Costs and Benefits of Crash Hedging

by

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Outline

1. Introduction/Motivation

Merton's (Classical) Portfolio Optimisation (1969, 1971):

- Investor has **logarithmic utility**, that is $U(x) = \ln(x)$.
- Investment opportunities are one risk–free asset (**bond**) and one risky asset (**stock**) with dynamics given by

$$\begin{aligned} dP_{0,0}(t) &= P_{0,0}(t)\, r_0\, dt\,, & P_{0,0}(0) = 1\,, \qquad \text{"bond"} \\ dP_{0,1}(t) &= P_{0,1}(t)\left[\mu_0\, dt + \sigma_0\, dW_0(t)\right], & P_{0,1}(0) = p_1\,, \qquad \text{"stock"} \end{aligned}$$

  with constant market coefficients μ_0, r_0, $\sigma_0 \neq 0$ and where W_0 is a Brownian Motion on a complete probability space $(\Omega, \mathcal{F}, P)$.

- X_0^π denotes the **wealth process** of the investor given the portfolio strategy π (which denotes the fraction invested in the risky asset). More specific, the wealth process satisfies

$$\begin{aligned} dX_0^\pi(t) &= X_0^\pi(t)\left[(r_0 + \pi(t)\left[\mu_0 - r_0\right])\, dt + \pi(t)\sigma_0\, dW_0(t)\right], \\ X_0^\pi(0) &= x. \end{aligned}$$

- With this, one can define the **performance function** for an arbitrary admissible portfolio strategy $\pi(t)$

$$\begin{aligned}\mathcal{J}_0(t,x,\pi) &:= \mathbb{E}\left[\ln\left(X_0^{\pi,t,x}(T)\right)\right] \\ &= \ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2\right] ds\right] .\end{aligned}$$

- Here,

$$\Psi_0 := r_0 + \frac{1}{2}\left(\frac{\mu_0 - r_0}{\sigma_0}\right)^2 = r_0 + \frac{\sigma_0^2}{2}\left(\pi_0^*\right)^2 \quad \text{and} \quad \pi_0^* := \frac{\mu_0 - r_0}{\sigma_0^2}$$

will be called the **utility growth potential** or **earning potential** and the **optimal portfolio strategy**, respectively.

- The **portfolio optimisation problem** is given by

$$\sup_{\pi(\cdot)\in A(x)} \mathcal{J}_0(t,x,\pi) =: \nu_0(t,x) \quad [= \ln(x) + \Psi_0(T-t)] \ , \qquad (1)$$

where ν_0 is called **value function**.

Merton's Portfolio Optimisation with Jumps (1976):

- In this case the dynamics of the risky asset changes to

$$dP_J(t) \;=\; P_J(t)\left[\mu_0\,dt + \sigma_0\,dW_0(t) - k\,dN(t)\right] ,$$

  where N is a Poisson process with intensity $\lambda > 0$ on $(\Omega, \mathcal{F}, P)$ and $k > 0$ is the crash or jump size.

