Computation of conditional expectations in jump-diffusion setting

Applied to pricing and hedging of financial products

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Option pricing and hedging

S: stock price process

T: time of maturity

r: risk-free interest rate

Φ: payoff function

- European option

$$P^{Eu}(t, S_t) = e^{-r(T-t)}\mathbb{E}[\Phi(S_T)|S_t],$$
$$\Delta^{Eu}(t, S_t) = \frac{\partial}{\partial S_t}P^{Eu}(t, S_t),$$

- American option pricing and hedging

Conditional expectations $\mathbb{E}[f(S_t)|S_s = \alpha], \ \alpha \in \mathbb{R}^+, s < t$

- Bayes rule

$$\mathbb{E}[f(S_t)|S_s = \alpha] = \frac{\mathbb{E}[f(S_t)\delta_0(S_s - \alpha)]}{\mathbb{E}[\delta_0(S_s - \alpha)]}$$

- For continuous stock price processes S:

$$\mathbb{E}[f(S_t)|S_s = \alpha] = \frac{\mathbb{E}[f(S_t)H(S_s - \alpha)\pi]}{\mathbb{E}[H(S_s - \alpha)\pi]},$$

$H(x) = \mathbf{1}_{\{x>0\}} + c$, $c \in \mathbb{R}$

See Fournié et al. (1999) ,Fournié et al. (2001).

Methods

- For discontinuous stock price processes:

$$\mathbb{E}[f(S_t)|S_s = \alpha] = \frac{\mathbb{E}[f(S_t)H(S_s - \alpha)\pi]}{\mathbb{E}[H(S_s - \alpha)\pi]}, \quad s < t\,,$$

  $H(x) = \mathbf{1}_{\{x>0\}} + c$, based on two methods:

  - Conditional density method
  - Malliavin approach

Outline

I- Conditional density method

1. Representations
2. Optimal weights
3. Delta

II- Malliavin calculus

1. Representations
2. Delta

III- Numerical methods and results

1. Americain options
2. Monte Carlo simulation
3. Numerical example

Outline

I- **Conditional density method**

1.Representations

Conditional density method for $\mathbb{E}[f(F)|G=\alpha]$

F and G are two random variables such that

$$F = g_1(X, Y) \qquad \text{and} \qquad G = g_2(U, V),$$

- (X, U) independent of (Y, V)
- X and U allowed to be dependent, joint density $p_{(X,U)}$

Under the necessary assumptions

$$\mathbb{E}[f(F)|G=\alpha] = \frac{\mathbb{E}[f(F)H(G-\alpha)\pi_{(X,U)}]}{\mathbb{E}[H(G-\alpha)\pi_{(X,U)}]},$$

where $H = \mathbf{1}_{\{x>0\}} + c$ and

$$\pi_{(X,U)} = -\frac{\partial}{\partial u} \log p_{(X,U)}(X, U).$$

See Benth et al. (2010).

CDM applied to an exponential Lévy process

Observe $S = S_0 e^L$, where L is a Lévy process with decomposition

$$L_t = at + bW_t + \widetilde{N}_t, \quad t \in [0, T]$$

W is a standard Brownian motion,

$\widetilde{N}$ is a compound Poisson process, independent of W

$$(X, U) = (bW_t, bW_s)$$

For any Borel measurable function f and positive number α,

$$\mathbb{E}[f(S_t)|S_s = \alpha] = \frac{\mathbb{E}[f(S_t)H(S_s - \alpha)\pi]}{\mathbb{E}[H(S_s - \alpha)\pi]},$$

where

$$\pi = \frac{tW_s - sW_t}{bs(t-s)}$$

Examples of stock price processes

$$\mathbb{E}[f(S_t)|S_s = \alpha], \qquad s < t$$

- Geometric Brownian motion $\longrightarrow$ regular density method
- Additive model $S = A + B$, where A is an Ornstein-Uhlenbeck process
- Approximation of jump process by replacing the small jumps by a scaled Brownian motion, e.g. NIG process

See Asmussen and Rosinski (2001).

