Optimal market making strategies under inventory constraints

Etienne CHEVALIER
Université d'Evry

6^{th} AMaMeF and Banach Center Conference
June 10-15, 2013

Joint work with :
M'hamed Gaigi, ENIT, Tunis
Vathana Ly Vath, ENSIIE & Université d'Evry
Mohamed Mnif, ENIT, Tunis

Motivations : Market making under constraints

- Liquidity takers :
  - $\rightarrow$ trade only through market order
  - $\rightarrow$ pay liquidity costs
- Liquidity takers and providers :
  - $\rightarrow$ trade in a limit order book through market and limit order
  - $\rightarrow$ pay less liquidity costs but have some inventory risk.
- Market makers :
  - $\rightarrow$ trade in a dealer market as a single or representative market maker
  - $\rightarrow$ face liquidity and inventory constraints.

Motivations : Liquidity costs for price takers

- Liquidity costs for price takers
  - Transaction costs due to bid-ask spread :

    $\rightarrow$ Shreve and Soner (1994) ; Korn (1998) ; Framstad, Oksendal and Sulem (2001),...

  - Price impact for large trades : Almgren and Chriss (2001)

    $\rightarrow$ Supply curves : Cetin, Jarrow, Protter (2004) ; Alfonsi, Fruth and Schied (2010),...

    $\rightarrow$ Impact functions : Bank and Baum (2004) ; Ly Vath, Mnif and Pham (2007) ; Kharroubi, Pham (2010) ; Roch (2011)...

Motivations : Liquidity in limit order book market

- Use limit orders instead of market orders.
  - Liquidation problems :

    →Guéant, Lehalle and Tapia (2011) ; Bayraktar and Ludkovski (2012) ; Bouchard, Lehalle and Dang (2011)
  - Market making/Portfolio management problems :

    → Avellaneda and Stoikov (2008) ; Guilbaud and Pham (2013)

Motivations : Market making under constraints

- A market maker in a dealer market faces some constraints
  - Provide liquidity
  - Set "reasonable" prices and spread
  - Cash and stock holdings constraints

►Ho, Stoll (1981) ; Huang, Simchi-Levi and Song (2012) ; Guéant, Lehalle, and Tapia (2012)

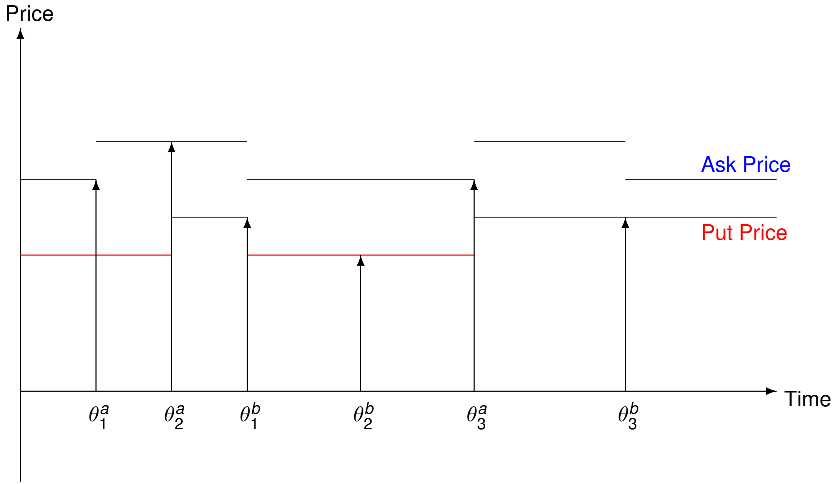
Market making strategies

- We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions.

- When the i^{th} buying (resp. selling) order arrives at the $\mathbb{F}$-stopping time θ_i^a (resp. θ_i^b) :
  - **Provide liquidity** : The market maker has to sell (resp. buy) an asset at the ask (resp. bid) price denoted by P^a (resp. P^b).
  - **Set Bid and Ask prices** : The market maker may either keep the bid and ask prices constant or increase (resp. decrease) one or both of them by one tick (δ).

- We consider a control $\alpha := (\epsilon_t^a, \epsilon_t^b, \eta_t^a, \eta_t^b)_{0 \leq t \leq T}$ $\mathbb{F}$-predictable process where the random variables $\epsilon_t^a, \epsilon_t^b, \eta_t^a, \eta_t^b$ are valued in $\{0, 1\}$.