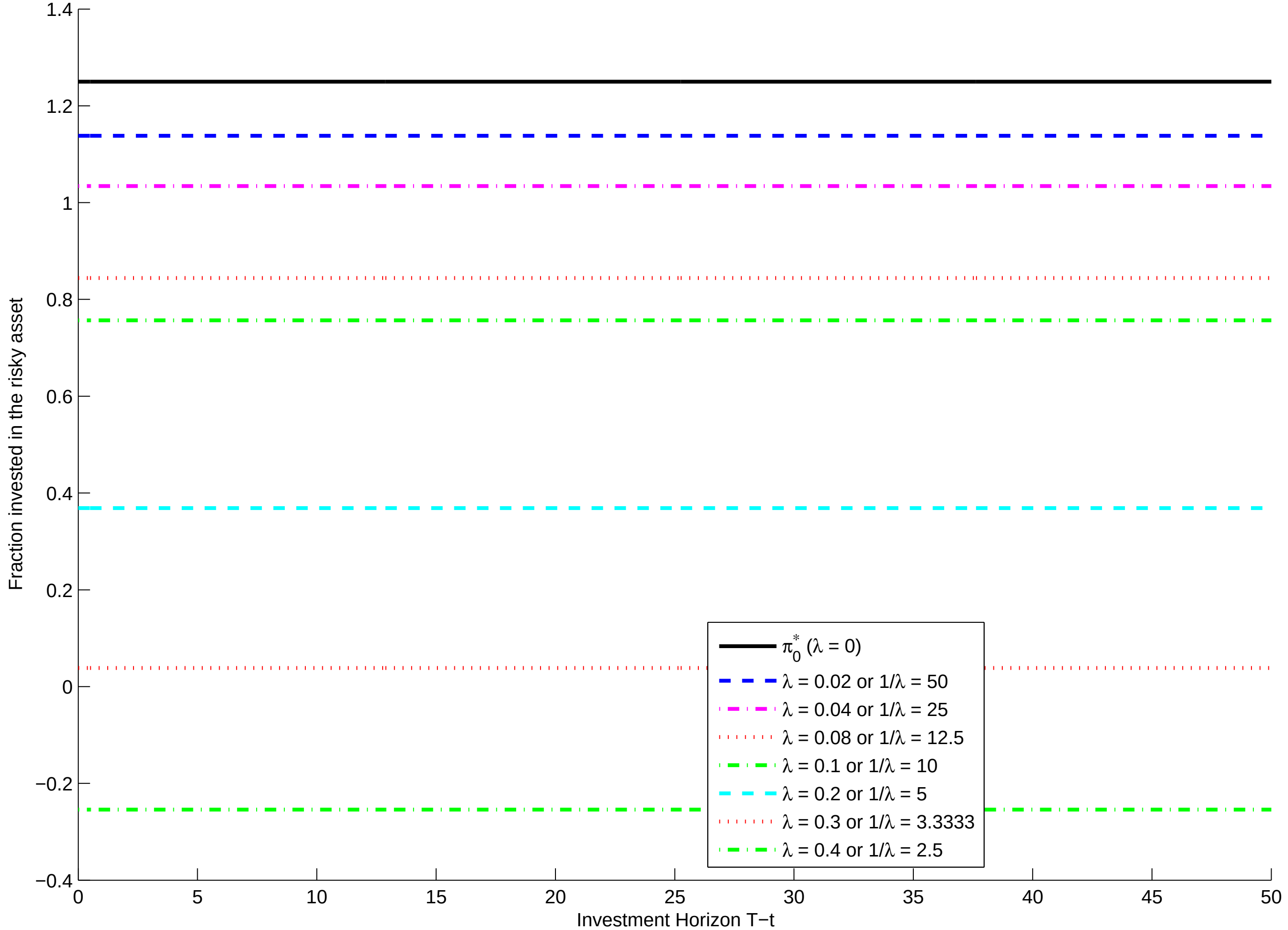
- The performance function is given in this setting as

$$\mathcal{J}_J\,(t, x, \pi)$$
$$= \;\ln(x) + \mathbb{E}\left[\int_t^T \left[\Psi_0 - \frac{\sigma_0^2}{2}\left(\pi(s) - \pi_0^*\right)^2 - \ln\left(1 - \pi(s)k\right)\lambda\right] ds\right] .$$

- The optimal portfolio strategy computes to

$$\pi_J^* = \frac{1}{2}\left(\pi_0^* + \frac{1}{k}\right) - \sqrt{\frac{1}{4}\left(\pi_0^* - \frac{1}{k}\right)^2 + \frac{\lambda}{\sigma_0^2}} .$$

Examples of Merton's Optimal Portfolio Strategies



This Figure is plotted with $\pi_0^* = 1.25$, $\sigma_0 = 0.25$, $r = 0.05$, $k = 0.25$, and $T = 50$. This implies that $\lambda_0 = \frac{\sigma_0^2 \pi_0^*}{k} = 0.3125$, $\Psi_0 \approx 0.098828$, and $\frac{1}{k^*} = 4$.

Alternative: Worst Case Scenario Portfolio Optimisation (Korn and Wilmott, 2002):

- In **normal times:** same set up as in the classical Merton case.
- At **crash time:** stock price falls by a factor of $k \in [k_*, k^*]$.

  Consequence: The wealth process $X_0^\pi(t)$ at crash time τ satisfies:

$$
\begin{aligned}
X_0^\pi(\tau-) &= (1-\pi(\tau))\, X_0^\pi(\tau-) + \pi(\tau) X_0^\pi(\tau-) \\
\Longrightarrow &\quad (1-\pi(\tau)\, X_0^\pi(\tau-) + \pi(\tau) X_0^\pi(\tau-)\,(1-k) \\
= &\quad \boxed{\boldsymbol{(1-\pi(\tau)k)\cdot X_0^\pi(\tau-) = X_0^\pi(\tau)}.}
\end{aligned}
$$

- **Main disadvantage:** Need to know the maximal possible *number of crashes* N that can happen at most and need to know the *worst crash size* k^* that can happen.

