A moment matching market implied calibration

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6th General AMaMeF and Banach Center Conference, Warsaw

June 14th, 2013
Motivation

“A derivative pricing model is said to be calibrated to a set of benchmark instruments if the value of these instruments, computed in the model, correspond to their market prices.” (Encyclopedia of Quantitative Finance)
Motivation II

- Standard calibration problem: minimize distance $f$ between market $\{P_i, \ i = 1, \ldots, M\}$ and model prices $\{\hat{P}_i, \ i = 1, \ldots, M\}$ of liquid derivatives
- typically perfect match not possible
- **optimal match:**

$$p^*: f(\{P_i\}, \{\hat{P}_i\}, p^*) \leq f(\{P_i\}, \{\hat{P}_i\}, p'), \quad p^*, p' \in p; \quad (1)$$

where $p = \text{model parameter set}$
- common choice of $f$:

$$f(\{P_i\}, \{\hat{P}_i\}, p) = \text{RMSE}(\{P_i\}, \{\hat{P}_i\}, p) = \sqrt{\sum_{i=1}^{M} \frac{(P_i - \hat{P}_i(p))^2}{M}}$$
Motivation III

- use $p^*$ to price exotic (illiquid) derivatives
- different $p^*$ $\Rightarrow$ different prices for exotics and different hedge ratios

Sources of calibration risk:

- Which error function $f$?

  ‘For complex products [...] banks must explicitly assess the need for valuation adjustments to reflect two forms of model risk: the model risk associated with using a possibly incorrect valuation methodology; and the risk associated with using unobservable (and possibly incorrect) calibration parameters in the valuation model.” (Basel Committee on Banking Supervision)
Motivation IV

- inverse problem (1)
  - ill-posed
  - instable solution for small changes in $\{P_i, \ i = 1, \ldots, M\} \Rightarrow$ significant impact on price of exotics/structured products
  - typically RMSE = non-convex function of $p \Rightarrow$ solution of (1) depends on initial values of $p$
  - computation time significant (need for numerical methods to compute model prices)

- potential solutions: relative entropy with respect to a prior model as
  - selection criterion
  - regularization of (1)

But solution still depends on choice of the prior and on starting values of $p$

$\Rightarrow$ alternative methodology: **moment matching market implied calibration**
Moment matching market implied calibration: concept I

\[ \mathbb{E}[g(S_T)] = g(\kappa) + g'(\kappa)(\exp((r - q)T)S_0 - \kappa) + \exp(rT) \left( \int_0^K g''(K)P(K,T)dK \right) + \int_\kappa^\infty g''(K)C(K,T)dK \]

\{ variance, skewness, kurtosis, ...\}
Moment matching market implied calibration: concept II

- $p^* = p$ compatible with market implied moments at one $T$
- market implied moments: vanilla option payoffs spanning formula (VIX, SKEW methodology)
- generalize CBOE methodology to extract 2nd to $(N + 1)$th moments from option surface ($N =$ size of $p$)
- match 2nd to $(N + 1)$th model and market implied moments
- $\Rightarrow$ moment matching calibration problem $\equiv$ system of $N$ algebraic equations which gives directly $p^*$ in terms of the moments observed in the market

**Applications:**
- calibration on one maturity
- deriving starting values for standard calibration problem
- deriving prior model
any twice differentiable payoff can be expanded as a weighted sum of vanilla option payoffs:

\[
\mathbb{E}[g(S_T)] = g(\kappa) + g'(\kappa) \exp(rT)(C(\kappa, T) - P(\kappa, T)) + \exp(rT) \left( \int_0^\kappa g''(K)P(K, T)dK + \int_\kappa^\infty g''(K)C(K, T)dK \right),
\]

where \(C(K, T)\) and \(P(K, T)\) are prices of European call and put options with maturity \(T\) and strike \(K\)

\(\kappa\) is some strike level

\(\Rightarrow\) closed-form expression for \(N\)th moment of \(X_T = \log\left(\frac{S_T}{S_0}\right)\):

\[
g(S_T) = \left(\log\left(\frac{S_T}{S_0}\right)\right)^N
\]
For $N \geq 2$

$$
\mathbb{E} \left[ \left( \log \frac{S_T}{S_0} \right)^N \right] = \left( \log \frac{K_0}{S_0} \right)^N + N \left( \log \frac{K_0}{S_0} \right)^{N-1} \left( \frac{F_0}{K_0} - 1 \right) + \exp(rT) \left( \int_0^{K_0} \frac{N}{K^2} \left( (N - 1) \left( \log \frac{K}{S_0} \right)^{N-2} - \left( \log \frac{K}{S_0} \right)^{N-1} \right) P(K, T) dK \right. \\
+ \left. \int_{K_0}^{\infty} \frac{N}{K^2} \left( (N - 1) \left( \log \frac{K}{S_0} \right)^{N-2} - \left( \log \frac{K}{S_0} \right)^{N-1} \right) C(K, T) dK \right)
$$

