A Stochastic Reversible Investment Problem on a Finite-Time Horizon: Free-Boundary Analysis

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Outline

A simple Economic model

- A firm produces a single good in a stochastic economy on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$
- Finite time-horizon $[0, T]$
- Production rate $R(\cdot)$ depending on a stochastic production capacity $\{C(t)\}_{t \in [0,T]}$
- The manager controls C via a control $\{\nu(t)\}_{t \in [0,T]}$, i.e. $C \equiv C^{\nu}$
- Investment and disinvestment are allowed, i.e. $\nu = (\nu_+, \nu_-)$
- Price of the produced good, cost of investment and benefit from disinvestment are constant

The manager's optimisation problem

$$\sup_{\nu} \mathbb{E}\Big\{ \int_0^T e^{-\mu_F t} R(C^{\nu}(t))dt \\ - c_+ \int_0^T e^{-\mu_F t} d\nu_+(t) + c_- \int_0^T e^{-\mu_F t} d\nu_-(t) + e^{-\mu_F T} G(C^{\nu}(T)) \Big\} \qquad (1)$$

with $\mu_F > 0$ manager's discount factor, $c_+ > 0$ cost of investment, $c_- > 0$ benefit form disinvestment, $c_+ > c_-$ and G a terminal reward.

Where does the model come from?

Literature on singular stochastic control is huge... this work is mostly inspired by

CH09 M.B. Chiarolla, U.G. Haussmann (2009). *On a Stochastic Irreversible Investment Problem*. SIAM J. Control Optim. 48

GP05 X. Guo, H. Pham (2005). *Optimal Partially Reversible Investment with Entry Decision and General Production Function*, Stochastic Process. Appl. 115

New features

- CH09 have finite horizon but irreversible investment, i.e. $\nu \equiv \nu_+$ and $t \mapsto \nu_+(t,\omega)$ monotone increasing $\mathbb{P}$-a.e. $\omega \in \Omega$
- GP05 have reversible investment, random entry time but $T = +\infty$

What changes?

- Reversible investment is linked to Zero-Sum optimal stopping games (rather than canonical optimal stopping problems as in CH09)
- $T < +\infty$ implies that the *inaction* region of the manager is delimited by two curves (rather than two points as in GP05) which are the free-boundaries of the associated Zero-Sum game

Main Goals of this work

- Existence and uniqueness of an optimal strategy $\nu^* = (\nu^*_+, \nu^*_-)$ in the class of bounded variation controls
- Study of the associated Zero-Sum optimal stopping game (ZSG): existence of Nash equilibrium, optimal stopping times (τ^*, σ^*), free-boundary problem for its value function
- Analysis of the time dependent free-boundaries $\hat{y}_+, \hat{y}_-$ of the ZSG and their representation as unique solution pair of coupled non-linear, integral equations of Volterra type (in the spirit of Peskir-Shiryaev 2006)
- Characterisation of ν^* via the solution of a Skorokhod problem in the time-dependent interval $(\hat{y}_+(t), \hat{y}_-(t))$, $t \in [0, T]$

The controlled dynamics

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $\{W(t)\}_{t\geq 0}$ a 1-dim Brownian motion and $\{\mathcal{F}_t\}_{t\geq 0}$ its natural filtration augmented by $\mathbb{P}$-null sets.

Take μ_C, σ_C and f_C positive constants, then

$$\begin{cases} dC^{y,\nu}(t) = C^{y,\nu}(t)[-\mu_C\, dt + \sigma_C\, dW(t)] + f_C\, d\nu(t), \;\; t \geq 0, \\ C^{y,\nu}(0) = y > 0, \end{cases} \tag{2}$$

where $f_C\, d\nu$ accounts for the net effect of investment-disinvestment on the production capacity.

