Construction of discrete time shadow price

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Maximization expected utility under transaction costs

bid $\underline{S}$ and ask $\overline{S}$ prices

Tranasaction costs and frictionless markets

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Problem: existence of shadow price

Based on joint paper with Ł. Stettner

Recent papers on shadow price

Kallsen J., Muhle-Karbe J. [2010]

Kallsen J., Muhle-Karbe J. [2011]

Gerhold S., Muhle-Karbe J., Schachermayer W. [2011]

Gerhold S., Muhle-Karbe J., Schachermayer W. [2011]

Czichowsky Ch., Muhle-Karbe J., Schachermayer W. [2012]

Shadow price in different models

On a finite probability spaces with functional

$$\mathbf{E}\sum_{n=0}^{\infty} g_n(c_n)$$

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shadow price always exists. [Kallsen J., Muhle-Karbe J., (2011)]
However, in infinite probability spaces it can fail to exist. [Czichowsky Ch., Muhle-Karbe J., Schachermayer W. (2012)]

Introduction

Assume on a filtrated probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=0}^N, \mathbf{P})$ we are given: strictly positive adapted processes $\underline{S} = (\underline{S}_n)_{n=0}^N$ and $\overline{S} = (\overline{S}_n)_{n=0}^N$ such that $\overline{S}_n > \underline{S}_n$ and

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$$\begin{aligned} supp\mathbf{E}[(\underline{S}_{N-k}, \ldots, \underline{S}_N)|\mathcal{F}_{N-k}] = \{\underline{S}_{N-k}\} \times [0, \infty)^k, \\ supp\mathbf{E}[(\overline{S}_{N-k}, \ldots, \overline{S}_N)|\mathcal{F}_{N-k}] = \{\overline{S}_{N-k}\} \times [0, \infty)^k \end{aligned} \tag{1}$$

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Our position (x, y), where x is the amount on the bank account and y is the number of assets in our portfolio.

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where g is a strictly increasing and concave utility function, e.g. $g(c) = \ln c$ or $g(c) = c^\alpha$ with $\alpha \in (0, 1)$.

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For $(x, y) \in \mathbf{R}_+^2$ and $\underline{s}, \overline{s} \in \mathbf{R}_+$ such that $\overline{s} \geq \underline{s} \geq 0$ let

$$\mathbf{A}(x, y, \underline{s}, \overline{s}) := \{(c, l, m) \in [0, x + \underline{s}y] \times \mathbf{R}_+^2 : \\ \forall_{s \in [0,\infty)}\ x - c + \underline{s}m - \overline{s}l + s(y - m + l) \geq 0\}.$$

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Proposition

Let $(x, y) \in \mathbf{R}^2_+$ and $\underline{s}, \overline{s} \in \mathbf{R}_+$ such that $\overline{s} \geq \underline{s} \geq 0$. Then we have

(i) $\mathbf{A}(\rho x, \rho y, \underline{s}, \overline{s}) = \rho \mathbf{A}(x, y, \underline{s}, \overline{s})$, *for* $\rho \geq 0$,

(ii) *the set* $\mathbf{A}(x, y, \underline{s}, \overline{s})$ *is convex,*

(iii) *for* $\overline{s} > \underline{s} > 0$ *the set* $\mathbf{A}(x, y, \underline{s}, \overline{s})$ *is compact.*

Set of constraints and Hausdorff metric

Theorem

Let $(x_n, y_n, \underline{s}_n, \overline{s}_n)_{n=1}^{\infty}$ *be a sequence from* $\mathbf{R}_+^4$ *such that for all* $n \in \mathbf{N}$ *we have* $\overline{s}_n > \underline{s}_n > 0$*, which converges to* $(x, y, \underline{s}, \overline{s}) \in \mathbf{R}_+^4$ *such that* $\overline{s} > \underline{s} > 0$*. Then*

$$h(\mathbf{A}(x, y, \underline{s}, \overline{s}), \mathbf{A}(x_n, y_n, \underline{s}_n, \overline{s}_n)) \xrightarrow{n\to\infty} 0,$$

