The approximation of bonds and swaptions prices in a Black-Karasiński Model based on the Karhunen-Loève expansion

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Summary of results

- Explicit formulae for Karhunen-Loève Theorem of the Ornstein-Uhlenbeck process and Ornstein-Uhlenbeck bridge
- New idea of approximation technique applicable for particular functionals of Gaussian processes
- Numerically efficient and accurate semi-analytic approximative formulae for bonds and swaptions pricing in a Black-Karasiński model

Karhunen-Loève Theorem

Let $(X_t)_{t\in[a,b]}$ be a centered stochastic process with a covariance function K. Then X_t admits the expansion:

$$X_t = \sum_{n=0}^{\infty} \sqrt{\lambda_n} f_n(t) Z_n \quad \text{a.e.} \qquad \text{where} \qquad Z_n = \frac{1}{\sqrt{\lambda_n}} \int_a^b X_t f_n(t)\, dt, \quad n \geqslant 0$$

are orthogonal random variables with mean value equal to zero, $\mathbb{E}(Z_n^2) = 1$ and $\{f_n(t),\ n \geqslant 0\}$ form an orthonormal basis in $L^2([a,b])$ consisting of the eigenfunctions (corresponding to nonzero eigenvalues) of a Fredholm operator $\mathcal{F}_K$, associated with kernel K, i.e.:

$$\mathcal{F}_K f_n = \lambda_n f_n \quad \text{for each } n \geqslant 0. \qquad \text{where} \qquad \Big(\mathcal{F}_K h\Big)(\cdot) = \int_a^b K(\cdot, x) h(x)\, dx.$$

This series converges pointwise a.s. in the norm $\|Y(t)\| := \Big(\mathbb{E} Y^2(t)\Big)^{1/2}$ and moreover the convergence is uniform in $t \in [a,b]$.
In particular, if $(X_t)_{t\in[a,b]}$ is a centered Gaussian process, then Z_n's are independent $\mathcal{N}(0,1)$ random variables.

Optimal approximation property

Truncated series of Karhunen-Loève expansion give optimal approximation of the process in a sense of a mean-square error. More precisely, if we consider any orthonormal basis f_k of $L^2([a, b])$, we may decompose the process X as:

$$X_t = \sum_{n=0}^{\infty} f_n(t) Z_n \quad \text{a.e.} \qquad \text{where} \qquad Z_n = \int_a^b X_t f_n(t) dt$$

If we consider approximation error of such truncated representation

$$e_t = X_t - \sum_{n=0}^{m} f_n(t) Z_n$$

then the Karhunen-Loève expansion is the one that minimizes its norm

$$\int_a^b \|e_m\|^2 \, dt$$

Note also the fact that the conditional expectation (w.r.t. the given sigma-field) of the random variable is its closest approximation in the L^2 norm among all random variables measurable w.r.t. this sigma-field.

Approximation idea

Consider a centered stochastic process $(X_t)_{t\in[a,b]}$ and its Karhunen-Loève expansion. Let also $h\colon \mathbb{R} \to \mathbb{R}$ be a function such that $\mathbb{E}(|h(X_t)|)$ exists for each $t \in [a, b]$. Consider the integral:

$$I = \int_a^b h(X_t)\, dt$$

Then we will approximate it as

$$I \approx \int_a^b \mathbb{E}(h(X_t)|Z_0)\, dt$$

Note that while the original integral is a random variable of rather complex distribution, the approximated one is just a Borel function of a single normal variable.

Much easier to deal with! ...if we only can compute this function effectively.

Ornstein-Uhlenbeck process

We will consider the centered Ornstein-Uhlenbeck process $(X_t)_{t\in[0,T}$ given by an Itô equation

$$dX_t = -bX_t\,dt + dW_t$$

with an initial value $X_0 = 0$ and a real parameter $b > 0$.
It can be characterized also as a centred Gaussian process with a covariance function

$$K(s,t) := \frac{1}{2b}e^{-b|t-s|} - \frac{1}{2b}e^{-b(t+s)}$$

and variance function

$$V(s) := K(s,s) = \frac{1}{2b}\left(1 - e^{-2bs}\right).$$

Ornstein-Uhlenbeck bridge

We can consider also an Ornstein-Uhlenbeck bridge process $\left(\hat{X}_t\right)_{t\in[0,T]}$ defined as

$$\hat{X}_t := X_t - \frac{K(t,T)}{V(T)} X_T, \quad t \in [0,T]$$

which is also centered Gaussian process with a covariance function

$$\hat{K}(s,t) := K(s,t) - \frac{K(s,T)K(t,T)}{V(T)}$$

and the variance function

$$\hat{V}(t) := \hat{K}(t,t) = V(t)\left(1 - \frac{e^{-2b(T-t)}}{1-e^{-2bT}}\right).$$