- **Aim:** Find the best uniform **worst case bound**, e.g. solve

$$\sup_{\pi(\cdot)\in A_0(x)} \inf_{\substack{0\leq\tau\leq T\\ k\in K}} \mathbb{E}\left[U\left(X^{\pi}\left(T\right)\right)\right], \tag{2}$$

where the final wealth satisfies $X^{\pi}(T) = (1-\pi(\tau)k)\, X_0^{\pi}(T)$ in the case of a crash of size k at stopping time τ. Moreover, $K = \{0\} \cup [k_*, k^*]$. We call is also the **worst case scenario portfolio problem**.

Note: To avoid bankruptcy we require $\pi(t) < \frac{1}{k^*}$ for all $t \in [0, T]$.

- The **value function** to the above problem is defined via

$$\nu_c(t,x) := \sup_{\pi(\cdot)\in A(t,x)} \inf_{\substack{t\leq\tau\leq T,\\ k\in K}} \mathbb{E}\left[\ln\left(X^{\pi,t,x}(T)\right)\right]. \tag{3}$$

- A portfolio strategy $\widehat{\pi} \geq 0$ determined via the equation

$$\mathcal{J}_0\left(t, x, \widehat{\pi}\right) = \nu_1\left(t, x\left(1 - \widehat{\pi}(t)k^*\right)\right) \qquad \text{for all } t \in [0, T]$$

will be called a **crash indifference strategy**.

- There exists a unique crash indifference strategy $\widehat{\pi}$, which is given by the solution of the differential equation

$$\widehat{\pi}'(t) = \frac{\sigma_0^2}{2}\left(\widehat{\pi}(t) - \frac{1}{k^*}\right)\left(\widehat{\pi}(t) - \pi_0^*\right)^2, \tag{4}$$

$$\text{and} \quad \widehat{\pi}(T) = 0. \tag{5}$$

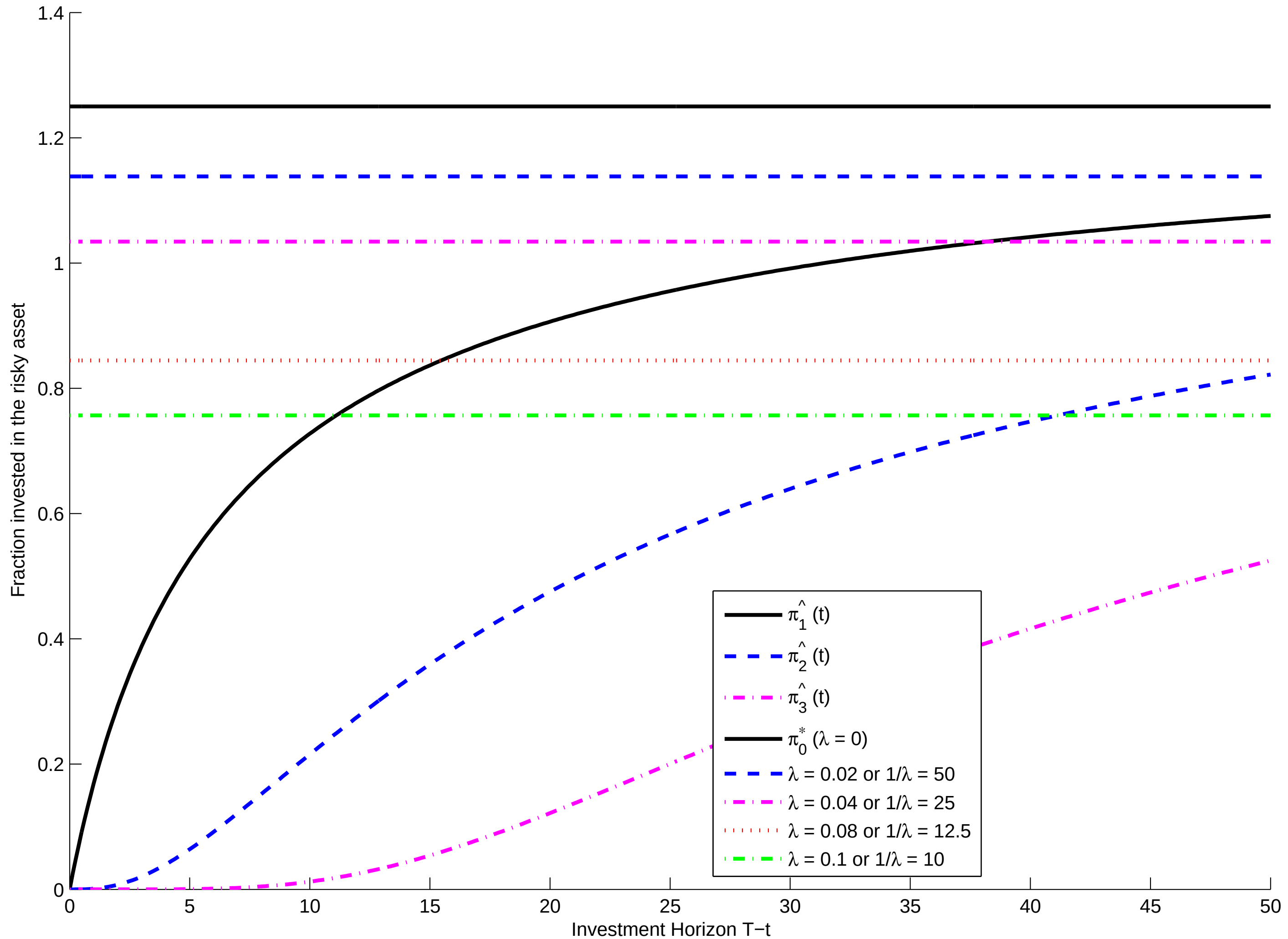
This crash indifference strategy is bounded by $0 \leq \widehat{\pi} \leq \min\{\pi_0^*, \frac{1}{k^*}\}$.