$$\nu \in \mathcal{S} := \{\nu : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+ \text{ of B.V., l.c., adapted s.t. } \nu(0) = 0, \; \mathbb{P}\text{-a.s.}\}$$

and $\nu := \nu_+ - \nu_-$ with $\nu_\pm \in \mathcal{S}$ and increasing (minimal decomposition). It can be proven that

$$C^{y,\nu}(t) = C^0(t)[y + \overline{\nu}(t)]$$

with

$$C^0(t) := e^{-(\mu_C + \frac{1}{2}\sigma_C^2)t + \sigma_C W(t)} \quad \text{and} \quad \overline{\nu}(t) := \int_0^t \frac{f_C}{C^0(s)}\, d\nu(s)$$

Production function

Standard Assumptions:

i) $C \mapsto R(C)$ is nondecreasing with $R(0) = 0$ and strictly concave

ii) R is twice continuously differentiable on $(0, \infty)$

iii) $R_c(C) := \frac{\partial}{\partial C} R(C)$ satisfies Inada conditions

$$\lim_{C \to 0} R_c(C) = \infty \qquad \& \qquad \lim_{C \to \infty} R_c(C) = 0.$$

A classical example is a Cobb-Douglas type, i.e. $R(C) = \alpha^{-1} C^{\alpha}$ for $\alpha \in (0,1)$

The *scrap* value

$G : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is concave, nondecreasing, continuously differentiable with

$$\frac{c_-}{f_C} \leq G_c(C) \leq \frac{c_+}{f_C} - \eta_o$$

for a fixed $\eta_o \in \left(0, \frac{c_+ - c_-}{f_C}\right)$.

The manager's problem

The firm's future total expected profit at time $t \in [0, T]$ is given by

$$\mathcal{J}_{t,y}(\nu) = \mathbb{E}\bigg\{ \int_0^{T-t} e^{-\mu_F s} R(C^{y,\nu}(s))ds + e^{-\mu_F (T-t)} G(C^{y,\nu}(T-t)) - c_+ \int_0^{T-t} e^{-\mu_F s} d\nu_+(s) + c_- \int_0^{T-t} e^{-\mu_F s} d\nu_-(s) \bigg\} \quad (3)$$

The value V of the optimal investment-disinvestment problem is

$$V(t,y) := \sup_{\nu \in \mathcal{S}^y_{t,T}} \mathcal{J}_{t,y}(\nu) \quad (4)$$

with $\mathcal{S}^y_{t,T} := \big\{ \nu \in \mathcal{S}$ restricted to $[0, T-t]$ and s.t. $y + \overline{\nu}(s) \geq 0$ $\mathbb{P}$-a.s for $s \in [0, T-t] \big\}$.

Theorem [Existence and uniqueness of an optimal control]

There exists a unique investment-disinvestment strategy $\nu^* \in \mathcal{S}^y_{t,T}$ optimal for (4).

Proof. Uniqueness by strict concavity of $R(\cdot)$ and hence of $V(t,\cdot)$; existence by an application of a version of Komlòs theorem by DeVDK09 (De Vallière-Denis-Kabanov (2009)).

- In what follows we consider a linear scrap value

$$G(C) = \kappa + \frac{c_-}{f_C} C$$

for some $\kappa \geq 0$

- We define a probability measure $\tilde{\mathbb{P}}$ by

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := e^{-\frac{1}{2}\sigma_C^2 t + \sigma_C W(t)}, \ t \geq 0$$

and a new Brownian motion under $\tilde{\mathbb{P}}$

$$\tilde{W}(t) := W(t) - \sigma_C t, \qquad t \geq 0$$

by Girsanov theorem

In our setting we can use a result by KW05 (I. Karatzas, H. Wang (2005)) (and a change of measure) to obtain

Theorem [From the BV control problem to a zero-sum game]

