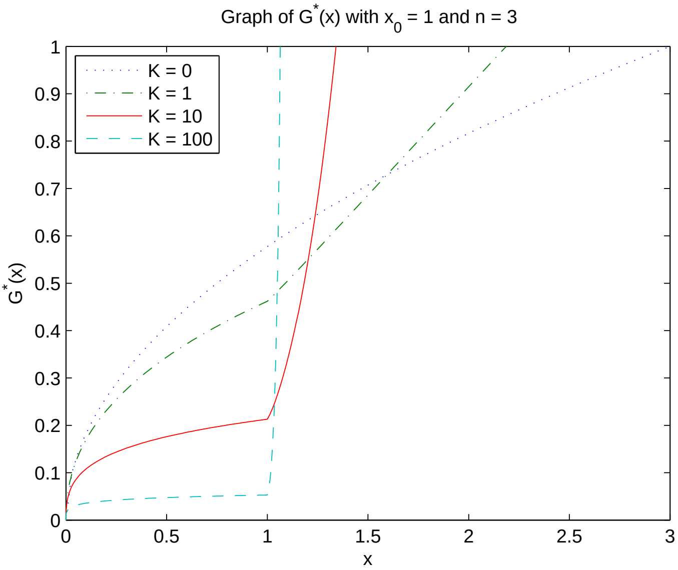
# Result

### Theorem 3

Let $r$, $\psi$, $\phi$, $\theta$ be as defined in Lemma 1. Then there exists a symmetric, atom-free Nash equilibrium for the problem for which $X^i_{\tau^i}$ has distribution $F$ where $F(x) = 0$ for $x \leq 0$, $F(x) = 1$ for $x \geq r$ and otherwise

$$F(x) = \begin{cases} \theta(\phi^{-1}(x))^{\frac{1}{n-1}} & \text{if } 0 < x < x_0, \\ \psi(x)^{\frac{1}{n-1}} & \text{if } x_0 \leq x < r. \end{cases}$$

# Example



Graph of G*(x) with x0 = 1 and n = 3
K = 0
K = 1
K = 10
K = 100
G*(x)
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3

# Stopping too soon

$M^i := M^i_{[\tau^i, H^i_0]} = \sup_{\tau^i \le t \le H^i_0} X^i_t.$

### Theorem 4

There exists a symmetric, atom-free Nash equilibrium for the problem for which $X^i_{\tau^i}$ has law $F(x)$, where for $x \ge 0$

$$F(x) = \min\left\{ \sqrt[N-1]{\frac{x}{Nx_0}}, 1 \right\}$$

with $N = n + K(n-1)$.

# Failure to stop at the best time

$M^i := M^i_{H^i_0} = \sup_{0 \le t \le H^i_0} X^i_t.$

**Theorem 5**

There exists a symmetric, atom-free Nash equilibrium for the problem for which $X^i_{\tau^i}$ has law $F(x)$, where for $x \geq 0$

$$F(x) = \min\left\{ \sqrt[n-1]{\frac{x}{n x_0}}, 1 \right\}.$$

Thank You!