Optimal Trading Stops

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Overview

1. Set Up
2. Optimal Stops with Constant P&L Drift
3. Optimal Stops with Unknown P&L Drift
4. Optimal Stops with Stochastic Drift
5. Numerical Examples

Set Up

- Underlying: single asset, time spreads, inter asset spreads, basket of assets
- When a position is entered a fixed number of contracts N is bought/sold
- When a position is exited all the outstanding N contracts are sold/bought back

Optimal Stops with Constant P&L Drift

- [Imkeller and Rogers] model the P&L of a position as a Brownian motion with constant drift

$$X_t = \sigma W_t + \mu t \tag{1}$$

  and assume that the cost of exiting the position at the random (stopping) time T is equal to c

- Consider the simple stopping strategy

$$T \equiv inf\{t : X_t = -a \quad or \quad X_t = b\} \tag{2}$$

- One approach to the exit problem is to maximise the expected utility of the P&L, i.e.

$$\phi = E[e^{-\rho T} U(X_T - c)] \tag{3}$$

  for some increasing and concave utility function $U(x)$

Optimal Stops with Constant P&L Drift

- If $f(x)$ is in $\mathcal{C}^2$, the process $M_t = e^{-\rho t} f(X_t)$ is a (Local) Martingale and the following equalities hold

$$f(x) = E^x[e^{-\rho T} U(X_T - c)] \tag{4}$$

$$\frac{1}{2}\sigma^2 f^{''}(x) + \mu f^{'}(x) - \rho f(x) = 0 \tag{5}$$

- It is thus sufficient to solve the ODE above with the appropriate boundary conditions in the interval $[-a, b]$ to obtain an explicit function for $\phi = f(0; a, b)$
- The optimal stop loss and target profit thresholds are obtained by maximising $f(0; a, b)$ as a function of the parameters a and b.

Example: CARA Utility

- If we choose the utility function to be

$$U(x) = 1 - \exp(-\gamma x) \tag{6}$$

we can find an explicit solution for the objective function ϕ

- The objective function can be written as

$$\phi \equiv f(0; a, b) = E[e^{-\rho T}] - e^{-\gamma c} E[e^{-\rho T - \gamma X_T}] \tag{7}$$

$$= L(\rho, 0) - e^{\gamma c} L(\rho, \gamma) \tag{8}$$

where

$$L(\rho, \gamma) = E[e^{-\rho T - \gamma X_T}] \tag{9}$$

- Solving the ODE, we obtain

$$L(\rho, \gamma) = \frac{e^{\gamma a}(e^{\beta b} - e^{\alpha b}) + e^{-\gamma b}(e^{\alpha a} - e^{-\beta a})}{e^{\alpha a + \beta b} - e^{-\alpha b - \beta a}} \tag{10}$$

Example: CARA Utility

- ... where α and β are equal to

$$\alpha = -\frac{\mu}{\sigma^2} + \frac{1}{\sigma}\sqrt{\frac{\mu^2}{\sigma^2} - 2\rho} \tag{11}$$

and

$$\beta = -\frac{\mu}{\sigma^2} - \frac{1}{\sigma}\sqrt{\frac{\mu^2}{\sigma^2} - 2\rho} \tag{12}$$

Optimal Stops with Unknown P&L Drift

- However, when the drift is deterministic and positive, it is not optimal to place stop losses. If the drift is deterministic and negative, it does not make sense to trade in the first place.
- [Imkeller and Rogers] suggest to let μ be a random variable with known distribution.
- The optimisation problem thus becomes

$$\phi(\mu; a, b) = \int E^{(\mu)}[e^{-\rho T} U(X_T - c)]\psi(\mu) d\mu \tag{13}$$

- As before we can solve for the stops a and b which maximise the function above.

