Linear Stochastic Volatility Model

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- Density in a lognormal SV model

Vanilla options in Linear SV

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Lognormal SV - the density approximate formula

Heston and extended Heston SV models

Assumptions

We consider a market defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $T < \infty$, satisfying the usual conditions. We assume:

$$B_t \equiv 1.$$

The price X of the underlying asset has a stochastic volatility Y, and is given by

$$dX_t = Y_t X_t dW_t, \tag{1}$$

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dZ_t, \tag{2}$$

where X_0, Y_0 are positive constants, the processes W, Z are correlated Brownian motions, $d\langle W, Z\rangle_t = \rho dt$ with $\rho \in (-1, 1)$, and $\mu : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are continuous functions such that there exists a unique strong solution of (2), which is positive and $\int_0^T Y_u^2 du < \infty$ $\mathbb{P}$-a.s.

Under these assumptions the process X has the form

$$X_t = X_0 e^{\int_0^t Y_u dW_u - \frac{1}{2}\int_0^t Y_u^2 du}, \tag{3}$$

and this is a unique strong solution of SDE (1) on $[0, T]$.

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3) **Heston model**

$$dS_t = \sqrt{\sigma_t} S_t dW_t^*,$$
$$d\sigma_t = \kappa(\nu - \sigma_t)dt + \eta\sqrt{\sigma_t} d\widetilde{W}_t,$$

where $W^*, \widetilde{W}$ are Brownian motions with constant correlation. ρ, κ, η, ν are constant. If $2\kappa\nu \geq 1$, then the process is strictly positive, so using Itô lemma we can write a SDE for $Y_t = \sqrt{\sigma_t}$.

Theorem 1

Fix $t \in [0, T]$. The distribution of X_t has the representation

$$\mathbb{P}(X_t \leq r) = \mathbb{E}\Phi\left(\frac{\ln \frac{r}{X_0} - \mu_Z(t)}{\sigma_Z(t)}\right), \tag{4}$$

where $r > 0$, Φ is the cumulative distribution function of $N(0, 1)$,

$$\mu_Z(t) = \rho \int_0^t Y_u dZ_u - \frac{1}{2} \int_0^t Y_u^2 du, \tag{5}$$

$$\sigma_Z^2(t) = (1 - \rho^2) \int_0^t Y_u^2 du. \tag{6}$$

Moreover, X_t has density function g_{X_t}, which has the representation

$$g_{X_t}(r) = \mathbb{E}\left[\frac{1}{r\sigma_Z(t)} \phi\left(\frac{\ln \frac{r}{X_0} - \mu_Z(t)}{\sigma_Z(t)}\right)\right], \tag{7}$$

where ϕ is the density function of $N(0, 1)$.

The idea of proof

We can represent W in the form $W_t = \rho Z_t + \sqrt{1-\rho^2} B_t$, where (B, Z) is the standard two-dimensional Wiener process. For fixed $r > 0$

$$\begin{aligned}
\mathbb{P}(X_t \le r) &= \mathbb{E}1_{\left\{X_0 \exp\left(\int_0^t Y_u dW_u - \frac{1}{2}\int_0^t Y_u^2 du\right) \le r\right\}} \\
&= \mathbb{E}\mathbb{E}\left[1_{\left\{\rho \int_0^t Y_u dZ_u + \sqrt{1-\rho^2}\int_0^t Y_u dB_u - \frac{1}{2}\int_0^t Y_u^2 du \le \ln \frac{r}{X_0}\right\}} \Big| \mathcal{F}_t^Z\right]. \\
&= \mathbb{E}\mathbb{P}\left(\mu_Z(t) + \sigma_Z(t) g \le \ln \frac{r}{X_0} \Big| \mathcal{F}_t^Z\right) \\
&= \mathbb{E}\Phi\left(\frac{\ln \frac{r}{X_0} - \mu_Z(t)}{\sigma_Z(t)}\right).
\end{aligned}$$

Remark:

The problem of finding the distribution of X_t, for fixed t, reduces to deriving the distribution of the vector $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$.