Additionally for any $t \in [0,T]$ the random vector $(X_T, \hat{X}_t)$ has a joint Gaussian distribution and by definition $\mathbb{E}(X_T \hat{X}_t) = 0$, hence $\hat{X}_s$ is independent of X_T.

Fredholm equation for Ornstein-Uhlenbeck kernels

Consider a Fredholm operator $\mathcal{F}$, associated with kernel K or $\hat{K}$, respectively. Elementary calculation shows that

$$\left(\mathcal{F}f\right)'' = -f + b^2\mathcal{F}f.$$

Hence eigenfunction f of $\mathcal{F}$ (with eigenvalue $\lambda \neq 0$) must fulfill linear differential equation:

$$-f + b^2\lambda f = \lambda f''.$$

with two additional boundary conditions:

$$f(0) = 0$$

and

$$f'(T) = -bf(T)$$

or

$$f(T) = 0$$

for the Ornstein-Uhlenbeck process or Ornstein-Uhlenbeck bridge, respectively. These differential equations can be easily solved, one can also prove that their solutions are solutions of the corresponding Fredholm equation.

Representation of the Ornstein-Uhlenbeck process

Let $(X_t)_{t\in[0,\tau]}$ be an Ornstein-Uhlenbeck process in the interval $[0,\tau]$. Then its Karhunen-Loève expansion has the form

$$X_t = \sum_{n=0}^{\infty} \sqrt{\lambda_n(\tau)} f_{n,\tau}(t) Z_n$$

where Z_n are independent, identically distributed normal $N(0,1)$ variables and $f_{n,\tau}, \lambda_n(\tau)$ are given by

$$f_{n,\tau}(t) = \sqrt{\frac{2}{\tau + b\lambda_n(\tau)}} \sin(\omega_n(\tau)t),$$

$$\lambda_n(\tau) = \frac{1}{b^2 + \omega_n(\tau)^2},$$

where $\omega_n(\tau)$ is the unique solution of the equation

$$\omega \operatorname{ctg}(\omega\tau) = -b$$

in the interval $\left(\left(n+\frac{1}{2}\right)\frac{\pi}{\tau}, (n+1)\frac{\pi}{\tau}\right),\ n \in \mathbb{N} \cup \{0\}.$

Representation of the Ornstein-Uhlenbeck bridge

Let $(\hat{X}_t)_{t\in[0,T]}$ be an Ornstein-Uhlenbeck bridge process in the interval $[0, T]$. Then its Karhunen-Loève expansion has the form

$$\hat{X}_t = \sum_{n=1}^{\infty} \sqrt{\hat{\lambda}_n(T)}\hat{f}_{n,T}(t)\hat{Z}_n$$

where $\hat{Z}_n$ are independent, identically distributed normal $N(0, 1)$ variables and $f_{n,T}, \hat{\lambda}_n(T)$ are given by

$$\hat{f}_{n,T}(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{n\pi t}{T}\right),$$

$$\hat{\lambda}_n(T) = \frac{T^2}{b^2 T^2 + n^2\pi^2}.$$

Black-Karasiński model

Consider the process $(r_t)_{t\geqslant 0}$ the short-term rate and of its logarithm $l_t :\ = \ln(r_t)$. In a single factor Black-Karasiński model l_t is assumed to follow the dynamics:

$$dl_t = \Big(a(t) - b\, l_t\Big)\, dt + \sigma\, dW_t,$$

where σ, b are positive constants, $a(t)$ is some deterministic function and $(W_t)_{t\geqslant 0}$ is a Wiener process under the spot measure.
In explicit terms

$$r_{u+t} = \exp\left(A(u, u+t) + e^{-bt}\ln(r_u) + \sigma X_t^{(u)}\right) = \bar{r}_{u,t}\exp\left(\sigma X_t^{(u)}\right)$$

where we denoted

$$A(u, u+v) := \int\limits_u^{u+v} e^{-b(u+v-s)} a(s) ds$$

$$\bar{r}_{u,v} := r_u^{e^{-bv}}\exp(A(u, u+v))$$

$$X_t^{(u)} := \frac{1}{\sigma}\Big(l_{u+t} - e^{-bt} l_u - A(u, u+t)\Big),$$

where the last process is an Ornstein-Uhlenbeck process, independent from $\mathcal{F}_u$.