where $h : \mathcal{H}(\mathbf{R}_+^3) \times \mathcal{H}(\mathbf{R}_+^3) \to \mathbf{R}_+$ *is a Hausdorff metric, i.e.*

$$h(A, B) := \max\{d(A, B), d(B, A)\}$$

for all $A, B \in \mathcal{H}(\mathbf{R}_+^3)$*.*

Bellman equations

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and inductively

$$w_{N-k}(x, y, \underline{s}, \overline{s}) := \sup_{(c,l,m)\in \mathbf{A}(x,y,\underline{s},\overline{s})} \mathbf{E}[$$

$$g(c) + \gamma w_{N-k+1}(x - c + \underline{s}m - \overline{s}l, y - m + l, \underline{S}_{N-k+1}, \overline{S}_{N-k+1})|\mathcal{F}_{N-k}]$$

for $k = 1, 2, \ldots, N$.

Bellman equations and original problem

Proposition

$$\mathbf{E}[w_0(x, y, \underline{s}, \overline{s})] = \sup \mathbf{J}^N_{(x,y,\underline{s},\overline{s})}(u).$$

with $\underline{S}_0 = \underline{s}$ *and* $\overline{S}_0 = \overline{s}$.

Strict concavity and its consequences

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The random mapping

$$(x, y) \longmapsto \mathbf{E}[w_{N-k+1}(x, y, \underline{S}_{N-k+1}, \overline{S}_{N-k+1}) | \mathcal{F}_{N-k}]$$

is strictly concave for $k = 1, 2, \ldots, N$.

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is strictly concave for $k = 1, 2, \ldots, N$. *Furthermore, for each* $(x, y) \in \mathbf{R}^2_+$ *and* $\underline{s}, \overline{s} \in \mathbf{R}_+$ *such that* $\overline{s} > \underline{s} > 0$ *there exists only one* $\mathcal{F}_{N-k}$*-measurable random variable* $(\hat{c}, \hat{l}, \hat{m})$ *which takes values in the set* $\mathbf{A}(x, y, \underline{s}, \overline{s})$ *and such that*

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Moreover, the random mapping

$$(x, y, \underline{s}, \overline{s}) \mapsto (\hat{c}(x, y, \underline{s}, \overline{s}), \hat{l}(x, y, \underline{s}, \overline{s}), \hat{m}(x, y, \underline{s}, \overline{s}))$$

is continuous on the set $\{(x, y, \underline{s}, \overline{s}) \in \mathbf{R}^4_+ : \overline{s} > \underline{s} > 0\}$.

Properties of the set of optimal strategies

For $k = 1, 2, ..., N$

$$\mathbf{NT}_{N-k}(\underline{s}, \overline{s}) := \{(x, y) \in \mathbf{R}^2_+ : w_{N-k}(x, y, \underline{s}, \overline{s}) = \sup_{c \in [0,x]} \mathbf{E}[g(c) + \gamma w_{N-k+1}(x - c, y, \underline{S}_{N-k+1}, \overline{S}_{N-k+1}) | \mathcal{F}_{N-k}]\},$$

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and

$$\mathbf{B}_{N-k}(\underline{s}, \overline{s}) := \{(x, y) \in \mathbf{R}^2_+ : w_{N-k}(x, y, \underline{s}, \overline{s}) = \sup_{(c,l,0) \in \mathbf{A}(x,y,\underline{s},\overline{s})} \mathbf{E}[g(c) +$$

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Local shadow price

At time moment $N-k$ one price $\tilde{s}$.

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$$\overline{\mathbf{B}}(x, y, \tilde{s}) := \{(c, K) :\in [0, x + \tilde{s}y] \times \mathbf{R} : x - c + \tilde{s}K \geq 0, y - K \geq 0\}$$

for $(x, y) \in \mathbf{R}^2_+$ and $\tilde{s} > 0$.