The value function $V(t,y)$ of the control problem (4) satisfies

$$\frac{\partial}{\partial y} V(t,y) = v(t,y), \quad (t,y) \in [0,T] \times (0,\infty),$$

where

$$v(t,y) := \inf_{\sigma\in[0,T-t]} \sup_{\tau\in[0,T-t]} \Psi(t,y;\sigma,\tau) = \sup_{\tau\in[0,T-t]} \inf_{\sigma\in[0,T-t]} \Psi(t,y;\sigma,\tau)$$

with

$$\Psi(t,y;\sigma,\tau) := \tilde{\mathbb{E}}\Big\{ \frac{c_+}{f_C} e^{-\bar{\mu}\sigma} \mathbb{I}_{\{\sigma\leq\tau\}} \mathbb{I}_{\{\sigma<T-t\}} + \frac{c_-}{f_C} e^{-\bar{\mu}\tau} \mathbb{I}_{\{\tau<\sigma\}} + e^{-\bar{\mu}(T-t)} \frac{c_-}{f_C} \mathbb{I}_{\{\tau=\sigma=T-t\}} + \int_0^{\tau\wedge\sigma} e^{-\bar{\mu}s} R_c(yC^0(s))ds \Big\}$$

and $\bar{\mu} := \mu_C + \mu_F$.

Remark: The control has "disappeared" and we deal now with an uncontrolled GBM $\{yC^0(t)\}_{t\geq 0}$. KW05 characterise Nash equilibrium in terms of the optimal control ν^*

Simple bounds: $c_-/f_C \le v(t,y) \le c_+/f_C$ for all $(t,y) \in [0,T] \times (0,\infty)$.

Theorem: [Continuity]

$(t,y) \mapsto v(t,y)$ is continuous on $[0,T] \times (0,\infty)$.

Proof. Penalisation method adapting arguments by Me80 (J.L. Menaldi (1980)) and St11 (L. Stettner (2011)).

Byproduct No. 1 of the proof of continuity:

Theorem [Optimal stopping times]

The stopping times

$$\begin{cases} \sigma^*(t,y) := \inf\{s \in [0,T-t) : v(t+s, yC^0(s)) \ge \frac{c_+}{f_C}\} \wedge (T-t), \\ \tau^*(t,y) := \inf\{s \in [0,T-t) : v(t+s, yC^0(s)) \le \frac{c_-}{f_C}\} \wedge (T-t), \end{cases} \tag{5}$$

are a saddle point for the ZSG.

Byproduct No. 2 of the proof of continuity:

Proposition [Semi-harmonic characterisation]

For $(t,y) \in [0,T] \times (0,\infty)$ arbitrary but fixed and $\rho \in [0, T-t]$ any stopping time, v satisfies

$$[SUB] \quad v(t,y) \le \tilde{\mathbb{E}}\Big\{ e^{-\bar{\mu}(\rho\wedge\tau^*)} v(t+\rho\wedge\tau^*, yC^0(\rho\wedge\tau^*)) + \int_0^{\rho\wedge\tau^*} e^{-\bar{\mu}s} R_c(yC^0(s))\, ds \Big\}$$

$$[SUP] \quad v(t,y) \ge \tilde{\mathbb{E}}\Big\{ e^{-\bar{\mu}(\sigma^*\wedge\rho)} v(t+\sigma^*\wedge\rho, yC^0(\sigma^*\wedge\rho)) + \int_0^{\sigma^*\wedge\rho} e^{-\bar{\mu}s} R_c(yC^0(s))\, ds \Big\}$$

$$[MG] \quad v(t,y) = \tilde{\mathbb{E}}\Big\{ e^{-\bar{\mu}(\rho\wedge\sigma^*\wedge\tau^*)} v(t+\rho\wedge\sigma^*\wedge\tau^*, yC^0(\rho\wedge\sigma^*\wedge\tau^*)) + \int_0^{\rho\wedge\sigma^*\wedge\tau^*} e^{-\bar{\mu}s} R_c(yC^0(s))\, ds \Big\}$$

cf. for instance Pe09 (G. Peskir (2009)) for more details on semi-harmonic characterisation of ZSGs

Continuation and stopping regions

- The *continuation region* is the open set

$$\mathcal{C} := \left\{(t,y) \in [0,T] \times (0,\infty) : \frac{c_-}{f_C} < v(t,y) < \frac{c_+}{f_C}\right\}$$