Representation of a market making strategies



Prices and spread dynamics

- **Bid and Ask processes** : For $c \in \{a, b\}$, the dynamics of P^c evolves according to the following equations

$$\text{for } i \in \mathbb{N}^*, \quad \begin{cases} dP_t^c = 0 & \text{for } \xi_i < t < \xi_{i+1} \\ P^c_{\theta_i^b} = P^c_{\theta_i^{b-}} - \delta \epsilon^c_{\theta_i^b} \\ P^c_{\theta_i^a} = P^c_{\theta_i^{a-}} + \delta \eta^c_{\theta_i^a}, \end{cases}$$

  where $(\xi_i)_{i \geq 0}$ is the sequence of transaction times.

- **Mid price and spread processes** : We set $P := \frac{P^a + P^b}{2}$ and $S := P^a - P^b$. For all $i \in \mathbb{N}^*$, the dynamics of the process (P, S) is given by

$$\begin{cases} dP_t = 0, & \text{for } \xi_i < t < \xi_{i+1} \\ P_{\theta_i^b} = P_{\theta_i^{b-}} - \frac{\delta}{2}(\epsilon^a_{\theta_i^b} + \epsilon^b_{\theta_i^b}) \\ P_{\theta_i^a} = P_{\theta_i^{a-}} + \frac{\delta}{2}(\eta^a_{\theta_i^a} + \eta^b_{\theta_i^a}). \end{cases} \quad \text{and} \quad \begin{cases} dS_t = 0, & \text{for } \xi_i < t < \xi_{i+1} \\ S_{\theta_i^b} = S_{\theta_i^{b-}} - \delta(\epsilon^a_{\theta_i^b} - \epsilon^b_{\theta_i^b}) \\ S_{\theta_i^a} = S_{\theta_i^{a-}} + \delta(\eta^a_{\theta_i^a} - \eta^b_{\theta_i^a}). \end{cases}$$

Cash and stock holdings dynamics

- **Cash holdings** : We denote by $r > 0$ the instantaneous interest rate. The bank account evolves according to the following equations

$$\text{for } i \in \mathbb{N}^*, \quad \left\{ \begin{array}{lll} dX_t & = & rX_t dt, \\ X_{\theta_i^b} & = & X_{\theta_i^{b-}} - P^b_{\theta_i^{b-}} \\ X_{\theta_i^a} & = & X_{\theta_i^{a-}} + P^a_{\theta_i^{a-}}, \end{array} \right. \quad \text{for } \xi_i < t < \xi_{i+1},$$

- **Stock holdings** : The number of shares held by the market maker at time $t \in [0, T]$ is denoted by Y_t, and evolves according to the following equations

$$\text{for } i \in \mathbb{N}^*, \quad \left\{ \begin{array}{lll} dY_t & = & 0, \\ Y_{\theta_i^b} & = & Y_{\theta_i^{b-}} + 1 \\ Y_{\theta_i^a} & = & Y_{\theta_i^{a-}} - 1 \end{array} \right. \quad \text{for } \xi_i < t < \xi_{i+1}$$

Regime switching

- **Liquidity regimes** :
  Let I be a continuous time, time homogeneous, irreductible Markov chain with m states.
  The generator of the chain I under $\mathbb{P}$ is denoted by $A = (\vartheta_{i,j})_{i,j=1,\dots m}$. Here $\vartheta_{i,j}$ is the constant intensity of transition of the chain L from state i to state j.

- **Market orders arrivals** : Let two Cox processes N^a and N^b.
  The intensity processes associated with N^a and N^b are defined, for $t \geq 0$, by $\lambda^a(I_t, P_t, S_t)$ and $\lambda^b(I_t, P_t, S_t)$ where λ^a and λ^b are positive deterministic functions, bounded and defined on $\{1, ..., m\} \times \frac{\delta}{2}\mathbb{N} \times \delta\mathbb{N}$.

  We define θ_k^a (resp. θ_k^b) as the k^{th} jump time of N^a (resp. N^b), which corresponds to the k^{th} buy (resp. sell) market order.