2.Optimal weights

- For any π in $\mathcal{W} := \left\{\pi : \mathbb{E}[\pi|\sigma(F,G)] = \pi_{(X,U)}\right\}$
  it is clear
  $$\mathbb{E}[f(F)H(G-\alpha)\pi] = \mathbb{E}[f(F)H(G-\alpha)\pi_{(X,U)}]$$
- In this set $\mathcal{W}$, the variance
  $$\text{Var}\big(f(F)H(G-\alpha)\pi\big) = \mathbb{E}\big[(f(F)H(G-\alpha)\pi)^2\big] - \mathbb{E}\big[f(F)H(G-\alpha)\pi\big]^2$$
  is minimized for $\pi = \pi_{(X,U)}$
- Different weight denominator
  $$\mathbb{E}[\delta_0(G-\alpha)] = \mathbb{E}[H(G-\alpha)\pi_{(X,U)}] = \mathbb{E}[H(G-\alpha)\pi_U],$$
  $$\text{where } \pi_U = -\frac{\partial}{\partial u}\log p_U(U)$$

3.Delta

$$\mathbb{E}[f(F)|G=\alpha] = \frac{\mathbb{E}[f(F)H(G-\alpha)\pi_{(X,U)}]}{\mathbb{E}[H(G-\alpha)\pi_{(X,U)}]} =: \frac{\mathbb{A}_{F,G}[f](\alpha)}{\mathbb{A}_{F,G}[1](\alpha)}$$

$$\frac{\partial}{\partial\alpha}\mathbb{E}[f(F)|G=\alpha] = \frac{\mathbb{B}_{F,G}[f](\alpha)\mathbb{A}_{F,G}[1](\alpha) - \mathbb{A}_{F,G}[f](\alpha)\mathbb{B}_{F,G}[1](\alpha)}{\mathbb{A}^2_{F,G}[1](\alpha)}$$

where
$$\begin{aligned}\mathbb{A}_{F,G}[\cdot](\alpha) &= \mathbb{E}[\cdot(F)H(G-\alpha)\pi_{(X,U)}]\\ \mathbb{B}_{F,G}[\cdot](\alpha) &= \frac{\partial}{\partial\alpha}\mathbb{A}_{F,G}[\cdot](\alpha)\\ &= \mathbb{E}[\cdot(F)H(G-\alpha)\{\pi^*_{(X,U)} - \pi^2_{(X,U)}\}]h'(\alpha)\end{aligned}$$

$$\begin{aligned}\pi_{(X,U)} &= -\frac{\partial}{\partial u}\log p_{(X,U)}(X,U)\\ \pi^*_{(X,U)} &= -\frac{\partial^2}{\partial u^2}\log p_{(X,U)}(X,U)\end{aligned}$$

Outline

II- Malliavin approach

1. Malliavin calculus

- Standard Brownian motion W, pure jump process J, then Lévy process $L = \Gamma(W, J)$
- The Malliavin derivative $D_{r,0}$, $r \in [0, T]$ of a Lévy process is essentially a derivative with respect to the Brownian part

$$\text{e.g. } D_{r,0}(W_t + J_t) = \mathbf{1}_{r \leq t}$$

- Skorohod integral of an adapted process u:

$$\delta(u) = \int_0^T u_r dW_r$$

- Chain rule, Integration by parts, Duality formula

See Nualart (1995) and solé et al. (2007).

2.Representations
First representation

F and G two random variables such that

$$F = g_1(X^c, X^d) \qquad \text{and} \qquad G = g_2(U^c, U^d) \tag{1}$$

Theorem
Assume f differentiable and F and G as described in (1).
For a Skorohod integrable process u satisfying

$$\mathbb{E}\Big[\int_0^T D_{t,0}Gu_t dt|\sigma(F,G)\Big] = 1,$$

it holds

$$\mathbb{E}[f(F)|G=\alpha] = \frac{\mathbb{E}\Big[f(F)H(G-\alpha)\delta(u) - f'(F)H(G-\alpha)\int_0^T D_{t,0}Fu_t dt\Big]}{\mathbb{E}\big[H(G-\alpha)\delta(u)\big]}$$