- The two *stopping regions* are the closed sets

$$\mathcal{S}_+ := \left\{(t,y) \in [0,T] \times (0,\infty) : v(t,y) = \frac{c_+}{f_C}\right\}$$

$$\mathcal{S}_- := \left\{(t,y) \in [0,T] \times (0,\infty) : v(t,y) = \frac{c_-}{f_C}\right\}$$

Proposition [Existence of the free-boundaries]

For any $t \in [0,T]$, there exist $\hat{y}_+(t) < \hat{y}_-(t)$ such that

$$\mathcal{C}_t = (\hat{y}_+(t), \hat{y}_-(t)) \subset [0,\infty)$$

$$\mathcal{S}_{+,t} = [0, \hat{y}_+(t)] \qquad \& \qquad \mathcal{S}_{-,t} = [\hat{y}_-(t), \infty)$$

Proof. Follows from $y \mapsto v(t,y)$ monotone.

Remark: optimal stopping times (σ^*, τ^*) are first entry times of $(t, yC^0(t))_{t \geq 0}$ to $\mathcal{S}_+$ and $\mathcal{S}_-$, respectively.

The free-boundary problem for v

Properties above + standard arguments (cf. G. Peskir, A. Shiryaev (2006)) imply that $v \in C^{1,2}$ inside the continuation region $\mathcal{C}$ and it solves

$$\begin{cases} [MG] \quad \big(\partial_t + \mathcal{L} - \bar{\mu}\big) v(t,y) = -R_c(y) & \text{for } \hat{y}_+(t) < y < \hat{y}_-(t),\ t \in [0,T) \\ [SUP] \quad \big(\partial_t + \mathcal{L} - \bar{\mu}\big) v(t,y) \le -R_c(y) & \text{for } y > \hat{y}_+(t),\ t \in [0,T) \\ [SUB] \quad \big(\partial_t + \mathcal{L} - \bar{\mu}\big) v(t,y) \ge -R_c(y) & \text{for } y < \hat{y}_-(t),\ t \in [0,T) \\ \frac{c_-}{f_C} \le v(t,y) \le \frac{c_+}{f_C} & \text{in } [0,T] \times (0,\infty) \\ v(t,\hat{y}_\pm(t)) = \frac{c_\pm}{f_C} & t \in [0,T) \quad \text{(continuous-pasting)} \\ v(T,y) = \frac{c_-}{f_C} & y > 0 \end{cases}$$

with

$$\mathcal{L}f := \frac{1}{2}\sigma_C^2 y^2 f'' + \big(\hat{\mu}_C + \sigma_C^2/2\big) y f' \qquad \text{for } f \in C_b^2((0,\infty))$$

the infinitesimal generator of $\{C^0(t)\}_{t\ge 0}$ under $\tilde{\mathbb{P}}$.

We expect that the *smooth-pasting* holds at the two boundaries. It will be proved later.

There is a close link to HJB equation!

It follows by semi-harmonic characterisation that

$$t \mapsto v(t,y) \text{ decreasing for each } y \in (0,\infty)$$

Proposition [Some properties of $(\hat{y}_+, \hat{y}_-)$]

i) $\hat{y}_+(t)$ and $\hat{y}_-(t)$ are decreasing;
ii) $\hat{y}_+(t)$ is left-continuous and $\hat{y}_-(t)$ is right-continuous;
iii) $0 < \hat{y}_+(t) < R_c^{-1}(\frac{\bar{\mu}c_+}{f_C})$, for $t \in [0,T)$;
iv) $\lim_{t\uparrow T} \hat{y}_+(t) =: \hat{y}_+(T) = 0$;
v) $0 < R_c^{-1}(\frac{\bar{\mu}c_-}{f_C}) < \hat{y}_-(t) < +\infty$, for $t \in [0,T)$;
vi) $\lim_{t\uparrow T} \hat{y}_-(t) =: \hat{y}_-(T-) = R_c^{-1}(\frac{\bar{\mu}c_-}{f_C})$.

Theorem [Continuity of the free-boundaries]

$t \mapsto \hat{y}_+(t)$ and $t \mapsto \hat{y}_-(t)$ are continuous on $[0,T]$.