- The optimal portfolio strategy for an investor, who wants to maximize her worst case scenario portfolio problem, is given by

$$\bar{\pi}(t) := \min\left\{\widehat{\pi}(t), \pi_0^*\right\} \qquad \text{for all } t \in [0, T]. \tag{6}$$

$\bar{\pi}$ will be named the **optimal crash hedging strategy**.

Examples of Worst Case Optimal Portfolio Strategies



This Figure is plotted with $\pi_0^* = 1.25$, $\sigma_0 = 0.25$, $r = 0.05$, $k = 0.25$, and $T = 50$. This implies that $\frac{\sigma_0^2 \pi_0^*}{k} = 0.3125$, $\Psi_0 \approx 0.098828$, and $\frac{1}{k^*} = 4$.

2. Literature Review

Worst Case Scenario Approach – Some References:

- Hua and Wilmott (1997) [→ Binomial Model Derivative Pricing],
- Korn and Wilmott (2002), [→ Portfolio Optimisation],
- Korn (2005) [→ Optimal Investment for Insurances],
- Korn and M. (2005) [→ Stochastic Control Approach],
- M. (2006) [→ Changing Market Coefficients after a Crash],
- Korn and Steffensen (2007) [→ Stochastic Differential Game],
- Seifried (2010) [→ Martingale Approach],
- Korn, M., Steffensen (2012) [→ Optimising Reinsurance],
- *Belak, M., Sass (2013)* [→ Transaction Costs],
- *Desmettre, Korn, Seifried (2013)* [→ Infinite Time Consumption Problem].

Remark: The worst case scenario optimisation problem is also known as *Wald's Maximin approach* (Wald 1945, 1950), which is a well–known concept in decision theory. There, this approach is known as **robust optimisation** (e.g. Bertsimas et al. (2011)) [$\rightarrow$ usually involves optimisation procedure done by a computer].

- Mataramvura and Oksendal (2008), Oksendal and Sulem (2006, 2009, 2011) [$\rightarrow$ Compute optimal strategies directly].

[$\rightarrow$ perturbation analysis].

3 Explicit and Implicit Solutions

The following abbreviations will be used in order to get more concise formulae.

$$\Delta := \sqrt{\frac{2}{\sigma_0^2}\left[\Psi_0 - \Psi_1\right]}, \qquad \kappa := \pi_0^* - \frac{1}{k^*},$$

$$\Theta(t) := \sigma_0^2 \pi_0^* (T-t), \qquad \eta := \frac{1}{2}\left[\pi_0^* + \frac{1}{k^*}\right].$$

Proposition 3.1
With these conventions one has the following characterizations for the solutions of the differential equation (4) with the terminal condition (5).