Density in Lognormal SV model

Theorem 2

In a log-normal stochastic volatility model the density function of the price X_t of the underlying asset has the form

$$g_{X_t}(r) =$$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\left[\frac{1}{rY_0\sqrt{y\frac{1-\rho^2}{\sigma^2}}}\phi\left(\frac{\ln\frac{r}{X_0}-f(x,y)+Y_0^2y\frac{1-\rho^2}{\sigma^2}}{Y_0\sqrt{y\frac{1-\rho^2}{\sigma^2}}}\right)\right]G_{t\sigma^2}(x,y)dydx,$$

where

$$f(x,y)=\frac{\rho}{\sigma}Y_0[e^x-1]-\frac{\rho^2}{2\sigma^2}Y_0^2y \tag{8}$$

$$G_t(x,y)=\exp\left(-\frac{x}{2}-\frac{t}{8}-\frac{1+e^{2x}}{2y}\right)\theta\left(\frac{e^x}{y},t\right)\frac{1}{y}, \tag{9}$$

and the function θ is defined, using hyperbolic functions, by the formula

$$\theta(r,t) = \frac{r}{\sqrt{2\pi^3 t}} e^{\frac{\pi^2}{2t}} \int_0^\infty e^{\frac{-\xi^2}{2t} - r\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi.$$

Theorem 2 follows form Theorem 1 and the distribution of the vector $(\int_0^t Y_u dZ_u, \int_0^t Y_u^2 du)$ obtained by Matsumoto and Yor.

Vanilla options in Linear SV

We provide representations for the arbitrage prices of European call and put options in Linear SV. These formulae generalize the famous Black-Scholes formulae as well as the result of Hull and White for a stochastic volatility model with uncorrelated noises.

It is worth mentioning that representations of option prices in Linear SV were presented earlier in work of Romano and Touzi. However the formulae were presented in a bit different setting of correlation structure and under some strong assumptions of bounded SDE's coefficients.

Theorem 3

The prices of European call and put options have the following representations:

$$\mathbb{E}[X_t - K]^+ = X_0 \mathbb{E}\left[e^{\mu_Z(t) + \frac{\sigma_Z^2(t)}{2}} \Phi(d_1(t))\right] - K\mathbb{E}\Phi(d_2(t)),$$

$$\mathbb{E}[K - X_t]^+ = K\mathbb{E}\Phi(-d_2(t)) - X_0 \mathbb{E}\left[e^{\mu_Z(t) + \frac{\sigma_Z^2(t)}{2}} \Phi(-d_1(t))\right],$$

$$d_1(t) = \frac{\ln \frac{X_0}{K} + \mu_Z(t) + \sigma_Z^2(t)}{\sigma_Z(t)}, \qquad d_2(t) = d_1(t) - \sigma_Z(t),$$

and $\mu_Z(t)$ and $\sigma_Z^2(t)$ are given by (5) and (6).

Connection between distribution of the asset price and prices of put options

The linear stochastic volatility model has conditionally the structure of Black-Scholes model, so vanilla options prices inherit some special properties of Black-Scholes that enable us to find a probabilistic representation for a density function in terms of prices of put options.

Theorem 4

In a linear stochastic volatility model with $X_0 = x$ we have, for $r \geq 0$,

$$\mathbb{P}(X_t \leq r) = \frac{\partial}{\partial r}\mathbb{E}(r - X_t)^+, \tag{10}$$

$$g_{X_t}(r) = \frac{\partial^2}{\partial r^2}\mathbb{E}(r - X_t)^+. \tag{11}$$

In the next corollary we find that the Laplace transform of X_t for $\lambda > 0$ is equal to a price of put option with random strike.

Corollary

In a linear stochastic volatility model we have, for any $\lambda > 0$,

$$\mathbb{E}e^{-\lambda X_t} = \lambda \mathbb{E}(T_\lambda - X_t)^+, \tag{12}$$

where T_λ is exponential random variable with parameter λ independent of X.

Proof.

We have, by (11),

$$\mathbb{E}e^{-\lambda X_t} = \int_0^\infty e^{-\lambda r} \frac{\partial^2}{\partial r^2} \mathbb{E}(r - X_t)^+ dr = \lambda \int_0^\infty \lambda e^{-\lambda r} \mathbb{E}(r - X_t)^+ dr, \tag{13}$$

where we in the second equality we have integrated by parts and fact that $\frac{\partial}{\partial r}\mathbb{E}(r - X_t)^+|_{r=0} = 0$.