Approximation of saving account

Consider the value of saving account (starting at the time $T \geqslant 0$):

$$\beta_T(\tau) := \exp\Big(\int_T^{T+\tau} r(s)\, ds \Big)$$

Now are going to approximate the logarithm of saving account according to our idea with the first term of the Karhunen-Loève expansion of the Ornstein-Uhlenbeck process. In order to do that we first compute directly:

$$\mathbb{E}\Big(r_{T+t} | Z_0 \Big) = \bar{r}_{T,t} F_\tau(t, Z_0)$$

where

$$F_\tau(t,z) := \exp\left(\sigma \sqrt{\lambda_0(\tau)} f_{0,\tau}(t) z - \frac{\sigma^2}{2} \Big(V(t) - \lambda_0(\tau) f_{0,\tau}^2(t) \Big) \right),$$

Thence we obtain

$$\beta_T(\tau) \approx \exp\Big(\int_0^{\tau} \bar{r}_{T,t} F_\tau(t, Z_0)\, dt \Big)$$

Approximation of bond price

In consequence the price of a zero-coupon bond $B(T, T+\tau)$ at time $T \geqslant 0$, with maturity $T+\tau$, can be approximated with the formula

$$B(T, T+\tau) = \mathbb{E}\Big(\frac{1}{\beta_T(\tau)}|\mathcal{F}_T\Big) \approx \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \exp\left(-\bar{r}_{T,t} \int\limits_0^\tau F_\tau(t,z)\, dt\right) e^{-\frac{z^2}{2}}\, dz \quad ,$$

This approximation can be further simplified by approximating exponential in the definition of function F_τ with a quadratic function and calculating integral over z effectively:

$$B(T, T+\tau) \approx \exp\left(-(c_0+c_2) + \frac{c_1^2}{2(1+2c_2)} + c_2^2\right),$$

where

$$h_{0,\tau}(t) := \exp\left(\frac{\sigma^2}{2}\Big(V(t) - \lambda_0(\tau) f_{0,\tau}(t)^2\Big)\right)$$

$$c_j := \frac{1}{j!}\Big(\sigma\sqrt{\lambda_0(\tau)}\Big)^j \int\limits_0^\tau \bar{r}_{T,t} h_{0,\tau}(t) f_{0,\tau}(t)^j\, dt \qquad j = 1, 2, 3.$$

Numerical results

In order to test the accuracy of approximations presented in the previous section, several prices of zero coupon bonds have been computed. It was assumed for simplicity that $a(\cdot)$ is a constant function of the form $a(t) = b \ln(r_{\text{avg}})$, where $r_{\text{avg}} = 5\%$.
For the sake of clarity and comparability, results are presented in the form of yields-to-maturity (with a continuous compounding convention), not the actual prices of bonds. In order to examine the dependence of results on different parameters, yields to maturity were calculated for the following:

- maturities: 2, 5, 10 and 20 years
- values of r_0: 2%,4% and 8%
- values of b: 0,05, 0,1 and 0,2
- values of σ: 15% and 30%

Below we present a comparison of results of such prices obtained from Approximation 1, Approximation 2 and from a Monte-Carlo simulations approach, which provides benchmark values.
As we can see, the errors of approximations are very small with an order of at most few basis points no matter the length of maturities or which parameter sets are examined. In addition, they are comparable with numerical errors from the Monte Carlo approach.