i) If $\Psi_1 = \Psi_0$ *and* $\frac{1}{k^*} = \pi_0^*$ *(that is* $\Delta = 0 = \kappa$*), then*

$$\widehat{\pi}(t) = \pi_0^* - \frac{\pi_0^*}{\sqrt{\left(\pi_0^*\right)^2 \sigma_0^2 (T-t) + 1}}.$$

ii) If $\Psi_1 = \Psi_0$ *and* $\frac{1}{k^*} \neq \pi_0^*$ *(that is* $\Delta = 0$*), then*

$$\frac{\sigma_0^2}{2}\kappa^2 (T-t) = \ln\left(\frac{\pi_0^* - \widehat{\pi}(t)}{\pi_0^*\left[1 - \widehat{\pi}(t) k^*\right]}\right) + \frac{\kappa}{\pi_0^*}\,\frac{\widehat{\pi}(t)}{\widehat{\pi}(t) - \pi_0^*}. \tag{7}$$

4 Approximating Implicit Solutions

Proposition 4.1
In the situation of $\Psi_1 = \Psi_0$ and $\pi_0^* \neq \frac{1}{k^*}$ (see Proposition 3.1), there exists three different explicit approximations of equation (7):

(a) An approximation of $\widehat{\pi}(t)$ in equation (7) is given by

$$\tilde{\pi}_a(t) = \frac{\pi_0^*}{2} + \frac{1}{2k^*} + \frac{1}{\Theta(t)} - \sqrt{\left(\frac{\pi_0^*}{2} + \frac{1}{2k^*} + \frac{1}{\Theta(t)}\right)^2 - \frac{\pi_0^*}{k^*}}\,. \tag{8}$$

This approximation holds always if $0 \leq \pi_0^* < \frac{1}{k^*}$. If $\pi_0^* > \frac{1}{k^*}$, then the approximation holds for those $t \in [0, T]$ for which the inequality

$$\widehat{\pi}(t) \leq \frac{\pi_0^*}{2\pi_0^* k^* - 1}$$

is satisfied. In the case of $\pi_0^* > \frac{1}{k^*}$, the error of this approximation has the following upper bound

$$|\varepsilon_{\tilde{\pi}_a}(t)| \leq \frac{1}{2}\left(\frac{\widehat{\pi}(t)}{\pi_0^*}\right)^2 \cdot \frac{\left(\pi_0^* k^* - 1\right)^2}{\left(1 - \widehat{\pi}(t)k^*\right)^2}\,. \tag{9}$$

(b) Another explicit approximation of $\widehat{\pi}(t)$ is given by

$$\tilde{\pi}_b(t) = \pi_0^* - \frac{k^*\pi_0^*}{k^* + \frac{1}{2}\Theta(t)}\,. \tag{10}$$

This approximation holds always if $\pi_0^* > \frac{1}{k^*}$. *If* $\pi_0^* < \frac{1}{k^*}$*, then the approximation holds for those* $t \in [0,T]$ *for which the inequality*

$$\widehat{\pi}(t) \le \frac{\pi_0^*}{2 - \pi_0^* k^*}$$

is satisfied. In the case of $\pi_0^* < \frac{1}{k^*}$*, the error of this approximation has the following bound*

$$|\varepsilon_{\tilde{\pi}_b}(t)| \le \frac{\widehat{\pi}^2(t)}{2} \cdot \frac{\left(1 - \pi_0^* k^*\right)^2}{\left(\pi_0^* - \widehat{\pi}(t)\right)^2}\,. \tag{11}$$

(c) *Yet another explicit approximation of* $\widehat{\pi}(t)$ *is given by*

$$\begin{aligned}\tilde{\pi}_c(t) &= \pi_0^* \left[1 - \frac{\frac{1}{4}\Theta(t)\kappa + \sqrt{1 + \frac{1}{2}\Theta(t)\left[\pi_0^* + \frac{1}{k^*}\right] + \frac{1}{16}\Theta^2(t)\kappa^2}}{1 + \frac{1}{2}\Theta(t)\left[\pi_0^* + \frac{1}{k^*}\right]}\right] \\ &= \pi_0^* \left[1 - \frac{\frac{1}{4}\Theta(t)\kappa + \sqrt{\left(1 + \frac{1}{4}\Theta(t)\kappa\right)^2 + \frac{\Theta(t)}{k^*}}}{1 + \frac{1}{2}\Theta(t)\left[\pi_0^* + \frac{1}{k^*}\right]}\right]\,. \end{aligned} \tag{12}$$

Lemma 4.2

We have the following inequalities for the approximations in Proposition 4.1:

$$\tilde{\pi}_b(t) \geq \hat{\pi}(t) \geq \tilde{\pi}_a(t) \qquad \textit{and}$$

$$\hat{\pi}(t) \left\{ \begin{array}{ll} < \tilde{\pi}_c(t) & \textit{if } \pi_0^* > \frac{1}{k^*} \\ > \tilde{\pi}_c(t) & \textit{if } \pi_0^* < \frac{1}{k^*} \end{array} \right\}$$

for all $t \in [0, T]$ with strict inequality applying for all $t < T$.

