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Convex risk measures for càdlàg processes on Orlicz hearts

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Outline

Basic setting

Consider an incomplete financial market.
Let the interest rate be given by $\mathbf{0}$.

Time horizon: $T \in (0, \infty)$
Underlying space $(\Omega, \mathcal{F}, P; \mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]})$

$\mathbf{F}$ is supposed to satisfy the usual condition, that is,
$\mathbf{F}$ is right-continuous, $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$.

Young function and Orlicz space

$\Phi : \mathbf{R} \to [0, \infty]$ is called a **Young function**, if it is an even convex function with $\Phi(0) = 0$, $\Phi(x) \to \infty$ as $x \to \infty$ and $\Phi(x) < \infty$ for x in a neighborhood of 0.
Let Ψ be the conjugate function of Φ. $\Psi(y) := sup_x\{xy - \Phi(x)\}$.

Let $\mathcal{L}^0$ be the set of all measurable functions on $(\Omega, \mathcal{F})$.

1. **Orlicz space**: $L^\Phi := \{X \in \mathcal{L}^0 | E[\Phi(c|X|)] < \infty$ for some $c > 0\}$.
2. **Orlicz heart**: $M^\Phi := \{X \in \mathcal{L}^0 | E[\Phi(c|X|)] < \infty$ for any $c > 0\}$.
3. **Luxemburg norm**: $||X||_\Phi := \inf\left\{\lambda > 0 | E\left[\Phi\left(\left|\frac{X}{\lambda}\right|\right)\right] \leq 1\right\}$,
4. **Orlicz norm**: $||X||^*_\Psi := \sup\{E[XY] | \ Y \in L^\Psi, ||Y||_\Psi \leq 1\}$.

Henceforth, a Young function Φ is fixed.
For simplicity, Φ is assumed to be $\mathbf{R}$-valued, that is, continuous, and to satisfy $\Phi(x) > 0$ for any $x \in \mathbf{R} \backslash \{0\}$.

shortfall risk measure | convex risk measure on processes | convex risk measure on processes 2 | Examples

Examples

1. $\Phi(x) = |x|^p/p$, $\infty > p > 1$.
   In this case, $L^\Phi = M^\Phi = L^p$ and $\Psi(x) = |x|^q/q$, where $1/p + 1/q = 1$.
2. $\Phi(x) = e^{|x|} - |x| - 1$ or $\Phi(x) = e^{|x|} - 1$.
   $L^\infty \subset M^\Phi \subset L^\Phi \subset L^p$.

Shortfall risk measure

Let $\mathcal{X} = M^{\Phi}$.

$\mathcal{U}$: a convex cone set, including $\mathbf{0}$, of attainable claims with zero initial endowment.

Let l be a nondecreasing convex function defined by

$$l(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ \Phi(x), & \text{otherwise.} \end{cases}$$

Shortfall risk with the **loss function** l for sellers is expressed by $E[l(-x - U + X)]$ when the price of a claim X and the hedging strategy are given by $x \in \mathbb{R}$ and $U \in \mathcal{U}$, respectively.

orlicz space

Shortfall risk measure 2

Fix $\delta > 0$. Define

$$\mathcal{A}^0 := \{X \in M^\Phi | E[l(-X)] \leq \delta\}.$$

For any $X \in M^\Phi$, define

$$\rho_l(X) := \inf\{x \in \mathbb{R} | \exists U \in \mathcal{U} \text{ s.t. } x + U + X \in \mathcal{A}^0\}.$$

$$\rho_l(-X) := \inf\{x \in \mathbb{R} | \exists U \in \mathcal{U} \text{ s.t. } x + U - X \in \mathcal{A}^0\}.$$

Proposition 1 (A, 2011)