Proof. Follows from PDE + probabilistic arguments (cf. DeA13)

A technical Assumption [Needed to prove smooth-pasting at $\hat{y}_-$]

For any $y_o > R_c^{-1}(\bar{\mu} c_- / f_C)$ there exists $\delta_o := \delta_o(y_o)$ such that

$$\tilde{\mathbb{E}}\left\{ \int_0^T e^{-\bar{\mu}s} \inf_{\{y:|y-y_o|\le\delta_o\}} R_{cc}(yC^0(s))\, ds \right\} > -\infty. \tag{6}$$

Since R_{cc} is continuous away from zero and C^0 is a GBM, it works for most of the examples. Benchmark example $R(C) = \alpha^{-1} C^{\alpha}$, $\alpha \in (0,1)$.

Theorem [Smooth-pasting]

It holds

$$v_y(t, \hat{y}_-(t)-) = 0, \qquad t \in [0,T) \tag{7}$$

$$v_y(t, \hat{y}_+(t)+) = 0, \qquad t \in [0,T) \tag{8}$$

Proof. (7) follows from standard arguments + (6).

(8) requires ad hoc arguments inspired by Pe07 (G. Peskir (2007)).

From an application of the so-called *local time-space calculus* Pe05 (G. Peskir (2005)) we obtain the following

Theorem [Integral equations for v, $\hat{y}_+$ and $\hat{y}_-$]

Pt.1. The value function v has the following representation

$$v(t,y) = e^{-\bar{\mu}(T-t)}\frac{c_-}{f_C} + \int_0^{T-t} e^{-\bar{\mu}s}\tilde{\mathbb{E}}\Big\{R_c(yC^0(s))\mathbb{I}_{\{\hat{y}_+(t+s)<yC^0(s)<\hat{y}_-(t+s)\}}\Big\}ds$$
$$+\frac{\bar{\mu}}{f_C}\int_0^{T-t} e^{-\bar{\mu}s}\Big[c_+\tilde{\mathbb{P}}\Big(yC^0(s)<\hat{y}_+(t+s)\Big)+c_-\tilde{\mathbb{P}}\Big(yC^0(s)>\hat{y}_-(t+s)\Big)\Big]ds$$

1. Set $y := \hat{y}_\pm(t)$
2. Use $v(t,\hat{y}_\pm(t)) = c_\pm/f_C$

to find equations for the free-boundaries.

Theorem [Integral equations for v, $\hat{y}_+$ and $\hat{y}_-$]

Pt.2. $\hat{y}_+$ and $\hat{y}_-$ are continuous, decreasing curves solving

$$\frac{c_-}{f_C} = F_1(t, \hat{y}_-(t), \hat{y}_-(t+\cdot), \hat{y}_+(t+\cdot)) \tag{9}$$

$$\frac{c_+}{f_C} = F_2(t, \hat{y}_+(t), \hat{y}_-(t+\cdot), \hat{y}_+(t+\cdot)) \tag{10}$$

for suitable functionals F_1, F_2 and given boundary conditions.

The good news is we can find numerical solutions to (9) and (10). Another good news is...

Theorem [Uniqueness]

The couple $(\hat{y}_+(t), \hat{y}_-(t))$ is the unique solution of the integral equations above in the class of continuous and decreasing functions.

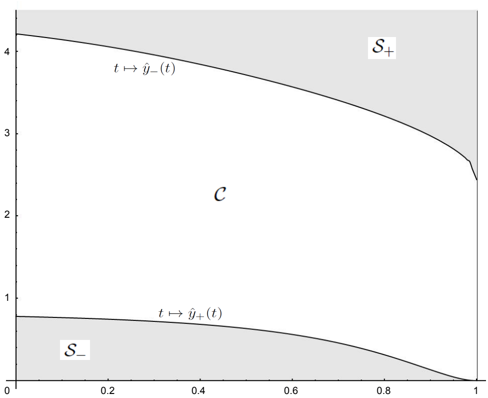

Figure: A computer drawing of the free-boundaries obtained by numerical solution of integral equations with $R_c(y) = 1/\sqrt{y}$, $\bar{\mu} = 0.8$, $\mu_C = 0.2$, $\sigma_C = 1$, $f_C = 1$, $c_+ = 1$, $c_- = 0.8$ and $T = 1$. The lower line represents $\hat{y}_+$ and the upper line represents $\hat{y}_-$.