Option prices in Lognormal SV model

Theorem 5

In a lognormal SV model the prices of vanilla options are given by

$$\mathbb{E}[X_t - K]^+$$
$$= \int_{-\infty}^{\infty}\int_0^{\infty}\Big[X_0 e^{f(x,y)}\Phi(d_1(x,y)) - K\Phi(d_2(x,y))\Big] G_{t\sigma^2}(x,y)\,dydx,$$
$$\mathbb{E}[K - X_t]^+$$
$$= \int_{-\infty}^{\infty}\int_0^{\infty}\Big[K\Phi(-d_2(x,y)) - X_0 e^{f(x,y)}\Phi(-d_1(x,y))\Big] G_{t\sigma^2}(x,y)\,dydx,$$

where f, G are given by (8) and (9), and

$$d_1(x,y) = \frac{\ln\frac{X_0}{K} + f(x,y)}{Y_0\sqrt{y\frac{1-\rho^2}{\sigma^2}}} + \frac{Y_0}{2}\sqrt{\frac{1-\rho^2}{\sigma^2}y},$$

$$d_2(x,y) = d_1(x,y) - \frac{Y_0}{2}\sqrt{\frac{1-\rho^2}{\sigma^2}y}.$$

Lognormal SV model

Sin and later Jourdain proved that in the lognormal stochastic volatility model the price process X is a martingale if and only if $\rho \leq 0$. Using our results we can present a simple proof of this fact.

Theorem 6

In the log-normal stochastic volatility model the price process X is a martingale if and only if $\rho \leq 0$.

Sufficiency

Fix any $t \geq 0$.

$$\mathbb{E}X_t = x\mathbb{E}e^{\int_0^t Y_u dW_u - \frac{1}{2}\int_0^t Y_u^2 du} = x\mathbb{E}e^{\rho\int_0^t Y_u dZ_u - \frac{\rho^2}{2}\int_0^t Y_u^2 du}.$$

Since

$$e^{\rho(Y_t - Y_0) - \frac{\rho^2}{2}\int_0^t Y_u^2 du} \leq xe^{-\rho Y_0},$$

the local martingale under the expectation is bounded, so it is a true martingale.

Necessity

Suppose that $\rho > 0$, $Y_0 = 1$. Suppose, contrary to our claim, that X is a martingale. Then $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} := e^{\rho \int_0^t Y_u dZ_u - \frac{\rho^2}{2} \int_0^t Y_u^2 du}$ is a new probability measure. The process $\hat{B}_s = B_s - \rho \int_0^s Y_u du$ for $s \leq t$ is a standard Brownian motion under $\mathbb{Q}$, by the Girsanov theorem. As $Y_s = e^{B_s - s/2}$, the Itô lemma implies

$$0 < e^{\hat{B}_t - B_t} = 1 + \int_0^t e^{\hat{B}_u - B_u} d(\hat{B}_u - B_u) = 1 - \rho \int_0^t e^{\hat{B}_u - u/2} du.$$

In result,

$$1 = \mathbb{Q}\Big(e^{\hat{B}_t - B_t} > 0\Big) = \mathbb{Q}\Big(1 - \rho \int_0^t e^{\hat{B}_u - u/2} du > 0\Big).$$

Contradiction. The process X can not be a martingale.

Lognormal SV - the density approximate formula

The log-normal stochastic volatility model is a special case of SABR model (parameter $\beta = 1$) for which the formula for Black–Scholes implied volatility is given by

$$\sigma(r,x,t) = \sigma \ln(x/r)\Big(1 + t(\sigma\rho y/4 + \sigma^2(2-3\rho^2)/24)\Big)$$
$$\times \Bigg(\ln\bigg(\sqrt{1 - 2\rho\sigma\ln(x/r)/y + (\sigma\ln(x/r)/y)^2} + \sigma\ln(x/r)/y - \rho\bigg)$$
$$- \ln(1-\rho)\Bigg)^{-1}.$$