Assume $\rho_I(0) > -\infty$.
ρ_I is an **R**-valued convex risk measure on M^Φ satisfying the following:

$$\rho_I(X) = \max_{Q \in \mathcal{P}^\Psi} \left\{ E_Q[-X] - \inf_{\lambda > 0} \frac{1}{\lambda} \left\{ \delta + E\left[\Psi\left(\lambda \frac{dQ}{dP} \right) \right] \right\} \right\},$$

where

$$\mathcal{M}^\Psi := \{Q \ll P | dQ/dP \in L^\Psi, E_Q[U] \leq 0 \text{ for any } U \in \mathcal{U}\},$$

$$\mathcal{A}^1 := \{X^1 \in M^\Phi | \exists U \in \mathcal{U} \text{ s.t. } X^1 + U \geq 0\}.$$

Theorem 3 Corollary

Motivation

Shortfall risk measure for American claims.

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In this talk, we consider convex risk measures for unbounded processes in the Orlicz heart framework.

Convex risk measure on processes

Let $\mathcal{X}$ be the set of all adapted càdlàg processes.
$X^* := \sup_{t\in[0,T]} |X_t|$ for any $X \in \mathcal{X}$
$\mathcal{R}^\Phi := \{X \in \mathcal{X} | X^* \in M^\Phi\}$.

orlicz space

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◂ orlicz space

Define an order $\preceq$ between two processes in $\mathcal{X}$ as follows:
for any $X^1, X^2 \in \mathcal{X}$,

$$X^1 \preceq X^2 \overset{\text{def.}}{\iff} X^1_t \leq X^2_t \text{ a.s. for any } t \in [0, T].$$

Remark that the following hold:

$$\begin{aligned} X^1 \preceq X^2 &\iff P(X^1_t \leq X^2_t \text{ for any } t \in [0, T]) = 1 \\ &\iff X^1_\tau \leq X^2_\tau \text{ a.s. for any } \tau \in \mathcal{T} \\ &\implies (X^1)^* \leq (X^2)^* \text{ a.s.,} \end{aligned}$$

where $\mathcal{T}$ is the set of all stopping times on $[0, T]$.

Convex risk measure on processes 2

Defining a norm $||\cdot||_{\mathcal{R}^\Phi}$ on $\mathcal{R}^\Phi$ as

$$||X||_{\mathcal{R}^\Phi} := ||X^*||_\Phi$$

for any $X \in \mathcal{R}^\Phi$.

we can see that $(\mathcal{R}^\Phi, ||\cdot||_{\mathcal{R}^\Phi})$ is a Banach lattice with order $\preceq$.

Remark that $\mathcal{R}^\Phi$ is not always order continuous.

orlicz space

Convex risk measure on $\mathcal{R}^\Phi$

Definition

A functional $\rho : \mathcal{R}^\Phi \to (-\infty, +\infty]$ is called a convex risk measure on $\mathcal{R}^\Phi$, if it satisfies the following four conditions:

1. **Properness**: $\rho(0) \in \mathbf{R}$,
2. **Monotonicity**: $\rho(X) \geq \rho(Y)$ for any $X, Y \in \mathcal{R}^\Phi$ with $X \leq Y$,
3. **Translation invariance**: $\rho(X + m) = \rho(X) - m$ for $X \in \mathcal{R}^\Phi$ and $m \in \mathbf{R}$,
4. **Convexity**: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ for any $X, Y \in \mathcal{R}^\Phi$ and $\lambda \in [0, 1]$.

Moreover, if a convex risk measure ρ satisfies

5. **Positive homogeneity**: $\rho(\lambda X) = \lambda\rho(X)$ for any $\lambda \geq 0$,

then ρ is called a coherent risk measure.

Let $(\mathcal{R}^\Phi)'$ be the dual space of $\mathcal{R}^\Phi$, that is, the space of all continuous linear functionals on $\mathcal{R}^\Phi$.

$(\mathcal{R}^\Phi)'_+ := \{J \in (\mathcal{R}^\Phi)' | J(X) \geq 0$ for any $X \geq 0\}$.

$(\mathcal{R}^\Phi)'_1 := \{J \in (\mathcal{R}^\Phi)'_+ | J(1) = 1\}$.