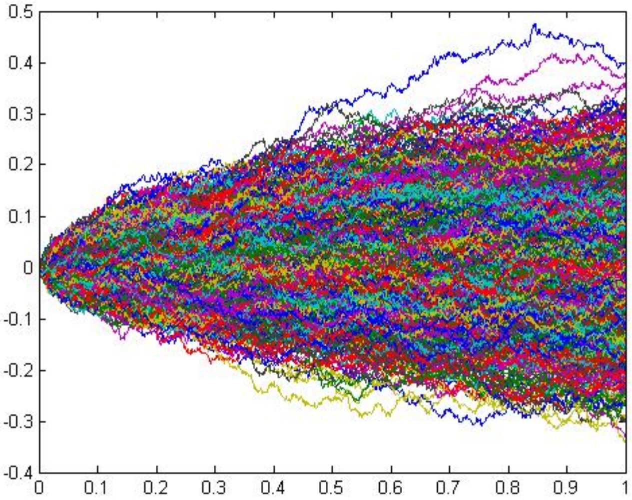
Optimal Stops with Stochastic Drift

- Assume now that the drift of the P&L changes over time in a stochastic fashion.
- For example, the drift may be high and positive when the trade is entered and weaken over time (or even become negative) as other market participants spot the same opportunity or other exogenous factors start affecting the price of the asset.
- In order to capture such a behaviour, we can model the P&L as a Markov-modulated diffusion (MMD)

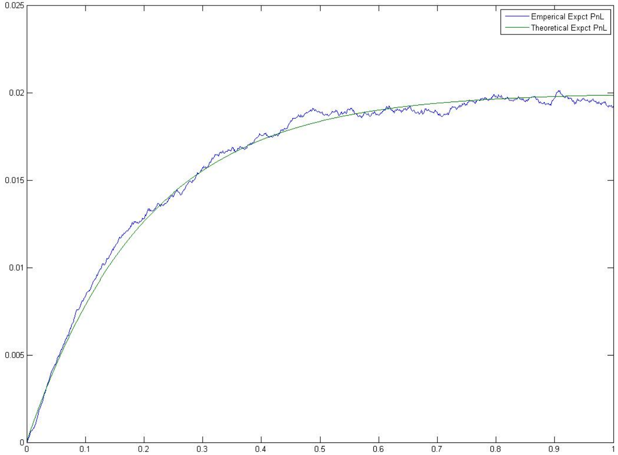
$$dX_t = \mu(y_t)dt + \sigma(y_t)dW_t \tag{14}$$

  where y_t is a continuous time Markov chain with infinitesimal generator Q, independent from W_t
- For example $y_t \in \{1, 2\}$ and $\mu(1) = \bar{\mu} > 0$ and $\mu(2) = 0$, i.e. the P&L has an initial positive drift which dies out at the random time when the chain changes state. In this case $y_t = 2$ is an absorbing state.

Simulated P&L



Expected P&L $E[X(t)]$



- In order to solve the optimisation problem

$$\phi = E^x[e^{-\rho T} U(X_T - c)] \tag{15}$$

when X_t is a MMD, consider the function $f(x, y) \in \mathcal{C}^{2,0}$.

- Applying Ito's formula to the function $\tilde{f} \equiv e^{-\rho t} f(X_t, y_t)$ we obtain

$$\begin{aligned} d(e^{-\rho t} f(X_t, y_t)) &= e^{-\rho t}(\mu(y_t) f_x(X_t, y_t) + \frac{1}{2}\sigma^2(y_t) f_{xx}(X_t, y_t) \\ &+ (Qf)(X_t, y_t) - \rho f(X_t, y_t))dt + dM_t^f \end{aligned}$$

where M_t^f is a local Martingale.

Optimal Stops with Stochastic Drift

- Since y_t can only take a finite number of values, with a slight abuse of notation we can think of $f(X_t)$ as a vector valued function with element i equal to $f_i(X_t) \equiv f(X_t, i)$
- For $\tilde{f}$ to be a local Martingale we require that

$$\frac{1}{2}\Sigma f^{''}(x) + Mf^{'}(x) + (Q - R)f(x) = 0$$

- Here Σ, M and R are diagonal matrices whose ith diagonal entry is equal to $\sigma(i)$, $\mu(i)$ and $\rho(i)$ respectively. Q is the infinitesimal generator of the chain.