Vanilla options only traded for a discrete set of strikes

$$
\mathbb{E} \left[ \left( \log \frac{S_T}{S_0} \right)^N \right] = \left( \log \frac{K_0}{S_0} \right)^N + N \left( \log \frac{K_0}{S_0} \right)^{N-1} \left( \frac{F_0}{K_0} - 1 \right) + \exp(rT)N \\
\sum_{i=1}^{M} \frac{\Delta K_i}{K_i^2} \left( (N - 1) \left( \log \frac{K_i}{S_0} \right)^{N-2} - \left( \log \frac{K_i}{S_0} \right)^{N-1} \right) Q(K_i), \quad (3)
$$
Matching of standardized moments I

- Market implied variance $v$, skewness $s$ and kurtosis $k$:

\[
\begin{align*}
v &= \mathbb{E}\left[X_T^2\right] - (\mathbb{E}[X_T])^2 \\
s &= \frac{\mathbb{E}\left[X_T^3\right] - 3\mathbb{E}[X_T]\mathbb{E}[X_T^2] + 2(\mathbb{E}[X_T])^3}{(\text{Var}(X_T))^{3/2}} \\
k &= \frac{\mathbb{E}\left[X_T^4\right] - 4\mathbb{E}[X_T]\mathbb{E}[X_T^3] + 6(\mathbb{E}[X_T])^2\mathbb{E}[X_T^2] - 3(\mathbb{E}[X_T])^4}{(\text{Var}(X_T))^2},
\end{align*}
\]

- Closed-form formula for the model moments if characteristic function of $X_T = \log\left(\frac{S_T}{S_0}\right)$ known in closed-form:

\[
\mathbb{E}\left[X_T^N\right] = i^{-N} \frac{d^N}{du^N} \phi_{X_T}(u)|_{u=0} \quad (4)
\]

- $\phi_{X_T}(u)$ available for a wide range of asset pricing models
Matching of standardized moments II

- infer $p^*$ by matching 2nd to $(N + 1)$th model and market implied moments/standardized moments
- mean of $S_T$ adjusted beforehand
Examples: Lévy driven models

Stock price model

\[ S_t = \frac{S_0 \exp((r - q)t + X_t)}{\mathbb{E}_Q[\exp(X_t)]]} = S_0 \exp((r - q + \omega)t + X_t) \quad (5) \]

where \( X = \{X_t, t \geq 0\} \) is a Lévy process

\( \omega = -\log(\phi_X(-i)) \) (convexity correction)

characteristic function of \( X_T \) known in closed-form for many popular Lévy processes

Numerical study for Meixner model (VG and NIG models: see paper)
Moment calibration problem

- Exponential VG, NIG and Meixner models: 3 parameters calibrated by solving

\[
\begin{align*}
\text{Var}^{\text{Market}}(X_T) &= \text{Var}^{\text{Model}}(X_T) \\
\text{Skewness}^{\text{Market}}(X_T) &= \text{Skewness}^{\text{Model}}(X_T) \\
\text{Kurtosis}^{\text{Market}}(X_T) &= \text{Kurtosis}^{\text{Model}}(X_T)
\end{align*}
\]

if (6) admits a solution which satisfies the domain conditions of the model parameter set \( p \)

- If (6) admits no solution \( \Rightarrow \) modified moment calibration problem (adjust market implied skewness and kurtosis)
Meixner model I

Characteristic function of the Meixner distribution $\text{Meixner}(\alpha, \beta, \delta)$ with parameters $\alpha > 0, \beta \in ]-\pi, \pi[ \text{ and } \delta > 0$

$$\phi_{\text{Meixner}}(u; \alpha, \beta, \delta) = \left( \frac{\cos \left( \frac{\beta}{2} \right)}{\cosh \left( \frac{\alpha u - i \beta}{2} \right)} \right)^{2\delta}$$