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for $(x,y) \in \mathbf{R}^2_+$ and $\tilde{s} > 0$. Define

$$v_{N-k}(x,y,\tilde{s}) := \sup_{(c,K)\in\overline{\mathbf{B}}(x,y,\tilde{s})} \mathbf{E}[g(c) + \gamma w_{N-k+1}(x-c+\tilde{s}K, y-K, \underline{S}_{N-k+1}, \overline{S}_{N-k+1})|\mathcal{F}_{N-k}].$$

Optimal strategies on shadow market

Proposition

There exists a unique $\mathcal{F}_{N-k}$-measurable random variable $(\tilde{c}(x,y,\tilde{s}), \tilde{K}(x,y,\tilde{s}))$, which takes values in the set $\overline{\mathbf{B}}(x,y,\tilde{s})$ which is an optimal one step strategy, i.e. for which

$$v_{N-k}(x,y,\tilde{s}) = \mathbf{E}[g(\tilde{c}) + \gamma w_{N-k+1}(x - \tilde{c} + \tilde{s}\tilde{K}, y - \tilde{K}, \underline{S}_{N-k+1}, \overline{S}_{N-k+1}) | \mathcal{F}_{N-k}].$$

Local shadow price - optimal strategies

For $\tilde{s} > 0$ and for $k = 1, ..., N$

$$\tilde{\mathbf{NT}}_{N-k}(\tilde{s}) := \{(x, y) \in \mathbf{R}^2_+ : v_{N-k}(x, y, \tilde{s}) =$$
$$= \sup_{c \in [0,x]} \mathbf{E}[g(c) + \gamma w_{N-k+1}(x - c, y, \underline{S}_{N-k+1}, \overline{S}_{N-k+1}) | \mathcal{F}_{N-k}]\},$$
$$\tilde{\mathbf{S}}_{N-k}(\tilde{s}) := \{(x, y) \in \mathbf{R}^2_+ : v_{N-k}(x, y, \tilde{s}) = \sup_{(c,K) \in \overline{\mathbf{B}}(x,y,\tilde{s}) \cap \mathbf{R}^2_+} \mathbf{E}[g(c) +$$
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and

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Local shadow price - definition

Definition

The family of $\mathcal{F}_{N-k}$-measurable random functions

$$\tilde{S}_{N-k} = \left\{ \tilde{S}_{N-k}(x, y, \underline{s}, \overline{s}) \right\}_{(x,y) \in \mathbf{R}^2_+ \setminus \{(0,0)\}, \overline{s} > \underline{s} > 0}$$

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is called local shadow price at time moment $N - k$ if for all $(x, y) \in \mathbf{R}_+^2 \setminus \{(0, 0)\}$ and $\underline{s}, \overline{s} \in \mathbf{R}_+$ such that $\overline{s} > \underline{s} > 0$ we have

$$\underline{s} \leq \tilde{S}_{N-k}(x, y, \underline{s}, \overline{s}) \leq \overline{s}$$

and

$$v_{N-k}(x, y, \tilde{S}_{N-k}(x, y, \underline{s}, \overline{s})) = w_{N-k}(x, y, \underline{s}, \overline{s}).$$

Applications of shadow price

Proposition

For $\underline{s}, \overline{s} \in \mathbf{R}_+$ *such that* $\overline{s} > \underline{s} > 0$ *and all* $\omega \in \Omega$ *we have*

$$\tilde{\boldsymbol{S}}_{N-k}(\underline{s})(\omega) = \boldsymbol{S}_{N-k}(\underline{s}, \overline{s})(\omega). \tag{3}$$

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Lemma

Let $\underline{s}, \overline{s} \in \mathbf{R}_+$ *be such that* $\overline{s} > \underline{s} > 0$. *Then for every* $\omega \in \Omega$ *we have*

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Construction of local shadow price

Lemma

Let $s_1, s_2 \in \mathbf{R}_+$ *be such that* $0 < s_1 \leq s_2$. *Then*

$$\tilde{\boldsymbol{S}}_{N-k}(s_1) \subseteq \tilde{\boldsymbol{S}}_{N-k}(s_2) \tag{6}$$

and

$$\tilde{\boldsymbol{B}}_{N-k}(s_2) \subseteq \tilde{\boldsymbol{B}}_{N-k}(s_1). \tag{7}$$

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Then $\underline{s}^*_{N-k}(x,y)$ *and* $\overline{s}^*_{N-k}(x,y)$ *are well defined* $\mathcal{F}_{N-k}-$*measurable random variables.*