$(\mathcal{R}^\Phi)'_1$ is nonempty.

Let $(\mathcal{R}^\Phi)'$ be the dual space of $\mathcal{R}^\Phi$, that is, the space of all continuous linear functionals on $\mathcal{R}^\Phi$.
$(\mathcal{R}^\Phi)'_+ := \{J \in (\mathcal{R}^\Phi)' | J(X) \geq 0$ for any $X \geq 0\}$.
$(\mathcal{R}^\Phi)'_1 := \{J \in (\mathcal{R}^\Phi)'_+ | J(1) = 1\}$.
$(\mathcal{R}^\Phi)'_1$ is nonempty.
By the **extended Namioka-Klee theorem**, any convex risk measure ρ defined on $\mathcal{R}^\Phi$ is represented as

$$\rho(X) = \max_{J \in (\mathcal{R}^\Phi)'_1} \left\{ J(-X) - a_\rho(J) \right\} \text{ for } X \in \mathrm{int}(\{\rho < \infty\}), \tag{1}$$

where a_ρ is defined as

$$a_\rho(J) := \sup_{X \in \mathcal{R}^\Phi} \{J(-X) - \rho(X)\}, \tag{2}$$

which is called the penalty function of ρ.

Let ρ be a $(-\infty, \infty]$-valued functional on $\mathcal{R}^{\Phi}$. ρ admits the representation

$$\rho(X) = \sup_{J \in (\mathcal{R}^{\Phi})'_1} \{J(-X) - a_\rho(J)\}, \tag{3}$$

if and only if ρ is a $\sigma(\mathcal{R}^{\Phi}, (\mathcal{R}^{\Phi})')$-l.s.c. convex risk measure.

Assumptions

Let Φ' be the left derivative of Φ, I the right-continuous inverse of Φ'. We moreover define

$$q_\Phi := \inf_{x>0} \frac{x\Phi'(x)}{\Phi(x)}, \quad p'_\Psi := \frac{q_\Phi}{q_\Phi - 1}, \quad \text{and} \quad p_\Psi := \sup_{y>0} \frac{yI(y)}{\Psi(y)}.$$

Note that $p'_\Psi \leq p_\Psi$.

Assumption 1

1. $p_\Psi < \infty$, which implies that Ψ is moderate growth function.
2. $\lim_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$.

Examples

We introduce Φs satisfying Assumption 1:

1. $\Phi(x) = |x|^p/p$, $p > 1$.
   In this case, $p_\Psi = p/(p-1)$ and $L^\Phi = M^\Phi$.
2. $\Phi(x) = e^{|x|} - |x| - 1$.
   We can say easily $p'_\Psi \leq p_\Psi \leq 2$.
   In this case, M^Φ is a proper subset of L^Φ.

orlicz space

We enumerate functions which do not satisfy Assumption 1:

1. $\Phi(x) = |x|$.
   The second assumption is not satisfied.
2. $\Phi(x) = e^{|x|} - 1$.
   We can see that $p'_\Psi = +\infty$.
3. $\Phi(x) = (|x| + 1)\log(|x| + 1) - |x|$.
4. $\Phi(x) = |x| - \log(|x| + 1)$.

As for the last two functions, we have $q_\Phi = 1$, that is, $p_\Psi = \infty$.

The space $\mathcal{D}$

$$\mathcal{D} := \Big\{(D^-, D^+) | D^{\pm} \text{ are right-continuous, integrable variation processes,}$$
$$V := \int_{(0,T]} |dD_t^-| + \int_0^T |dD_t^+| \in M^{\Psi},$$
$$D^- \text{ is a predictable process with } D_0^- = 0,$$
$$D^+ \text{ is a purely discontinuous optional process}\Big\}.$$

◂ randomized stopping time

Theorem 1

The following theorem as an extension of Theorem VII.65 in Dellacherie and Meyer.