We have

$$\mathbb{E}(r - X_t)^+ = r\Phi(-d_2) - x\Phi(-d_1),$$

where

$$d_1 = \frac{\ln(x/r) + t\sigma^2(r,x,t)/2}{\sigma(r,x,t)\sqrt{t}}, \quad d_2 = d_1 - \sigma(r,x,t)\sqrt{t}.$$

This allows us to obtain, using Theorem 4, the density function of X_t in the Hull-White stochastic volatility model

$$\begin{aligned} f(r) &= \frac{\partial^2}{\partial r^2}\mathbb{E}(r - X_t)^+ = \frac{\partial^2}{\partial r^2}\Big(r\Phi(-d_2) - x\Phi(-d_1)\Big) \\ &= \frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\Big(rd_2\Big(\frac{\partial d_2}{\partial r}\Big)^2 - 2\frac{\partial d_2}{\partial r} - r\frac{\partial^2 d_2}{\partial r^2}\Big) \\ &+ \frac{xe^{-d_1^2/2}}{\sqrt{2\pi}}\Big(d_1\Big(\frac{\partial d_1}{\partial r}\Big)^2 + \frac{\partial^2 d_1}{\partial r^2}\Big). \end{aligned} \tag{14}$$

In result, when we consider the Hull-White stochastic volatility model with parameter ρ calibrated to market prices of the options, the formula (14) gives the calibrated distribution of the asset price process.

Heston and extended Heston SV models

Let us recall that an extended CIR process is a process R given by

$$dR_t = \kappa(\theta(t) - R_t)dt + \sqrt{R_t}dZ_t, \tag{15}$$

where κ is a positive constant, $\theta : [0, \infty) \mapsto [0, \infty)$ is a continuous function and $R_0 \geq 0$. It is well known that $R_t \geq 0$. If $\theta(t) \equiv \theta > 0$, then we have the classical CIR process given by

$$dR_t = \kappa(\theta - R_t)dt + \sqrt{R_t}dZ_t. \tag{16}$$

Taking $Y_t^2 = R_t$, where R is a CIR or an extended CIR process, we consider the Heston stochastic volatility model and the extended Heston stochastic volatility model. We have to mention the results about explosions in Heston model and it's martingale property due to works of Leif, Andersen and Piterbarg or the recent work of Keller-Ressel. We expand considerations to extended Heston model and give another, new look at classical Heston model.

Theorem 7

In the Heston and extended Heston stochastic volatility models the process X is a martingale.

Theorem 8

Let $\rho \leq 0$. If the natural number k satisfies $k \leq \frac{1}{1-\rho^2}$, then the k-moment of X_t exists for $t \geq 0$ in the Heston and extended Heston models.

Proof

Fix $t \geq 0$. From the fact $R = Y^2$ and from (16) we have

$$\mathbb{E}X_t^k = x^k \mathbb{E}e^{k\int_0^t Y_u dW_u - \frac{k}{2}\int_0^t Y_u^2 du} = x^k \mathbb{E}e^{k\rho\int_0^t Y_u dZ_u - \left(\frac{k}{2} - \frac{k^2(1-\rho^2)}{2}\right)\int_0^t Y_u^2 du}.$$

By (15)

$$\int_0^t Y_u dZ_u = R_t - R_0 - \kappa\int_0^t \theta(u)du + \kappa\int_0^t R_u du$$

and $R_t \geq 0$.

Proof

In result

$$\mathbb{E}X_t^k = x^k e^{-k\rho R_0 - k\rho\kappa\int_0^t \theta(u)du}\mathbb{E}e^{k\rho R_t + k\rho\kappa\int_0^t R_u du - \left(\frac{k}{2} - \frac{k^2(1-\rho^2)}{2}\right)\int_0^t R_u du} \tag{17}$$

$$\leq x^k e^{-k\rho R_0 - k\rho\kappa\int_0^t \theta(u)du},$$

because $R_s \geq 0$, $\rho \leq 0$ and $k(1-\rho^2) \leq 1$. The result follows.