Parameters					Yields			Errors		
Maturity [Y]	r_0 [%]	a	b	σ [%]	MC [%]	Appr. 1 [%]	Appr. 2 [%]	MC [%]	Appr. 1 vs MC [%]	Appr. 2 vs Appr. 1 [%]
2	4.0	−0.30	0.10	15	4.1239	4.1234	4.1233	0.002	0.000	0.000
5	4.0	−0.30	0.10	15	4.2719	4.2728	4.2725	0.002	0.001	0.000
10	4.0	−0.30	0.10	15	4.4495	4.4502	4.4508	0.003	0.001	0.001
20	4.0	−0.30	0.10	15	4.6479	4.6502	4.6537	0.003	0.002	0.003
2	2.0	−0.30	0.10	15	2.2029	2.2033	2.2032	0.001	0.000	0.000
5	2.0	−0.30	0.10	15	2.4951	2.4936	2.4933	0.001	−0.002	0.000
10	2.0	−0.30	0.10	15	2.9210	2.9198	2.9197	0.002	−0.001	0.000
20	2.0	−0.30	0.10	15	3.5550	3.5406	3.5427	0.003	−0.014	0.002
2	8.0	−0.30	0.10	15	7.7238	7.7255	7.7253	0.003	0.002	0.000
5	8.0	−0.30	0.10	15	7.3610	7.3617	7.3618	0.004	0.001	0.000
10	8.0	−0.30	0.10	15	6.8797	6.8815	6.8836	0.004	0.002	0.002
20	8.0	−0.30	0.10	15	6.2600	6.2694	6.2752	0.004	0.009	0.006

Comparison of prices and differences between them obtained from Monte Carlo simulations, Approximation 1 and Approximation 2

Parameters					Yields			Errors		
Maturity [Y]	r_0 [%]	a	b	σ [%]	MC [%]	Appr. 1 [%]	Appr. 2 [%]	MC [%]	Appr. 1 vs MC [%]	Appr. 2 vs Appr. 1 [%]
2	4.0	−0.15	0.05	15	4.0847	4.0842	4.0840	0.002	−0.001	0.000
5	4.0	−0.15	0.05	15	4.1920	4.1911	4.1907	0.002	−0.001	0.000
10	4.0	−0.15	0.05	15	4.3280	4.3247	4.3256	0.003	−0.003	0.001
20	4.0	−0.15	0.05	15	4.4769	4.4755	4.4850	0.004	−0.001	0.009
2	4.0	−0.60	0.20	15	4.1993	4.1960	4.1959	0.001	−0.003	0.000
5	4.0	−0.60	0.20	15	4.4078	4.4075	4.4074	0.002	0.000	0.000
10	4.0	−0.60	0.20	15	4.6262	4.6223	4.6225	0.002	−0.004	0.000
20	4.0	−0.60	0.20	15	4.8280	4.8184	4.8191	0.002	−0.010	0.001
2	4.0	−0.30	0.10	30	4.2440	4.2418	4.2396	0.003	−0.002	−0.002
5	4.0	−0.30	0.10	30	4.5087	4.5071	4.5035	0.005	−0.002	−0.004
10	4.0	−0.30	0.10	30	4.7557	4.7572	4.7679	0.006	0.001	0.011
20	4.0	−0.30	0.10	30	4.9313	4.9328	4.9816	0.006	0.002	0.049

Comparison of prices and differences between them obtained from Monte Carlo simulations, Approximation 1 and Approximation 2

Approximation of conditional bond price

For further applications we will need yet another approximation of bond's price - conditional on the level of then underlying Ornstein-Uhlenbeck process at maturity, being a function of this level. Such approximation can be obtained the same way as unconditional one, using expansion of the Ornstein-Uhlenbeck bridge instead of Ornstein-Uhlenbeck process itself. As a result we get:

$$\hat{B}(0,T,x) := \mathbb{E}\Big(\frac{1}{\beta_0(T)}|X_t = x\Big) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\int_0^T \bar{r}_{0,t} \hat{F}_\tau(t,z,x)\,dt\right) e^{-\frac{z^2}{2}}\,dz$$

where

$$\hat{F}_\tau(t,z,x) := \exp\left(\sigma\sqrt{\hat{\lambda}_1(T)}\,\hat{f}_{1,T}(t)\,z - \frac{\sigma^2}{2}\left(\hat{V}(t) - \hat{\lambda}_1(\tau)\hat{f}_{1,T}(t)^2\right) + \sigma\frac{K(t,T)}{V(T)}x\right),$$

$$\hat{f}_{1,T}(t) = \sqrt{\frac{2}{T}}\sin\left(\frac{\pi t}{T}\right), \quad \hat{\lambda}_1(T) = \frac{T^2}{b^2 T^2 + \pi^2} \quad .$$