Representation 2

Theorem
Assume again the setting of Theorem 1 and in addition that u is a Skorohod integrable process, satisfying

$$\mathbb{E}\Big[\int_0^T D_{t,0}F u_t dt | \sigma(F,G)\Big] = 0.$$

Then we have the following representation

$$\mathbb{E}[f(F)|G=\alpha] = \frac{\mathbb{E}[f(F)H(G-\alpha)\delta(u)]}{\mathbb{E}[H(G-\alpha)\delta(u)]}$$

Malliavin method applied to linear SDE

Stock price process modeled by the linear SDE

$$\begin{cases} dS_t &= \alpha S_{t-}dt + \beta S_{t-}dW_t + \int_{\mathbb{R}_0}(e^z - 1)S_{t-}\widetilde{N}(dt, dz), \\ S_0 &= x \end{cases}$$

In Benth et al. (2001), it is shown $S_t = S_t^c e^{S_t^d}$

$$\mathbb{E}[f(S_t)|S_s = \alpha] = \frac{\mathbb{E}\left[f(S_t)H(S_s-\alpha)\frac{1}{S_s}\left(\frac{W_s}{s\beta}+1\right) - f'(S_t)H(S_s-\alpha)\frac{S_t}{S_s}\right]}{\mathbb{E}\left[H(S_s-\alpha)\frac{1}{S_s}\left(\frac{W_s}{s\beta}+1\right)\right]}$$

$$\mathbb{E}[f(S_t)|S_s = \alpha] = \frac{\mathbb{E}\left[f(S_t)H(S_s-\alpha)\frac{1}{S_s}\left(\frac{tW_s - sW_t}{s(t-s)\beta}+1\right)\right]}{\mathbb{E}\left[H(S_s-\alpha)\frac{1}{S_s}\left(\frac{tW_s - sW_t}{s(t-s)\beta}+1\right)\right]}$$

Examples of stock price processes

- Geometric Brownian motion, see Bally et al. (2005)
- Jump-diffusion model
- Stochastic differential equations, in particular:
  - Linear SDE
  - Stochastic volatilty model

3. Delta

Theorem

Consider the same setting as in Theorem 2 and that the Skorohod integral $\delta(u)$ is $\sigma(F,G)$-measurable. Then

$$\frac{\partial}{\partial\alpha}\mathbb{E}[f(F)|G=\alpha] = \frac{\mathbb{B}_{F,G}[f](\alpha)\mathbb{A}_{F,G}[1](\alpha) - \mathbb{A}_{F,G}[f](\alpha)\mathbb{B}_{F,G}[1](\alpha)}{\mathbb{A}^2_{F,G}[1](\alpha)},$$

where

$$\mathbb{A}_{F,G}[\cdot](\alpha) = \mathbb{E}[\cdot(F)H(G-\alpha)\delta(u)],$$
$$\mathbb{B}_{F,G}[\cdot](\alpha) = \mathbb{E}\Big[\cdot(F)H(G-\alpha)\Big\{-\delta^2(u) + \int_0^T u_r D_{r,0}\delta(u)dr\Big\}\Big]$$

Outline

III- Numerical methods and results

1. American options

Pricing algorithm for American options

- Payoff function Φ, stock price process S with initial value x
- Approximated by Bermudan option, discretization of time interval into n periods with step size $\varepsilon = \frac{T}{n}$
- Priced iteratively via the Bellman dynamic programming principle:

$$P_{n\varepsilon}(S_{n\varepsilon}) \equiv \Phi(S_{n\varepsilon}) = \Phi(S_T),$$
$$P_{k\varepsilon}(S_{k\varepsilon}) = \max\left\{\Phi(S_{k\varepsilon}), e^{-r\varepsilon}\mathbb{E}\left[P_{(k+1)\varepsilon}(S_{(k+1)\varepsilon})\middle|S_{k\varepsilon}\right]\right\},$$