Formula (17) gives a form of the k-moment of X in terms of the Laplace transform $\mathbb{E}e^{-\lambda R_t - \gamma\int_0^t R_u du}$ for $\lambda \geq 0$ and $\gamma > 0$. For the CIR process the form of this transform is well known. In the next theorem we generalize this result and present an explicite form of Laplace transform for an extended CIR process. This, in particular, enables us to use (17) to find an explicite form of the k-moment of X.

Laplace transform

Theorem 7

Let R be an extended CIR process. For $\lambda \geq 0,\ \gamma > 0,\ t \geq 0$
$\lambda > \sqrt{\kappa^2 + 2\gamma} - \kappa$ we have

$$\mathbb{E}e^{-\lambda R_t - \gamma \int_0^t R_u du} = e^{-R_0 f(t) - \kappa \int_0^t \theta(s) f(s) ds}, \tag{18}$$

where

$$f(t) = \frac{\kappa + \sqrt{\kappa^2 + 2\gamma} + ce^{\sqrt{\kappa^2+2\gamma}t}(\sqrt{\kappa^2 + 2\gamma} - \kappa)}{ce^{\sqrt{\kappa^2+2\gamma}t} - 1}, \tag{19}$$

$$c = \frac{\lambda + \kappa + \sqrt{\kappa^2 + 2\gamma}}{\lambda + \kappa - \sqrt{\kappa^2 + 2\gamma}} > 1. \tag{20}$$

Using Theorem 7 we obtain an alternative proof of the well-known result for a classical CIR process where $\theta(t) \equiv \theta > 0$.

Corollary

For a classical CIR process R and for $\lambda \geq 0, \; \gamma \geq 0, \; t \geq 0$ $\lambda > \sqrt{\kappa^2 + 2\gamma} - \kappa$ we have

$$\mathbb{E} e^{-\lambda R_t - \gamma \int_0^t R_u du} = e^{-R_0 f(t) + \theta\kappa t(\kappa + \sqrt{\kappa^2 + 2\gamma})} \left(c e^{\sqrt{\kappa^2+2\gamma} t} - 1 \right)^{-2\kappa\theta},$$

where f is given by (19) *and c is given by* (20).

Approximation

We can approximate the price of put option in an extended Heston model in the case $\rho \leq 0$ using last corollary and Theorem 7. Indeed, for $\lambda > 0$ we see from (13) that

$$\int_0^\infty e^{-\lambda u}\mathbb{E}(u - X_t)^+ du = \frac{1}{\lambda^2}\mathbb{E}e^{-\lambda X_t}. \tag{21}$$

If $\rho \leq 0$ and $n \leq \frac{1}{1-\rho^2}$ for $i \leq n$ we can compute $\mathbb{E}X_t^i$ using (17). Now, we use the following approximation

$$\mathbb{E}e^{-\lambda X_t} \approx \sum_{i=0}^n \frac{(-\lambda)^i}{i!}\mathbb{E}X_t^i.$$

In result from (21) we have

$$\int_0^\infty e^{-\lambda u}\mathbb{E}(u - X_t)^+ du \approx \frac{1}{\lambda^2}\sum_{i=0}^n \frac{(-\lambda)^i}{i!}\mathbb{E}X_t^i = \sum_{i=0}^n \frac{(-\lambda)^{i-2}}{i!}\mathbb{E}X_t^i.$$

Bibligraphy

- Hull, J., White, A. *The pricing of options on assets with stochastic volatilities.* J. Finance 42, 281-300 (1987).
- Heyde C., Wong B. *On changes of measure in stochastic volatility models*. Int. J. Stoch. Anal, Volume 2006, Article ID 18130, 1-13 (2006).
- Jeanblanc M., Yor M., Chesney M. *Mathematical Methods for Financial Markets*. Springer-Verlag London 2009.
- Leblanc B. *Une approche unifiée pour une forme exacte du prix d'une option dans les différents modèles à volatilité stochastique.* Stochastics and Stochastics Reports 57, 1-35 (1996).
- Revuz D., Yor M. *Continous Martingales and Brownian Motion*. Springer-Verlag (3rd ed.). 2005.
- Sin C. *Complications with stochastic volatility models*. Adv. in Appl. Probab. 30, 256-268, (1998).

Thank you for attention !