Approximation of conditional bond price

This can be approximated further to

$$\hat{B}(0,T,x) \approx \exp\left(-(\hat{c}_0 + \hat{c}_2) + \frac{\hat{c}_1^2}{2(1+2\hat{c}_2)} + \hat{c}_2^2\right),$$

where

$$\hat{h}_{1,T}(t,x) := \exp\left(\frac{\sigma^2}{2}\left(\hat{V}(t) - \hat{\lambda}_1(T)\hat{f}_{1,T}(t)^2\right) + \sigma\frac{K(t,T)}{V(T)}x\right)$$

$$\hat{c}_j := \frac{1}{j!}\left(\sigma\sqrt{\hat{\lambda}_1(T)}\right)^j \int_0^T \bar{r}_{0,t}\hat{h}_{1,T}(t,x)\hat{f}_{1,T}(t)^j\,dt \qquad j = 1,2,3.$$

Swaption pricing

Consider a payer/receiver swaption with strike S and expiry date T on an underlying swap rate paid at $t = T + k\delta, k = 1, ..., n$. It is equivalent to the put/call option on a coupon-paying bond and can be priced under a spot measure as

$$Swpt = \mathbb{E}\left(\frac{1}{\beta_0(T)} 1_{\{\omega \geqslant \omega C(T,n,S)\}} \omega \Big(1 - C(T, n, S) \Big) | \mathcal{F}_0 \right)$$

where $\omega = +/-1$ corresponds to the payer/receiver swaption and

$$C(T, n, S) := B(T, T + n\delta) + S \sum_{k=1}^{n} B(T, T + k\delta) \ .$$

Swaption pricing

But in a single factor model bond's price is a decreasing function of the short rate, so if we denote this dependence as $C(T, n, S) = C(T, n, S; r_T)$, we can rewrite previous formula as

$$Swpt = \mathbb{E}\left(\frac{1}{\beta_0(T)} 1_{\{\omega r_T \geqslant \omega r(S)\}} \omega \left(1 - C(T, n, S; r_T)\right) | \mathcal{F}_0\right)$$

where $r(S)$ is the unique solution of the equation

$$C(T, n, S; r(S)) = 1 \ .$$

Moreover we can first take this expectation conditioned on X_T, which gives

$$Swpt = \mathbb{E}\left(1_{\{\omega r_T \geqslant \omega r(S)\}} \hat{B}(0, T, X_T) \omega \left(1 - C(T, n, S; r_T)\right) | \mathcal{F}_0\right) \ .$$

Technical approximations

To obtain tractable formula we should make a few addional approximations:

- take the second approximation of $\hat{B}(0, T, x)$ and use it to find approximation of $r(S)$ numerically
- expanding into Taylor series derive approximation of the form

$$\hat{B}(0, T, x) \approx \exp\left(m_0(S) + m_1(S)\sigma x + m_2(S)\sigma^2 x^2\right)$$

- similarly, using approximation for the zero-coupon bond and Taylor series compute coefficients of approximation of the form:

$$\tilde{C}(T, n, S) \approx u_0(S) + u_1(S) r_T + u_2(S) r_T^2$$

... then put it all together into the swaption pricing formula.
At the end we get:

Approximation of swaption price

The price of a payer/receiver swaption can be approximated by the formula

$$\begin{aligned}\omega\Gamma^{-1}Swpt \approx\ & \Big(1 - u_0(S)\Big)\Phi\Big(\omega d_0\Big) \\ & - u_1(S)\bar{r}_{0,T}\exp\left(\left(m_1(S) + \frac{1}{2}\right)\frac{\sigma^2}{\vartheta^2}\right)\Phi(\omega d_1) \\ & - u_2(S)\bar{r}_{0,T}^2\exp\left(2\Big(m_1(S)+1\Big)\frac{\sigma^2}{\vartheta^2}\right)\Phi(\omega d_2)\end{aligned}$$

where:

$$\vartheta := \sqrt{\frac{1 - 2m_2(S)\sigma^2 V(T)}{V(T)}}$$

$$d_j := \frac{m_1(S) + j}{\vartheta}\sigma - \frac{\vartheta}{\sigma}\ln\left(\frac{\tilde{r}(S)}{\bar{r}_{0,T}}\right), \quad j = 0, 1, 2,$$

$$\Gamma = \frac{1}{\vartheta\sqrt{V(T)}}\exp\left(m_0(S) + \frac{1}{2}\left(\frac{m_1(S)\sigma}{\vartheta}\right)^2\right)$$