1. For any $(D^-, D^+) \in \mathcal{D}$, defining a functional J as
$$J(X) := E\left[\int_{(0,T]} X_{t-} dD_t^- + \int_0^T X_t dD_t^+\right] \text{ for any } X \in \mathcal{R}^\Phi, \tag{4}$$
we have $J \in (\mathcal{R}^\Phi)'$. Moreover, $||J|| \leq 2||V||_\Psi$, where $||J|| := \sup_{X \in \mathcal{R}^\Phi, ||X||_{\mathcal{R}^\Phi} \leq 1} |J(X)|$.
2. For any $J \in (\mathcal{R}^\Phi)'$, there exists $(D^-, D^+) \in \mathcal{D}$ uniquely, which satisfies (4). Furthermore, we have $||V||_\Psi \leq p'_\Psi ||J||$.

Theorem 2

Any $J \in (\mathcal{R}^\Phi)'_1$ is represented as

$$J(X) = E\left[\int_{(0,T]} X_{t-}L_{t-}dK^-_t + \int_0^T X_t L_t dK^+_t\right] \text{ for any } X \in \mathcal{R}^\Phi, \tag{5}$$

where processes L, K^- and K^+ satisfy the following conditions:

1. L is a nonnegative local martingale with $L_0 = 1$.
2. Both $K^\pm$ are right-continuous nondecreasing processes. K^- is a predictable process with $K^-_0 = 0$, and K^+ is a purely discontinuous optional process. Denoting $K := K^- + K^+$, $0 \leq K \leq 1$ holds.
3. $L_t = 1 + \int_{(0,t]} 1_{\{K_{s-}<1\}} dL_s$ and $K_t = \int_0^t 1_{\{L_s>0\}} dK_s$.

In addition, we can take K so that $K_T = 1$, and a triplet (L, K^-, K^+) with $K_T = 1$ is essentially unique.

Theorem 3

Canonical density

By Theorem 2, each $J \in (\mathcal{R}^{\Phi})'_1$ is corresponding to a triplet (L, K^-, K^+) with $K_T = 1$ satisfying conditions 1–3 in Theorem 2.

Furthermore, $V_J := \int_{(0,T]} L_{t-} dK_t^- + \int_0^T L_t dK_t^+$ is said the **canonical density** of J.

Theorem 3 Optimal stopping

Randomized stopping times 1

Let $\mathcal{R}^1$ be the space $\mathcal{R}^\Phi$ corresponding to the case where $\Phi(x) = |x|$.

A positive linear continuous functional $\gamma \in (\mathcal{R}^1)'$ is called a **randomized stopping time**, if there exists a pair of adapted nondecreasing right-continuous processes (A^-, A^+) such that A^- is a predictable process with $A_0^- = 0$, A^+ is a purely discontinuous optional process, $A_T^- + A_T^+ = 1$, and

$$\gamma(X) = E\left[\int_{(0,T]} X_{t-} dA_t^- + \int_0^T X_t dA_t^+\right] \text{ for any } X \in \mathcal{R}^1.$$

Set $X_\gamma := \int_{(0,T]} X_{t-} dA_t^- + \int_0^T X_t dA_t^+$.

Denote by Γ the set of all randomized stopping times.

◂ $\mathcal{D}$

Randomized stopping times 2

Any stopping time $\tau \in \mathcal{T}$ is clearly a randomized stopping time, that is, $\mathcal{T} \subset \Gamma$.

Note that Γ forms a convex set.

$\Gamma = \{J \in (\mathcal{R}^\Phi)'_1 | V_J = 1\}$ holds.

Corollary

For any $J \in (\mathcal{R}^\Phi)'_1$, there exist a nonnegative local martingale L with $L_0 = 1$, and a randomized stopping time $\gamma \in \Gamma$ such that

$$J(X) = E[(XL)_\gamma] \text{ for any } X \in \mathcal{R}^\Phi.$$

That is, each $J \in (\mathcal{R}^\Phi)'_1$ has a corresponding pair (L, γ).