Admissible strategies

- **Liquidity constraints** : Let $K > 0$, the market maker has to use controls such that

$$P_t - S_t/2 > 0 \quad \text{and} \quad 0 < S_t \leq K \times \delta, \quad \text{for } 0 \leq t \leq T.$$

- **Inventory and cash constraints** : Let $x_{min} < 0$ and $y_{min} \leq y_{max}$. We introduce the following notations :

$$\begin{aligned} \mathbb{S} &= (x_{min}, +\infty) \times \{y_{min}, ..., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, ..., K\}, \\ \mathcal{S} &= \{(t, x, y, p, s) \in [0, T] \times \mathbb{S} : p - \frac{s}{2} \geq \delta\}. \end{aligned}$$

For a control α, we define the liquidation time :

$$\tau^{t,i,z,\alpha} := \inf\{u \geq t : X_u^{t,i,x,\alpha} \leq x_{min} \text{ or } Y_u^{t,i,y,\alpha} \in \{y_{min} - 1, y_{max} + 1\}\}$$

- **Admissible strategies** : Let $(t, z) := (t, x, y, p, s) \in \mathcal{S}$, the strategy $\alpha = (\epsilon_u^a, \epsilon_u^b, \eta_u^a, \eta_u^b)_{t \leq u \leq T}$ is admissible, if the processes $\epsilon^a, \epsilon^b, \eta^a, \eta^b$ are valued in $\{0, 1\}$ and for all $u \in [t, T]$, $(u, Z_{u-}^{t,i,z,\alpha}) \in \mathcal{S}$.
We denote by $\mathcal{A}(t, z)$ the set of all these admissible policies.

Objective function

- **Portfolio liquidation** : If the market maker decides (or has) to liquidate her portfolio, then she actually gets

$$Q(t,y,p,s) = (p - \text{sign}(y)\frac{s}{2})f(t,y),$$

where $f : [0,T] \times \mathbb{R} \to \mathbb{R}_+$, non-linear in y and such that

$$f(t,y) \leq f(t,y^{'}) \text{ if } y^{'} \leq y \quad \text{and} \quad yf(t^{'},y) \leq yf(t,y) \text{ if } t^{'} \leq t.$$

- **Utility and penalty functions** : Let $\gamma > 0$ and $U(x) = 1 - e^{-\gamma x}$ on $\mathbb{R}$. We set

$$U_L = U \text{o} L \quad \text{where } L(t,x,y,p,s) = x + yQ(t,y,p,s).$$

Let g a bounded positive function defined on $\{y_{min}, ..., y_{\max}\}$.

- **Objective function** : We consider the functions $(v_i)_{i\in\{1,\ldots,m\}}$ defined on $\mathcal{S}$ by

$$v_i(t,z) := \sup_{\alpha\in\mathcal{A}(t,z)} J_i^{\alpha}(t,z)$$

where we have set

$$J_i^{\alpha}(t,z) \quad := \quad \mathbb{E}\left[U_L(T\wedge\tau^{t,i,z,\alpha}, Z^{t,i,z,\alpha}_{(T\wedge\tau^{t,i,z,\alpha})-}) - \int_t^{T\wedge\tau^{t,i,z,\alpha}} g(Y_s^{t,i,y,\alpha})ds\right].$$

Analytical properties and dynamic programming principle

Objective functions bounds

- Let $(t, z) := (t, x, y, p, s) \in \mathcal{S}$. From monotonicity of f,

$$L(t,z) \geq x_{min} + y_{min} f(0, y_{min})(p - \frac{K\delta}{2}).$$

Proposition

There exist C_1, C_2 and C_3 positive constants such that

$$1 - C_1 - C_2 e^{C_3 p} \leq v_i(t,z) \leq 1, \quad \forall (i,t,z) := (i,t,x,y,p,s) \in \{1,...,m\} \times \mathcal{S},$$

Uniform continuity of the objective functions

Hölder continuity of the criteria functions

Let $i \in \{1, ..., m\}$, $(t, z) := (t, x, y, p, s) \in \bar{\mathcal{S}}$ and (t', x') in $[0, T] \times (x_{min}, +\infty)$.
For all $\alpha \in \mathcal{A}(t \wedge t', z)$ such that $\alpha_{||[t \wedge t', t \vee t']} = 0$, we have $\alpha \in \mathcal{A}(t, z) \cap \mathcal{A}(t', z')$ with $z' = (x', y, p, s)$ and, if (t^{prime}, x') is close enough to (t, x), then

$$| J_i^{\alpha}(t, z) - J_i^{\alpha}(t', z') | \quad \leq \quad K_2(p) \left(\psi(re^{rT} \mid x'(t - t') \mid) + \psi(x' - x) + \mid t' - t \mid \right).$$

where $K_2(p) > 0$ and ψ an Hölder continuous function on $\mathbb{R}$.