- Since $f(x)$ is bounded in the interval $[-a, b]$, it follows from the optional stopping theorem that

$$f_i(x) = E^{x,i}[e^{-\rho T} U(X_T - c)]$$

where $y_0 = i$ is the initial state of the chain and we have used the boundary conditions

$$\begin{cases} f_i(-a) = U(-a - c) & i \in \{1, \dots, n\} \\ f_i(b) = U(b - c) & i \in \{1, \dots, n\} \end{cases}$$

Optimal Stops with Stochastic Drift

- The system of ODEs above admits solutions of the form

$$f(x) = ve^{-\lambda x} \tag{16}$$

where v is a n dimensional vector

- Substituting (16) into the system (13) and re-arranging we obtain

$$\lambda^2 v - 2\lambda\Sigma^{-1}Mv + 2\Sigma^{-1}(Q-R)v = 0$$

- The quadratic eigenvalue problem (15) can be reduced to a canonical eigenvalue problem

$$\begin{pmatrix} 2\Sigma^{-1}M & -2\Sigma^{-1}(Q-R) \\ I & 0 \end{pmatrix}\begin{pmatrix} h \\ v \end{pmatrix} = \lambda\begin{pmatrix} h \\ v \end{pmatrix}$$

where

$$h = \lambda v$$

Optimal Stops with Stochastic Drift

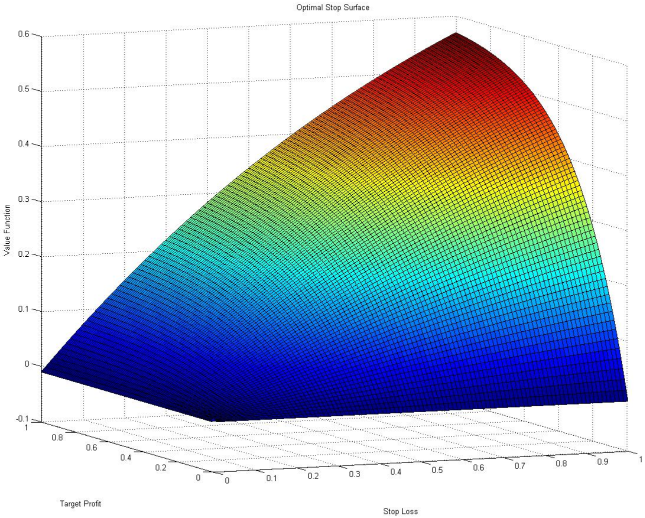
- This is a standard eigenvalue problem which admits n solutions in the positive half plane and n in the negative half plane
- The solution to our ODE system will thus take the form

$$f(x) = \sum_{i=1}^{2n} w_i v_i e^{-\lambda_i x}$$

- The $2n$ coefficients w_i can be derived by solving the system
$$\begin{cases} \sum_{i=1}^{2n} w_i v_i e^{-\lambda_i b} = \bar{U}(b-c) \\ \sum_{i=1}^{2n} w_i v_i e^{\lambda_i a} = \bar{U}(-a-c) \end{cases}$$
- Here $\bar{U}(z)$ is an n dimensional vector with ith entry equal to $U(z)$.