Optimal stopping risk measure

$$\mathcal{E}(X) := \sup_{\tau \in \mathcal{T}} E[-X_\tau] \text{ for any } X \in \mathcal{R}^\Phi.$$

$\mathcal{E}$ is rewritten as $\mathcal{E}(X) = \sup_{\gamma \in \Gamma} E[-X_\gamma]$.

Moreover, since $\mathcal{E}$ is **R**-valued, we can rewrite it as $\mathcal{E}(X) = \max_{\gamma \in \Gamma} E[-X_\gamma]$.

The penalty function $a_{\mathcal{E}}$ is given as

$$a_{\mathcal{E}}(J) = \begin{cases} 0, & \text{if } V_J = 1, \\ \infty, & \text{if } V_J \neq 1, \end{cases}$$

where V_J is the canonical density of J.

Canonical density

Greatest coherent risk measure

1. $\mathbb{M}_1(X) := \sup_{Q\in\mathcal{P}^\Psi} E_Q[\sup_{t\in[0,T]}(-X_t)]$,
2. $\mathbb{M}_2(X) := \sup_{J\in(\mathcal{R}^\Phi)'_1} J(-X)$,
3. $\mathbb{M}_3(X) := \sup_{Q\in\mathcal{P}^\Psi,\tau\in\mathcal{T}} E_Q[-X_\tau] = \sup_{Q\in\mathcal{P}^\Psi,\gamma\in\Gamma} E_Q[-X_\gamma] = \sup_{J\in\mathcal{J}} J(-X)$,

where $\mathcal{P}^\Psi$ is the set of all probability measures $Q \ll P$ with $dQ/dP \in M^\Psi$.

Note that $\mathbb{M}_1 \geq \mathbb{M}_2 \geq \mathbb{M}_3$ holds true, and
$\mathbb{M}_2$ is the greatest $\sigma(\mathcal{R}^\Phi, (\mathcal{R}^\Phi)')$-l.s.c. coherent risk measure.
Let us see $\mathbb{M}_1 \geq \mathbb{M}_2$.
For any $J \in (\mathcal{R}^\Phi)'_1$, there exists a $Q^J \in \mathcal{P}^\Psi$ with $dQ^J/dP = V_J$.
Thus, we have $J(-X) \leq E_{Q^J}[\sup_{t\in[0,T]}(-X_t)] \leq \mathbb{M}_1(X)$, namely,
$\mathbb{M}_1(X) \geq \sup_{J\in(\mathcal{R}^\Phi)'_1} J(-X)$.

We presume an investor selling an American option H.
She is supposed to control her shortfall risk at the moment when H is exercised.
If she selects $C \in \mathcal{C}$ as her hedging strategy and H is exercised at a stopping time τ, her final cash-flow is $x + C_\tau - H_\tau$, and its shortfall is $(-x - C_\tau + H_\tau) \vee 0$.

Denoting her loss function by l, her shortfall risk is described by $E[l(-x - C_\tau + H_\tau)]$.
We assume that $l(x) = 0$ if $x \leq 0$, and $l(x) = \Phi(x)$ if $x > 0$.

Given the limitation of shortfall risk which she can endure as $\delta > 0$, the least price for H which she can accept is given by

$$\inf\{x \in \mathbf{R} | \text{ there exists } C \in \mathcal{C} \text{ such that } x + C - H \in \mathcal{A}^0\}, \tag{6}$$

where $\mathcal{A}^0 := \{X \in \mathcal{R}^\Phi | \sup_{\tau \in \mathcal{T}} E[l(-X_\tau)] \leq \delta\}$.

Regarding (6) as a functional, denoted by ρ_l, it is defined as

$$\rho_l(X) := \inf\{x \in \mathbf{R} | \text{ there exists } C \in \mathcal{C} \text{ such that } x + C + X \in \mathcal{A}^0\}$$

for any $X \in \mathcal{R}^\Phi$, and call it shortfall risk measure. Then, we have $\rho_l(-H) = (6)$.