Uniform continuity of the objective functions

Let $(i, y, p, s) \in \{1, ..., m\} \times \{y_{min}, .., y_{max}\} \times \frac{\delta}{2}\mathbb{N}^* \times \delta\{1, ..K\}$ such that $p - \frac{s}{2} > 0$.

The function $(t, x) \to v_i(t, x, y, p, s)$ is uniformly continuous on $[0, T] \times [x_{min}, +\infty)$.

Dynamic programming principle

Dynamic programming principle

Let $(i,t,z) := (i,t,x,y,p,s) \in \{1,...,m\} \times \mathcal{S}$. Let ν be a stopping time in $\mathcal{T}_{t,T}$, we have

$$\begin{aligned} v_i(t,z) &= \sup_{\alpha \in \mathcal{A}(t,z)} \mathbb{E}\Big[v_{I_{\nu\wedge\hat\theta}} \left(\nu \wedge \hat\theta,\ Z^{t,i,z,\alpha}_{\nu\wedge\hat\theta}\right) \mathbb{1}_{\{\nu\wedge\hat\theta<\hat\tau^\alpha\}} \\ &\quad + U_L\left(\hat\tau^\alpha, xe^{r(\hat\tau^\alpha - t)}, y, p, s\right) \mathbb{1}_{\{\hat\tau^\alpha \le \nu\wedge\hat\theta\}} - g(y)\left(\nu\wedge\hat\theta\wedge\hat\tau^\alpha - t\right)\Big], \end{aligned}$$

with $\hat\tau^\alpha = \tau^{t,i,z,\alpha} \wedge T$ and

$$\hat\theta = \inf\{u \ge t :\ N_u > N_{u-} \text{ or } N^{a,i,t,z}_u > N^{a,i,t,z}_{u-} \text{ or } N^{b,i,t,z}_u > N^{b,i,t,z}_{u-}\}.$$

Analytical properties of the objective function and dynamic programming principle

- Model and problem formulation
- Analytical properties and dynamic programming principle
- Viscosity characterization of the objective function
- Numerical illustrations

HJB equation (1)

- **Set of admissible controls** : We define the following set :

$$A(t,z) \quad := \quad \{\alpha = (\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in \{0,1\}^4 : \ \delta\varepsilon^b < p - \frac{s}{2}, \\ \delta \leq s - \delta(\varepsilon^a - \varepsilon^b) \leq K\delta \text{ and } \delta \leq s + \delta(\eta^a - \eta^b) \leq K\delta\}.$$

- **Transactions operators** : For all $(i,t,x,y,p,s) := (i,t,z) \in \{1,...,m\} \times \mathcal{S}$ and $\alpha := \{\varepsilon^a, \varepsilon^b, \eta^a, \eta^b) \in A(t,z)$, we introduce the two operators :

$$\mathcal{A}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{min},p,s) & \text{if } y = y_{min}, \\ v_i(t, x+p+\frac{s}{2}, y-1, p+\frac{\delta}{2}(\eta^a+\eta^b), s+\delta(\eta^a-\eta^b)) & \text{else.} \end{cases}$$

$$\mathcal{B}v_i(t,z,\alpha) = \begin{cases} U_L(t,x,y_{max},p,s), & \text{if } y = y_{max} \\ U_L(t,z) & \text{if } x < x_{min}+p-\frac{s}{2} \text{ or } x = x_{min}+p-\frac{s}{2} < 0 \\ v_i(t, x-p+\frac{s}{2}, y+1, p-\frac{\delta}{2}(\varepsilon^a+\varepsilon^b), s-\delta(\varepsilon^a-\varepsilon^b)) & \text{else.} \end{cases}$$