<table>
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<tr>
<th></th>
<th>$X_1 \sim \text{Meixner}(\alpha, \beta, \delta)$</th>
<th>$X_T \sim \text{Meixner}(\alpha, \beta, \delta T)$</th>
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<td>$\frac{\alpha^2 \delta T}{2 \cos^2 \left( \frac{\beta}{2} \right)}$</td>
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<td>$\sin \left( \frac{\beta}{2} \right) \sqrt{\frac{2}{\delta T}}$</td>
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<tr>
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<td>$3 + \frac{2-\cos(\beta)}{\delta}$</td>
<td>$3 + \frac{2-\cos(\beta)}{\delta T}$</td>
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</table>
Meixner model II

system (6) reduces to

\[
\begin{align*}
\beta &= 2 \arctan \left( \text{sign}(s) \sqrt{\frac{s^2}{2(k-3)-3s^2}} \right) \\
\alpha &= \sqrt{vs} \cot \left( \frac{\beta}{2} \right) \\
\delta &= \frac{2-\cos(\beta)}{T(k-3)}.
\end{align*}
\]

Meixner system (7) admits a solution \(\{\alpha > 0, \beta \in ]-\pi, \pi[, \delta > 0\}\) iff

\[6 + 3s^2 - 2k < 0.\]

existence domain of the Meixner moment calibration problem independent of the market implied variance

only depends on the market implied skewness and kurtosis
Moment calibration problem: existence domain

\[ 6 + 3s^2 - 2k < 0 \quad \supset \quad 9 + 5s^2 - 3k < 0 \]

Figure: Existence domain for the moment calibration problem (6) under the VG and Meixner exponential models (left) and under the NIG exponential model (right).
Modified moment calibration problem I

Method 1: least squares moment calibration problem

- replace \((s^M, k^M)\) by \((s^A_1, k^A_1 + \epsilon)\)
- minimize distance between adjusted and market implied couples skewness-kurtosis
- \(\equiv\) allocating same importance to skewness and kurtosis matching in a standard calibration optimizer

Figure: Calibration methods.
Modified moment calibration problem II

Method 2: matching the skew
- replace \((s^M, k^M)\) by 
  \((s^A_2 = s^M, k^A_2 + \epsilon)\)
- match market implied skewness
- adjust kurtosis

**Figure:** Calibration methods.
Bootstrapping market implied moment matching calibration I

- Calibration of Markov models with piecewise constant parameters between successive quoted option maturities
- Exploit additive property of cumulants of independent random variables
- Solve $M$ independent moment matching systems of $N$ equations, where $M$ is the number of maturities
- Examples: Lévy models with piecewise constant parameters
Stock price model:

\[ S_t = S_{T_{i-1}} \exp \left( (r - q + \omega_i)(t - T_{i-1}) + X_{t-T_{i-1}}^{(i)} \right), \quad t \in [T_{i-1}, T_i) \]

where \( X^{(i)} = \{X_t^{(i)}, t \geq 0\}, \ i = 1, \ldots, M \) are independent Lévy processes

- \( T_0 \equiv 0 \)
- \( \omega_i = -\log \left( \phi_{X_1^{(i)}}(-i) \right), \ i = 1, \ldots M \) (convexity corrections)

\[ S_t = S_0 \prod_{j=1}^{i-1} \exp \left( (r - q + \omega_j)(T_j - T_{j-1}) + X_{T_{j-1}-T_j}^{(j)} \right) \]
\[ \times \exp \left( (r - q + \omega_i)(t - T_{i-1}) + X_{t-T_{i-1}}^{(i)} \right), \quad t \in [T_{i-1}, T_i]. \]
Matching of the subprocesses standardized moments I

- match market implied and model moments of the subprocesses $X_t^{(i)}$, $t \in [T_{i-1}, T_i), i = 1, 2, \ldots, M$
- market implied formula (3): approximation of the moments of $X_t$ over the periods $[0, T_i]$
- need for similar approximations for the moments of $X_t$ over the subperiods $[T_{i-1}, T_i], i = 1, \ldots, M$
- $\Rightarrow$ consider cumulants as latent variables
- compute recursively the cumulants of $X^{(i)}$, $i = 1, \ldots, M$:

$$
\kappa_n \left( X_{T_i-T_{i-1}} \right) = \kappa_n \left( X_{T_i} \right) - \sum_{j=1}^{i-1} \kappa_n \left( X_{T_j-T_{j-1}} \right), \quad n = 2, \ldots, N + 1
$$
Matching of the subprocesses standardized moments II