$k = n-1, \ldots, 1, 0$

- Delta of the option $\Delta_0(x) := \partial_x P_0(x)$,

$$\Delta_\varepsilon(S_\varepsilon) = \begin{cases} \partial_\alpha \Phi(\alpha)\Big|_{\alpha=S_\varepsilon} & \text{if } P_\varepsilon(\alpha) < \Phi(\alpha), \\ e^{-r\varepsilon}\partial_\alpha \mathbb{E}\left[P_{2\varepsilon}(S_{2\varepsilon})|S_\varepsilon = \alpha\right]\Big|_{\alpha=S_\varepsilon} & \text{if } P_\varepsilon(\alpha) \geq \Phi(\alpha), \end{cases}$$

$$\Delta_0(x) = \mathbb{E}_x[\Delta_\varepsilon(S_\varepsilon)]$$

2. Monte Carlo estimation

- For the conditional expectations we make use of the representation

$$\mathbb{E}\big[P_{(k+1)\varepsilon}(S_{(k+1)\varepsilon})\big|S_{k\varepsilon}=S^q_{k\varepsilon}\big]=\frac{\mathbb{E}\big[P_{(k+1)\varepsilon}(S_{(k+1)\varepsilon})H(S_{k\varepsilon}-S^q_{k\varepsilon})\pi_k\big]}{\mathbb{E}\big[H(S_{k\varepsilon}-S^q_{k\varepsilon})\pi_k\big]}$$

and the Monte Carlo estimation

$$\mathbb{E}\big[.(S_{(k+1)\varepsilon})H(S_{k\varepsilon}-\alpha)\pi_k\big]\approx\frac{1}{N}\sum_{j=1}^{N}.(S^j_{(k+1)\varepsilon})H(S^j_{k\varepsilon}-\alpha)\pi^j_k$$

- Pricing algorithm: backward in time

$\Rightarrow$ backward simulation of the stock price process, to obtain a better efficiency

Possible for GBM model Bally et al. (2005), Merton model.

3. Variance reduction

- Localization technique
- Including a control variable

3. Numerical example

- American put option on a stock price given by the Merton model

$$S_t = S_0 \exp\left((r - \sigma^2/2)t + \sigma W_t + \sum_{i=1}^{N_t} Z_i\right)$$

W: standard Brownian motion
N: Poisson process with intensity μ
Z_i: i.i.d. $N(-\delta^2/2, \delta^2)$

- Weight in the representation

$$\pi_{\mathsf{CDM}} = \frac{tW_s - sW_t}{\sigma s(t-s)} \quad \text{and} \quad \pi_{\mathsf{MM}} = \frac{1}{S_s}\left(\frac{tW_s - sW_t}{\sigma s(t-s)} + 1\right)$$

- Parameter set, see Amin (1993)

T	S_0	r	σ^2	μ	δ^2
0.25	40	0.08	0.05	5	0.05

Numerical example

Results for $n = 10$ time periods and $N = 10000$ simulated paths

strike	Eur Opt	CDM	CDM+L	MM	MM+L	Amin
30	0.681	2.077	1.412	0.520	0.807	0.674
35	1.686	4.579	2.602	1.409	1.830	1.688
40	3.605	7.903	4.562	3.403	3.772	3.630
45	6.660	11.761	7.687	6.491	6.837	6.734
50	10.548	16.502	11.744	10.569	10.772	10.696

Increasing the number of simulated paths to $N = 30000$

strike	Eur Opt	CDM	CDM+L	MM	MM+L	Amin
35	1.693	1.5120	1.8998	2.120	1.757	1.688

Conclusion

For discontinuous stock price processes:

$$\mathbb{E}[f(S_t)|S_s = \alpha] = \frac{\mathbb{E}[f(S_t)H(S_s - \alpha)\pi]}{\mathbb{E}[H(S_s - \alpha)\pi]}, \quad s < t,$$

based on two methods

- Conditional density method
- Malliavin approach

- Work in progress
  - Multidimensional setting
  - Path-dependent payoff (Asian options)
  - Backward simulation of Lévy processes

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