The next two results are based on BKR09 (K. Burdzy, W. Kang, K. Ramanan (2009).

We solve the problem of finding ν s.t. $C^{y,\nu}$ is constrained between the two boundaries with a minimal effort

Theorem Pt.1 [The Skorokhod problem - Existence & Uniqueness]

Let $t \in [0,T]$ and $y > 0$ be arbitrary but fixed. Given $\hat{y}_+$ and $\hat{y}_-$ there exists a unique left-continuous adapted process of bounded variation $\overline{\nu}^* = \overline{\nu}^*_+ - \overline{\nu}^*_- \in \mathcal{S}$ such that

$$\begin{cases} C^{y,\overline{\nu}^*}(s) = C^0(s)[y + \overline{\nu}^*_+(s) - \overline{\nu}^*_-(s)], \quad s \in [0, T-t) \\ C^{y,\overline{\nu}^*}(0) = y, \\ \hat{y}_+(t+s) \le C^{y,\overline{\nu}^*}(s) \le \hat{y}_-(t+s), \quad \text{a.e. } s \in [0, T-t], \\ \int_0^{T-t} \mathbb{I}_{\{C^{y,\overline{\nu}^*}(s) < \hat{y}_-(t+s)\}} d\overline{\nu}^*_-(s) = 0, \qquad \int_0^{T-t} \mathbb{I}_{\{C^{y,\overline{\nu}^*}(s) > \hat{y}_+(t+s)\}} d\overline{\nu}^*_+(s) = 0 \end{cases}$$

hold $\tilde{\mathbb{P}}$-a.s.

Moreover, if $y \in [\hat{y}_+(t), \hat{y}_-(t)]$ then $\overline{\nu}^*_+(\omega,\cdot)$ and $\overline{\nu}^*_-(\omega,\cdot)$ are continuous. When $y < \hat{y}_+(t)$, then $\overline{\nu}^*_+(\omega,0+) = \hat{y}_+(t) - y$, $\overline{\nu}^*_-(\omega,0+) = 0$ and $C^{y,\overline{\nu}^*}(\omega,0+) = \hat{y}_+(t)$; when $y > \hat{y}_-(t)$, then $\overline{\nu}^*_-(\omega,0+) = y - \hat{y}_-(t)$, $\overline{\nu}^*_+(\omega,0+) = 0$ and $C^{y,\overline{\nu}^*}(\omega,0+) = \hat{y}_-(t)$.

Theorem Pt.2 [The Skorokhod problem - Characterisation]

The solution $\overline{\nu}^*$ is

$$\overline{\nu}^*(s+) = -\max\left\{\left[\left(y-\hat{y}_-(t)\right)^+ \wedge \inf_{u\in[0,s]}\left(\frac{yC^0(u)-\hat{y}_+(t+u)}{C^0(u)}\right)\right],\right.$$

$$\left.\sup_{r\in[0,s]}\left[\left(\frac{yC^0(r)-\hat{y}_-(t+r)}{C^0(r)}\right)\wedge \inf_{u\in[r,s]}\left(\frac{yC^0(u)-\hat{y}_+(t+u)}{C^0(u)}\right)\right]\right\}$$

for every $s \in [0, T-t)$.

A verification theorem

Applying Itô's formula for general semi-martingales to

$$e^{-\mu_F s} V(t+s, C^{y,\overline{\nu}^*}(s)) \qquad s \in [0, T-t]$$

under $\mathbb{P}$, using HJB equation and above results on Skorokhod problem... we finally prove optimality of $\overline{\nu}^*$.