- Market implied variance $v_i$, skewness $s_i$ and kurtosis $k_i$, $i = 1, \ldots, M$:

\[
\begin{align*}
V_i &= \kappa_2 \left( X_{T_i - T_{i-1}}^{(i)} \right) \\
S_i &= \frac{\kappa_3 \left( X_{T_i - T_{i-1}}^{(i)} \right)}{v_i^{3/2}} \\
k_i &= \frac{\kappa_4 \left( X_{T_i - T_{i-1}}^{(i)} \right)}{v_i^2} + 3,
\end{align*}
\]
Matching of the subprocesses standardized moments III

- solve successively the moment system

\[
\begin{align*}
\text{Var}^{\text{Model}}\left(X_{T_i-T_{i-1}}^{(i)}\right) &= v_i \\
\text{Skewness}^{\text{Model}}\left(X_{T_i-T_{i-1}}^{(i)}\right) &= s_i \\
\text{Kurtosis}^{\text{Model}}\left(X_{T_i-T_{i-1}}^{(i)}\right) &= k_i
\end{align*}
\]

(9)

if (9) admits a solution which satisfies the domain conditions of the parameter set \(p_i\) of the subprocess \(X^{(i)}\)

- if (9) admits no solution \(\Rightarrow\) modified moment calibration problem (i.e. replace \(s_i\) by \(s_i^{A_1}\) (or \(s_i^{A_2}\)) and \(k_i\) by \(k_i^{A_1}\) (or \(k_i^{A_2}\)))
Numerical study: calibration on one maturity

- Comparison with standard calibration problem: Carr-Madan formula

\[ \hat{C}(K, T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_{0}^{+\infty} \exp(-iv \log(K)) \varrho(v) dv, \]

where

\[ \varrho(v) = \exp(-rT) \mathbb{E}[\exp(i(v - (\alpha + 1)i) \log(S_T))] \frac{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v}{\alpha^2 + \alpha - v^2 + i(2\alpha + 1)v} \]

- Calibration of VG, NIG and Meixner models on each $T$-set of S&P 500 options quoted on 30/10/09
## Numerical study: calibration on one maturity II

Table: Standardized moments - Meixner exponential model (30/10/2009)

<table>
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<tr>
<th>T</th>
<th>Market Var</th>
<th>Market Skew</th>
<th>Market Kurt</th>
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### Numerical study: calibration on one maturity III

**Table:** Optimal parameter set - Meixner exponential model (30/10/2009)

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Numerical study: calibration on one maturity IV

- $v$ can always be reproduced under moment calibration problem (existence domain independent of $v$)
- standard calibration: difference between model and market implied var per annum of up to $> 15 \%$ ($\max \frac{\Delta \sigma}{\sigma} = 0.07336$)
- $v$, $s$ and $k$ can be matched for a wider set of $T$’s under VG and Meixner models
- if $(s, k) \notin$ existence domain, $s$ and $k$ still fitted relatively well
- standard calibration: significant discrepancies between model and market implied moments
- if $\not\exists$ solution for moment calibration problem (6), method 2 for the modified moment calibration problem gives a better fit of the volatility curve than method 1
- moment calibration procedure: more than 5000 x faster
### Numerical results: term structure Lévy models I

#### Table: Optimal parameter set - Meixner term structure model

<table>
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<th>Moment matching (1)</th>
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### Numerical results: term structure Lévy models II

#### Table: Optimal parameter set - Meixner term structure model

(30/10/2009)

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Numerical results: term structure Lévy models III

- Variance fit − Meixner (10/10/08)
- Skewness fit − Meixner (10/10/08)
- Kurtosis fit − Meixner (10/10/08)
- Variance fit − Meixner (30/10/09)
- Skewness fit − Meixner (30/10/09)
- Kurtosis fit − Meixner (30/10/09)
### Numerical results: term structure Lévy models IV

**Table**: Average precision and computation time for the different calibration methodologies.

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**Conclusion**

- market implied calibration based on moment matching
- extracting moments from option surface
- calibration in terms of closed-form formulae only $\Rightarrow$
  - no need of starting value for the model parameters
  - avoids to be stuck in local minima
  - almost instantaneous calibration
- applications
  - calibration on one maturity curve
  - calibration on whole set of maturities: bootstrapping calibration
  - providing starting values/prior model for standard calibration problems


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