Proposition 4

Assuming $\rho_l(0) > -\infty$, ρ_l is an **R**-valued convex risk measure on $\mathcal{R}^\Phi$.

Representations of ρ_l

Henceforth, we assume that $\rho_l(0) > -\infty$.
Since ρ_l is **R**-valued, Namioka-Klee theorem implies that ρ_l is represented as

$$\rho_l(X) = \max_{J \in (\mathcal{R}^\Phi)'_1} \left\{ J(-X) - a_{\rho_l}(J) \right\},$$

where a_{ρ_l} is the penalty function of ρ_l defined in (2).

We focus on estimating the penalty function a_{ρ_l}. First of all, we illustrate another description of a_{ρ_l}.

Lemma

We have $a_{\rho_l}(J) = \sup_{X \in \widetilde{\mathcal{A}}} J(-X)$ for any $J \in (\mathcal{R}^\Phi)'_1$, where
$\widetilde{\mathcal{A}} := \{X \in \mathcal{R}^\Phi |$ there exists $C \in \mathcal{C}$ such that $X + C \geq Y$ for some $Y \in \mathcal{A}^0\}$.

Denoting $\mathcal{A}^1 := \{X \in \mathcal{X} |$ there exists $C \in \mathcal{C}$ such that $C + X \geq 0\}$, we have $\widetilde{\mathcal{A}} := \{X^0 + X^1 | X^0 \in \mathcal{A}^0, X^1 \in \mathcal{A}^1\}$. Thus, we can rewrite $a_{\rho_I}(J)$ as

$$a_{\rho_I}(J) = \sup_{X^0 \in \mathcal{A}^0} J(-X^0) + \sup_{X^1 \in \mathcal{A}^1} J(-X^1). \tag{7}$$

Since $\mathcal{C}$ is cone, we have, for any $J \in (\mathcal{R}^\Phi)'_1$,

$$\sup_{X \in \mathcal{A}^1} J(-X) = \begin{cases} 0, & \text{if } J \in (\mathcal{R}^\Phi)'_{\mathcal{C}} \\ +\infty, & \text{otherwise,} \end{cases}$$

where $(\mathcal{R}^\Phi)'_{\mathcal{C}} = \{J \in (\mathcal{R}^\Phi)'_1 | J(C) \leq 0$ for any $C \in \mathcal{C}\}$. We can rewrite as

$$\rho_I(X) = \max_{J \in (\mathcal{R}^\Phi)'_{\mathcal{C}}} \left\{ J(-X) - \sup_{X^0 \in \mathcal{A}^0} J(-X^0) \right\}.$$

Hence, we estimate only the first term of the RHS of (7).

Theorem 3

For any $J \in (\mathcal{R}^{\Phi})'_1$, we have

$$\inf_{\lambda>0} \frac{1}{\lambda}\{\delta + E[l^*(\lambda L^*)]\} \geq \sup_{X\in\mathcal{A}^0} J(-X) \geq \inf_{\lambda>0} \frac{1}{\lambda}\{\delta + E[l^*(\lambda V_J)]\} \tag{8}$$

where l^* is the conjugate function of l, L is the corresponding local martingale to J in the sense of Theorem 2, and V_J is the canonical density of J.

Prop1 canonical density Theorem 2

Corollary

The shortfall risk measure ρ_l is evaluated as follows:

$$\sup_{J \in (\mathcal{R}^\Phi)'_C} \left\{ J(-X) - \inf_{\lambda > 0} \frac{1}{\lambda} \{\delta + E[l^*(\lambda L^*)]\} \right\}$$

$$\leq \quad \rho_l(X) \leq \sup_{J \in (\mathcal{R}^\Phi)'_C} \left\{ J(-X) - \inf_{\lambda > 0} \frac{1}{\lambda} \{\delta + E[l^*(\lambda V_J)]\} \right\}.$$

Prop1

Thank you for your attention!