HJB equation (2)

Let $(\varphi_i)_{1\leq i\leq m}$ a family of smooth functions defined on $\mathcal{S}$. We introduce the following operator associated with state $i \in \{1, .., m\}$:

$$\begin{aligned}\mathcal{H}_i(t, z, \varphi_i, \frac{\partial \varphi_i}{\partial x}) &= rx\frac{\partial \varphi_i}{\partial x} + \sum_{j\neq i} \gamma_{ij} \left(\varphi_j(t, x, y, p, s) - \varphi_i(t, x, y, p, s)\right) - g(y) \\ &+ \sup_{\alpha\in A(t,z)} \Big[\lambda_i^a(p, s) \left(\mathcal{A}\varphi_i(t, x, y, p, s, \alpha) - \varphi_i(t, x, y, p, s)\right) \\ &+ \lambda_i^b(p, s) \left(\mathcal{B}\varphi_i(t, x, y, p, s, \alpha) - \varphi_i(t, x, y, p, s)\right)\Big] = 0.\end{aligned}$$

We consider the HJB equation :

$$-\frac{\partial \varphi_i}{\partial t} - \mathcal{H}_i(t, z, \varphi_i, \frac{\partial \varphi_i}{\partial x}) = 0, \quad \text{for } (t, z) \in \mathcal{S}, \tag{1}$$

with the following boundary and terminal conditions :

$$v_i(t, x_{min}, y, p, s) = U_L(t, x_{min}, y, p, s) \tag{2}$$

$$v_i(T, x, y, p, s) = U_L(T, x, y, p, s) \tag{3}$$

Viscosity characterization of the objective function

Theorem :

The family of objective functions $(v_i)_{1 \leq i \leq m}$ is the unique family of functions such that

i) Continuity condition : For all $(i, y, p, s) \in \{1, .., m\} \times \{y_{min}, .., y_{max}\} \times \frac{\delta}{2}\mathbb{N} \times \delta\{1, .., K\}$, $(t, x) \to v_i(t, x, y, p, s)$ is continuous on $\{(t, x) \in [0, T) \times [x_{min}, +\infty) : (t, x, y, p, s) \in \mathcal{S}\}$.

ii) Growth condition : There exists C_1, C_2 and C_3 positive constants such that

$$1 - C_1 - C_2 e^{C_3 p} \leq v_i(t, x, y, p, s) \leq 1, \quad \text{on } \{1, .., m\} \times \mathcal{S}.$$

iii) Boundary conditions :

$$v_i(t, x_{min}, y, p, s) = U_L(t, x_{min}, y, p, s) \text{ and } v_i(T, x, y, p, s) = U_L(T, x, y, p, s).$$

iv) Viscosity solution : $(v_i)_{1 \leq i \leq m}$ is a viscosity solution of the system of variational inequalities (1) on $\{1, ..., m\} \times \mathcal{S}$.

Numerical illustrations

Numerical values

- Market values :
  $\rightarrow$ Initial conditions : $x = 5, \quad y = 2, \quad p = 1, \quad s = 0.02.$
  $\rightarrow$ $r = 0.05, \quad \delta = 0.02, \quad \lambda = 20.$
  $\rightarrow$ Impact function : $f(t, y) = \exp\left(0.09y(T - t)\right).$
  $\rightarrow$ Intensity functions :

$$\lambda_i^a(p, s) = \frac{\psi_i^a}{p} \exp\left(-s - 0.01(p-1)\right) \quad \text{and} \quad \lambda_i^b(p, s) = \psi_i^b p \exp\left(-s + 0.01(p-1)\right),$$

  with $\psi_1^a = 120, \quad \psi_2^a = 80, \quad \psi_1^b = 80, \quad \psi_2^b = 120.$
- Constraints :
  $\rightarrow$ $x_{min} = -20, \quad y_{min} = -10, \quad y_{max} = 10, \quad K = 5, \quad T = 1.$
  $\rightarrow$ Penalty function : $g(y) = y^2 \times 10^{-3}.$
  $\rightarrow$ Utility function : $U(l) = 1 - e^{-0.01l}$ i.e. $\gamma = 0.01.$
- Numerical values :
  $\rightarrow$ Localisation : $x_{max} = 20, \quad p_{min} = 1 - 20 \times \frac{\delta}{2}, \quad p_{max} = 1 + 20 \times \frac{\delta}{2}$
  $\rightarrow$ Discretization : $n_x = 40$ and $n_t = 20.$