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$$\underline{s}^*_{N-k}(x,y) = \overline{s}^*_{N-k}(x,y) =: \tilde{s}_{N-k}(x,y) \tag{10}$$

and the random mapping $(x,y) \mapsto \tilde{s}_{N-k}(x,y)$ *is continuous.*

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$$\forall_{(x,y)\in\mathbf{R}^2_+\setminus\{(0,0)\}}\ (x, y) \in \tilde{\boldsymbol{NT}}_{N-k}(\tilde{s}_{N-k}(x, y)). \tag{11}$$

Construction of local shadow price

Theorem

For all $(x, y) \in \mathbf{R}_+^2 \setminus \{(0,0)\}$ *and* $\underline{s}, \overline{s} \in \mathbf{R}_+$ *such that* $\overline{s} > \underline{s} > 0$ *let*

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$$\tilde{S}_{N-k}(x, y, \underline{s}, \overline{s}) := \begin{cases} \underline{s} & \text{on } \{(x,y) \in \boldsymbol{S}_{N-k}(\underline{s}, \overline{s})\} \\ \tilde{s}_{N-k}(x,y) & \text{on } \{(x,y) \in \boldsymbol{NT}_{N-k}(\underline{s}, \overline{s})\} \\ \overline{s} & \text{on } \{(x,y) \in \boldsymbol{B}_{N-k}(\underline{s}, \overline{s})\} \end{cases}, \qquad (12)$$

where the random mapping $\tilde{s}_{N-k}$ *is defined by (10).*

Construction of local shadow price

Theorem

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is $\mathcal{F}_{N-k}$*-measurable and is a local shadow price at time moment* $N-k$*. Furthermore, the optimal strategies at time moment* $N-k$ *are the same in both markets.*

Global shadow price

Definition

The family

$$\tilde{S} = \{\tilde{S}_n(x_n, y_n, \underline{S}_n, \overline{S}_n)\}_{n=0,\ldots,N,(x_0,y_0),\ldots,(x_N,y_N)\in\mathbf{R}^2_+\setminus\{(0,0)\}},$$

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where $(x, y) \in \mathbf{R}^2_+ \setminus \{(0,0)\}$ *is called global shadow price, if the mapping* $(x, y) \mapsto \tilde{S}_n(x, y, \underline{S}_n, \overline{S}_n)$ *is* $\mathcal{F}_n$ *- measurable and for every* $(x, y) \in \mathbf{R}_+ \setminus \{(0,0)\}$ *we have* $\underline{S}_n \leq \tilde{S}_n(x, y, \underline{S}_n, \overline{S}_n) \leq \overline{S}_n$ *for* $n = 0, 1, ..., N$ *and the expected value of the discounted utility in the market with price process* $\tilde{S}$ *and in the market with bid and ask price processes* $\underline{S}$ *and* $\overline{S}$ *respectively coincide.*

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Theorem

Let the family of processes $\tilde{S}$ *be defined as in (12).*

Global shadow price

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where $(x, y) \in \mathbf{R}_+^2 \setminus \{(0, 0)\}$ *is called global shadow price, if the mapping* $(x, y) \mapsto \tilde{S}_n(x, y, \underline{S}_n, \overline{S}_n)$ *is* $\mathcal{F}_n$ *- measurable and for every* $(x, y) \in \mathbf{R}_+ \setminus \{(0, 0)\}$ *we have* $\underline{S}_n \leq \tilde{S}_n(x, y, \underline{S}_n, \overline{S}_n) \leq \overline{S}_n$ *for* $n = 0, 1, \dots, N$ *and the expected value of the discounted utility in the market with price process* $\tilde{S}$ *and in the market with bid and ask price processes* $\underline{S}$ *and* $\overline{S}$ *respectively coincide.*

Theorem

Let the family of processes $\tilde{S}$ *be defined as in (12). Then* $\tilde{S}$ *is a global shadow price. Furthermore, the optimal strategies in the market with shadow price are the same as in the original market with bid and ask prices.*

Why shadow price is an important thing?

Markets with transaction costs in which we have a small investor are in fact illiquid markets, i.e. these are markets on which the price of an asset depends on the current position of investor.

The end