Financial markets contagion - the spatial approach

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Based on joint works with F.Durante and M.Pitera.

Overview

1. Motivation.

2. Toy model.

3. Copula based detection of contagion.

4. Empirical study.

Motivation

Financial contagion is usually referred to as a cross-market transmission of shocks or the general cross-market spillover effects. It can take place both during "good" times and "bad" times. Thus, contagion does not need to be related to crises. However, it is emphasized during crisis times. If present, it may mitigate the benefits of diversification precisely when those benefits are needed most and have serious consequences for investors. Therefore, understanding this highly nonlinear effect is of great interest not only to financial theorists but to practitioners as well.

Spatial approach (Bradley & Taqqu 2004)

"There is contagion from market **X** to market **Y** if there is more dependence between **X** and **Y** when **X** is doing badly than when **X** exhibits typical performance."

Copula approach (Durante & Jaworski 2010)

Since copulas are the universal tools to describe the dependence among random variables, we restated the above in the following manner:
"There is contagion from market **X** to market **Y** if the conditional copula of the market returns X and Y, when X is smaller than certain quantile, dominates the conditional copula when X is around its median."

Axiomatic definition of a bivariate copula

Definition 1 *A function*

$$C : [0,1]^2 \longrightarrow [0,1]$$

is called a copula if

$$\begin{array}{ll} (C1) & \forall u, v \in [0,1] \quad C(0,v) = 0, \;\; C(u,0) = 0; \\ (C2) & \forall u, v \in [0,1] \quad C(1,v) = v, \;\; C(u,1) = u; \\ (C3) & \forall u_1, u_2, v_1, v_2 \in [0,1], u_1 \leq u_2, v_1 \leq v_2 \\ & \qquad C(u_1,v_2) + C(u_2,v_1) \leq C(u_1,v_1) + C(u_2,v_2). \end{array}$$

Probabilistic definition of a bivariate copula

A bivariate *copula* is the restriction to the unit square $[0,1]^2$ of a distribution function whose univariate margins are uniformly distributed on $[0,1]$.

Sklar's Theorem

By the celebrated *Sklar's Theorem*, the joint distribution function $F_{X,Y}$ of a pair of random variables X, Y defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ can be written as a composition of a copula $C_{X,Y}$ and the univariate marginals F_X and F_Y, i.e. for all $(x, y) \in \mathbb{R}^2$,

$$F_{X,Y}(x, y) = C_{X,Y}(F_X(x), F_Y(y)).$$

Moreover, if F_X and F_Y are continuous, then the copula $C_{X,Y}$ is uniquely determined.

Sklar's Theorem - the special case

Let

$$U = F_X(X), \quad V = F_Y(Y).$$

If F_X and F_Y are continuous then

$$C_{X,Y}(u,v) = F_{U,V}(u,v), \quad \text{for} \quad (u,v) \in [0,1]^2.$$

Toy model

Let X_t and Y_t be the daily logarithmic returns from market indices.

$$\begin{aligned} X_t &= \sigma_X \varepsilon_t^1, \\ Y_t &= \sigma_Y \varepsilon_t^2 - \delta \left(\varepsilon_t^1 + 0.841 \right)^- , \end{aligned}$$

where ε_t^1 and ε_t^2 are independent random variables, $N(0,1)$, and $\sigma_X, \sigma_Y, \delta > 0$.

Then under the condition $X_t > -0.841 \cdot \sigma_X$ the returns X_t and Y_t are independent.
While under the condition $X_t < -0.841 \cdot \sigma_X$ the returns X_t and Y_t are positively dependent.

Market Contagion – formal definition

Let X_t and Y_t be the returns from indices of markets **X** and **Y**. We assume that they are stationary and ergodic.

Let $C_{[a,b]}$ denotes the copula of the conditional distribution of (X_t, Y_t) under the condition $F_X(X_t) \in [a, b]$.

Definition 2 *We say that there is contagion from market* **X** *to market* **Y** *with respect to intervals* $[0, \alpha]$ *and* $[\beta_1, \beta_2]$ *if*

$$\forall (u, v) \in [0, 1]^2 \quad C_{[0,\alpha]}(u, v) \geq C_{[\beta_1,\beta_2]}(u, v)$$

and

$$C_{[0,\alpha]} \neq C_{[\beta_1,\beta_2]}.$$

Univariate conditioning of copulas

Definition 3 *Let a bivariate copula C be the joint distribution function of random variables U, V which are uniformly distributed on the unit interval. For every $a, b \in [0, 1]$, $a < b$, we denote by $C_{[a,b]}$ the copula of the conditional distribution of U, V with respect to the condition $a \leq U \leq b$.*

Univariate conditioning of copulas

By the Sklar's Theorem, $C_{[\gamma]}$ admits the following characterization:

Proposition 1 *Let C be an bivariate copula. For every $a, b \in [0, 1]$, $a < b$, the conditional copula $C_{[a,b]}$ is a unique solution of the equation*

$$C_{[a,b]}\left(x, \frac{C(b, y) - C(a, y)}{b - a}\right) = \frac{C(a + (b - a)x, y)}{b - a}, \quad x, y \in [0, 1]^2. \quad (1)$$

Univariate conditioning of distributions

In Definition 3 one can replace the uniformly distributed random variables U, V by any pair of random variables having continuous distribution functions. Indeed:

Proposition 2 *If the copula C describes the interdependencies between random variables X, Y, then $C_{[a,b]}$ is the copula of the conditional distribution of X, Y with respect to the condition $q_1 \leq X \leq q_2$, where $\mathbf{P}(X \leq q_1) = a$ and $\mathbf{P}(X \leq q_2) = b$.*

Proposition 3 *Let* κ *be a measure of concordance. For any copulas* C_1 *and* C_2*, if*

$$\forall (u,v) \in [0,1]^2 \;\; C_1(u,v) \leq C_2(u,v)$$

then

$$\kappa(C_1) \leq \kappa(C_2).$$

Contagion index

Definition 4 *Let C be a copula and $\beta \in (0, 0.5)$.*

$$\Delta_\beta(C) = \rho(C_{[0,\beta]}) - \rho(C_{[\beta,1-\beta]}),$$

where ρ denotes the Spearman rank correlation.

For our "toy model" we have

$$\rho(C_{[0.2,0.8]}) = 0, \quad \rho(C_{[0,0.2]}) = \rho(Ga_{[0,0.2]}) > 0,$$

where Ga is a Gaussian copula with a positive Pearson correlation

$$r = \delta(\sigma_Y^2 + \delta^2)^{-0.5}.$$

Thus

$$\Delta_\beta(C) = \rho(Ga_{[0,0.2]}) > 0.$$

Choice of β

Paradigm:
There is no contagion for Gaussian copulas.

Facts:
For a Gaussian copula with the positive Pearson correlation

1. $\Delta(\beta)$ is a strictly increasing function.
2. $\lim_{\beta \to 0} \Delta(\beta) = -6/\pi \, \mathrm{arcsin}(r/2)$.
3. $\lim_{\beta \to 0.5} \Delta(\beta) > 0$.
4. Δ changes sign close to (but above) $\beta = 0.21$.

SMI versus Dow Jones

Data:
Daily returns.
From 12.11.1990 to 15.03.2013 (Jahoo Finance).
5500 observations.

Contagion index, $\beta = 0.2$:
DOW $\to$ SMI $\quad \Delta = 0.205685901$,
SMI $\to$ DOW $\quad \Delta = 0.124744469$.

Bibliography

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