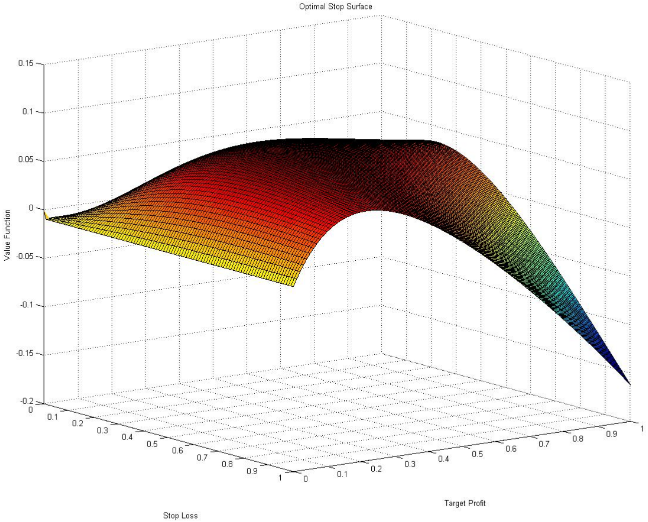
Numerical Examples

Constant drift - $\mu = 0.15$, $\sigma = 0.25$, $c = 0.01$

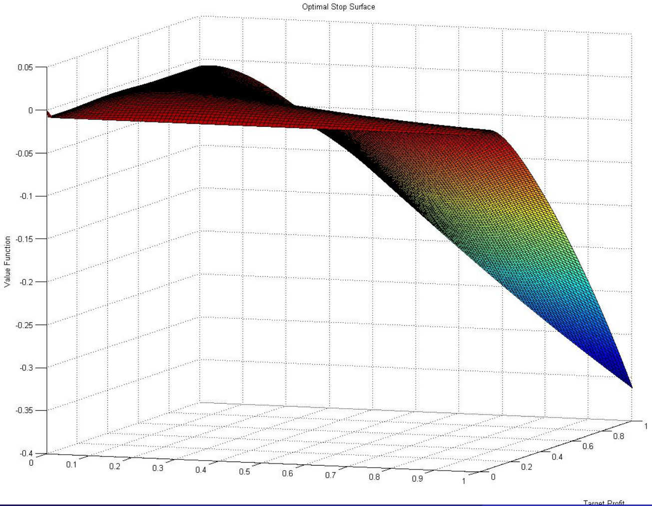


Numerical Examples

Signal with slow decay - $\mu(1) = 0.15$, $\mu(2) = 0$, $q = 0.5$, $c = 0.01$

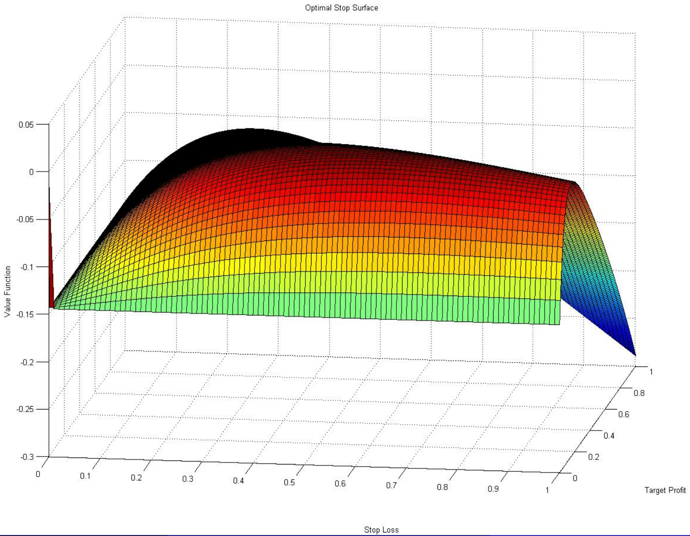


Signal with fast decay - $\mu(1) = 0.15$, $\mu(2) = 0$, $\sigma = 0.25$, $q = 2$, $c = 0.01$



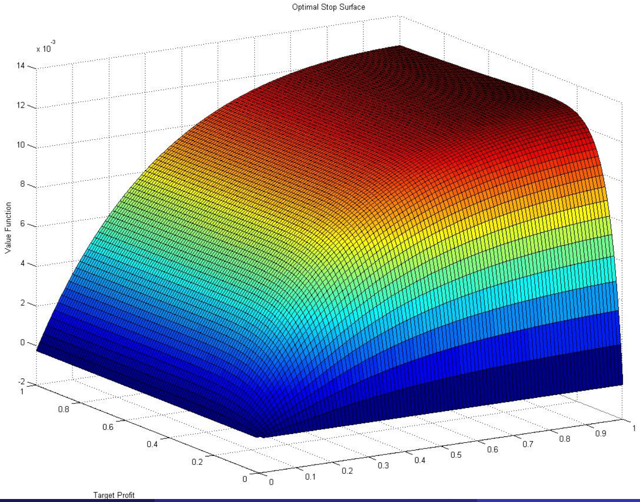
Numerical Examples

Illiquid Security - $\mu(1) = 0.15$, $\mu(2) = 0$, $\sigma = 0.25$, $q = 0.5$, $c = 0.15$



Numerical Examples

Low risk aversion - $\mu(1) = 0.15$, $\mu(2) = 0$, $\sigma = 0.25$, $q = 0.5$, $c = 0.15$, $\gamma = 0.05$



References

Imkeller and Rogers (2011)
Trading to Stop *Working Paper, University of Cambridge*

Di Graziano and Rogers (2005)
Barrier option pricing for assets with Markov-modulated dividends *Journal of Computational Finance, 9, 75-87*