A cash holdings path

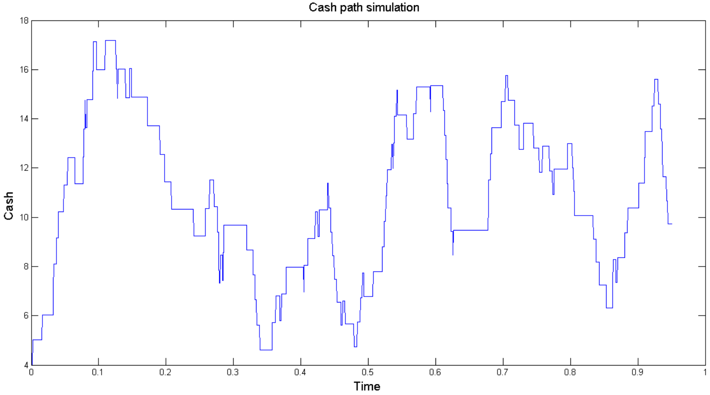


FIGURE : *A cash holdings path*

A stock holdings path

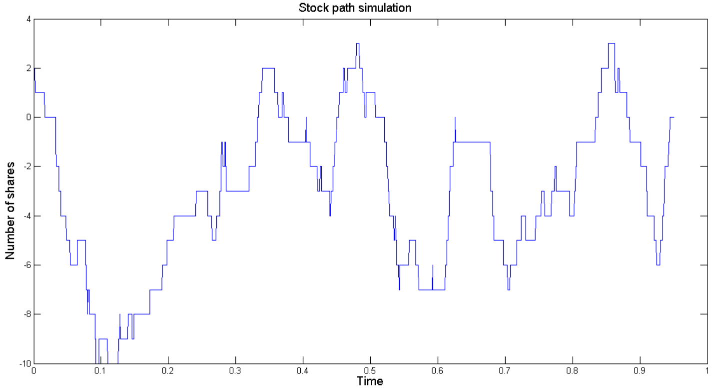


FIGURE : *A stock holdings path*

Bid and ask price paths

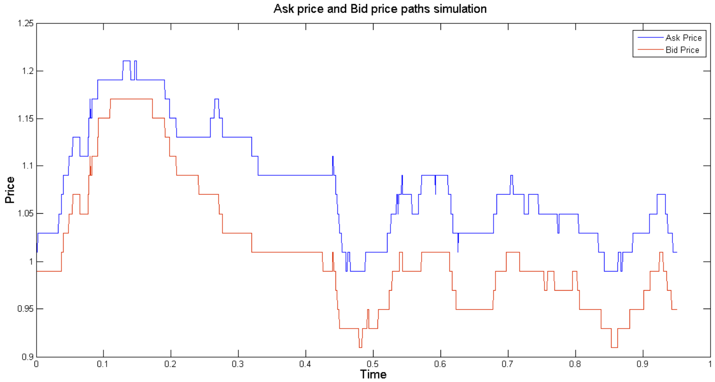


FIGURE : *Bid and ask price paths*

Liquidation value

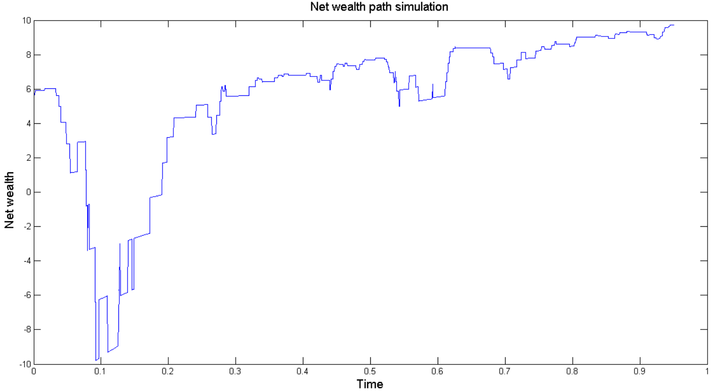


FIGURE : *A path of* $L(t, Z_t)$