Numerical results

Model parameters				Swaption parameters								
r_0 [%]	a	b	σ [%]	exp. [Y]	tenor [Y]	Fw. IRS [%]	ATM [%]	OTM 3.5% [%]	OTM 4% [%]	OTM 4.5% [%]	OTM 5% [%]	OTM 5.5% [%]
4.0	−0.2996	0.1	15	1	1	4.2108	0.0198	−0.032	0.0015	0.0274	0.0259	0.0305
4.0	−0.2996	0.1	15	1	2	4.2478	0.0244	−0.0457	0.0000	0.0366	0.0427	0.0442
4.0	−0.2996	0.1	15	1	5	4.3422	0.0122	−0.1006	−0.0198	0.0213	0.0427	0.0198
4.0	−0.2996	0.1	15	1	10	4.4528	−0.0076	−0.1951	−0.0579	0.0351	0.0396	0.0107
4.0	−0.2996	0.1	15	1	20	4.5627	−0.0091	−0.689	−0.1402	0.0213	0.0412	0.0076
2.0	−0.2996	0.1	15	2	1	4.2867	0.0108	0.0463	0.0259	0.0086	0.0075	0.0065
2.0	−0.2996	0.1	15	2	2	4.3206	0.0065	0.0593	0.041	0.0065	0.0129	0.0162
2.0	−0.2996	0.1	15	2	5	4.4054	−0.0226	0.0808	0.0625	0.0216	0.0172	0.0269
2.0	−0.2996	0.1	15	2	10	4.5018	−0.083	0.1153	0.0798	0.0259	−0.0183	0.0097
2.0	−0.2996	0.1	15	2	20	4.5967	−0.1099	0.1444	0.0873	0.0356	−0.0463	−0.0065
8.0	−0.2996	0.1	15	5	1	4.471	−0.0089	−0.002	0.0027	0.0055	0.0027	0.0143
8.0	−0.2996	0.1	15	5	2	4.4946	−0.0423	−0.0048	−0.0014	−0.0027	−0.0191	0.0027
8.0	−0.2996	0.1	15	5	5	4.5529	−0.137	−0.0089	−0.0095	−0.0191	−0.0839	−0.0293

Differences between implied volatilities calculated from swaption approximation prices and from Monte-Carlo simulation prices

Numerical results

Model parameters				Swaption parameters								
r_0 [%]	a	b	σ [%]	exp. [Y]	tenor [Y]	Fw. IRS [%]	ATM [%]	OTM 3.5% [%]	OTM 4% [%]	OTM 4.5% [%]	OTM 5% [%]	OTM 5.5% [%]
8.0	−0.2996	0.1	15	5	10	4.6172	−0.24	−0.032	−0.0273	−0.0382	−0.1663	−0.0641
8.0	−0.2996	0.1	15	5	20	4.6757	−0.2638	−0.0198	−0.0273	−0.032	−0.2038	−0.0593
4.0	−0.1498	0.05	15	10	1	4.6561	−0.0583	0.0111	0.013	0.0125	−0.054	−0.0419
4.0	−0.1498	0.05	15	10	2	4.6677	−0.0964	0.0145	0.013	0.0072	−0.0887	−0.0641
4.0	−0.1498	0.05	15	10	5	4.6952	−0.2058	0.0149	0.0096	−0.0072	−0.1894	−0.1205
4.0	−0.1498	0.05	15	10	10	4.7226	−0.3181	0.0048	−0.0034	−0.0255	−0.3037	−0.162
4.0	−0.1498	0.05	15	10	20	4.7431	−0.3476	0.0352	0.0000	−0.0246	−0.0983	−0.1533
4.0	−0.5991	0.2	15	20	1	4.7724	−0.0835	0.032	0.0337	0.0358	0.0358	−0.0631
4.0	−0.5991	0.2	15	20	2	4.7738	−0.1101	0.0327	0.0341	0.0341	0.0303	−0.0777
4.0	−0.5991	0.2	15	20	5	4.7756	−0.1916	0.0337	0.0327	0.0262	0.0095	−0.1217
4.0	−0.5991	0.2	15	20	10	4.7753	−0.2758	0.0358	0.0273	0.0157	−0.016	−0.1558
4.0	−0.5991	0.2	15	20	20	4.7695	−0.2931	0.061	0.0389	0.0184	−0.0252	−0.1391

Differences between implied volatilities calculated from swaption approximation prices and from Monte-Carlo simulation prices

Thank you for your attention