Conclusions

1. Started off a firm's manager investment-disinvestment problem on a finite time-horizon
2. Formulated a singular stochastic control problem
3. Proved existence and uniqueness of an optimal policy
4. Established a link with a Zero-Sum game
5. Studied the associated free-boundaries
6. Characterised the manager's optimal policy in terms of the free-boundaries

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Grazie.

Theorem [Integral equations for v, $\hat{y}_+$ and $\hat{y}_-$]

Pt.2. $\hat{y}_+$ and $\hat{y}_-$ are continuous, decreasing curves solving the coupled integral equations

$$\frac{c_-}{f_C} = e^{-\bar{\mu}(T-t)}\frac{c_-}{f_C} + \int_0^{T-t} e^{-\bar{\mu}s}\tilde{\mathbb{E}}\Big\{R_c(\hat{y}_-(t)C^0(s))\mathbb{I}_{\{\hat{y}_+(t+s)<\hat{y}_-(t)C^0(s)<\hat{y}_-(t+s)\}}\Big\}ds$$
$$+\frac{\bar{\mu}}{f_C}\int_0^{T-t} e^{-\bar{\mu}s}\Big[c_+\tilde{\mathbb{P}}\Big(\hat{y}_-(t)C^0(s)<\hat{y}_+(t+s)\Big)+c_-\tilde{\mathbb{P}}\Big(\hat{y}_-(t)C^0(s)>\hat{y}_-(t+s)\Big)\Big]ds$$

and

$$\frac{c_+}{f_C} = e^{-\bar{\mu}(T-t)}\frac{c_-}{f_C} + \int_0^{T-t} e^{-\bar{\mu}s}\tilde{\mathbb{E}}\Big\{R_c(\hat{y}_+(t)C^0(s))\mathbb{I}_{\{\hat{y}_+(t+s)<\hat{y}_+(t)C^0(s)<\hat{y}_-(t+s)\}}\Big\}ds$$
$$+\frac{\bar{\mu}}{f_C}\int_0^{T-t} e^{-\bar{\mu}s}\Big[c_+\tilde{\mathbb{P}}\Big(\hat{y}_+(t)C^0(s)<\hat{y}_+(t+s)\Big)+c_-\tilde{\mathbb{P}}\Big(\hat{y}_+(t)C^0(s)>\hat{y}_-(t+s)\Big)\Big]ds$$

for $t \in [0,T)$, with boundary conditions

$$\hat{y}_-(T) = R_c^{-1}\left(\frac{\bar{\mu}c_-}{f_C}\right) \quad \& \quad \hat{y}_+(T) = 0 \tag{11}$$

and such that

$$R_c^{-1}\left(\frac{\bar{\mu}c_-}{f_C}\right) < \hat{y}_-(t) < +\infty \quad \& \quad 0 < \hat{y}_+(t) < R_c^{-1}\left(\frac{\bar{\mu}c_+}{f_C}\right) \quad \text{for all } t \in [0,T). \tag{12}$$

The HJB equation

From the dynamic programming principle one has

$$\min\Big\{-R+\mu_F V-\mathcal{D}V-V_t\,,\,c_+/f_C-V_y\,,\,V_y-c_-/f_C\Big\}=0 \qquad (t,y)\in[0,T]\times\mathbb{R}_+ \tag{13}$$

$$V(T,y)=\frac{c_-}{f_C}y+\kappa \qquad y\in\mathbb{R}_+ \tag{14}$$

with

$$\mathcal{D}V:=\sigma_C^2/2y^2V_{yy}-\mu_C yV_y$$

if V is regular enough for (13) to be well defined.

Inside $\mathcal{C}$ we have

$$c_+/f_C-V_y>0,\quad V_y-c_-/f_C>0 \qquad \text{and} \quad V_t+\mathcal{D}V-\mu_F V=-R$$

From properties of v we have

$$V_t,\,V_y,\,V_{yy}\in L^\infty([0,T]\times\mathbb{R}_+)$$

therefore

1. (13) may be interpreted in a weak sense
2. the optimal strategy should be: "keep $(t,C_t^\nu)_{t\geq 0}$ inside $\mathcal{C}$ in a *minimal* way"