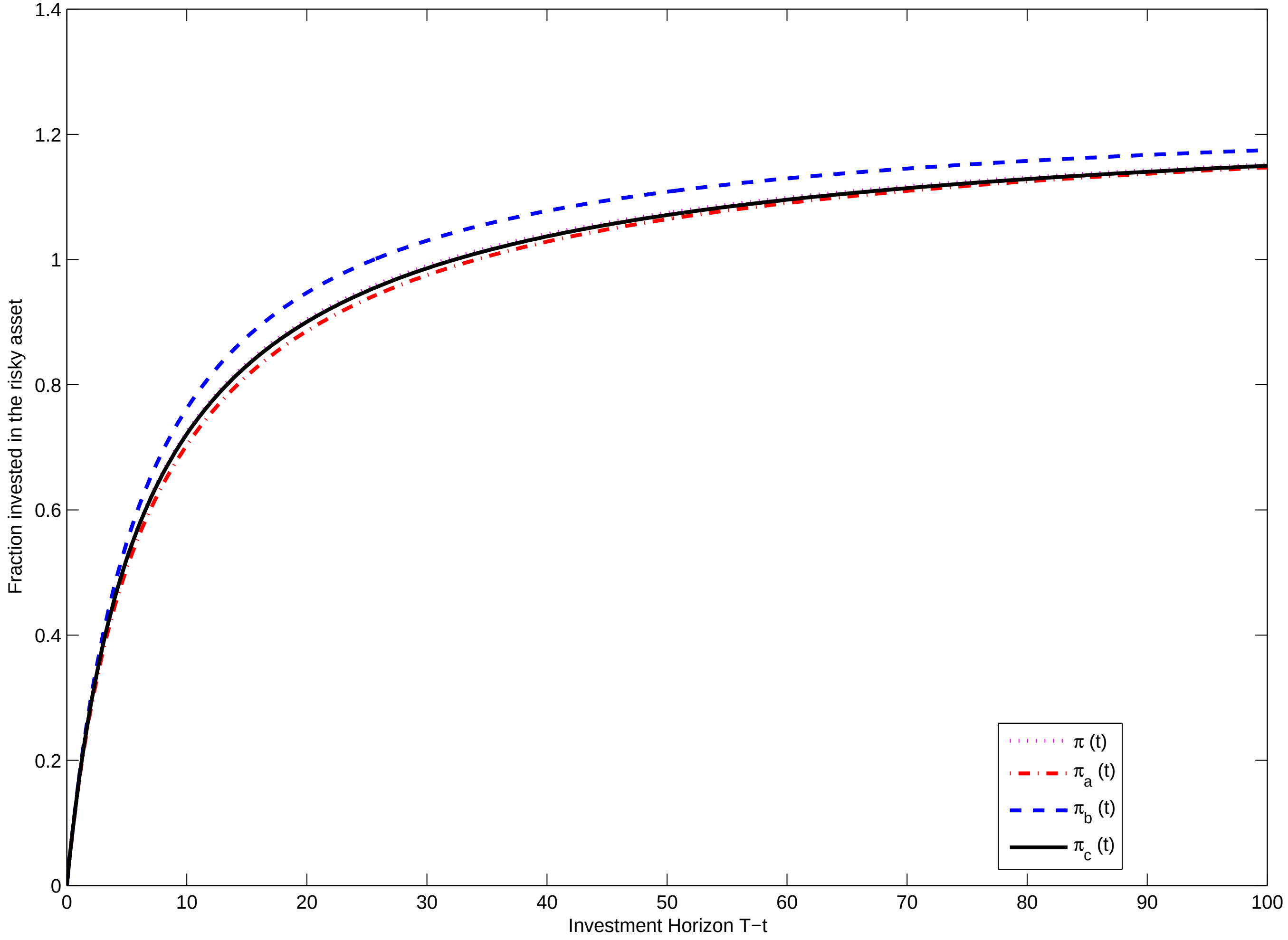
Proof: This follows directly from the corresponding properties of the different approximations for the logarithm:

$$x - 1 \geq \ln(x) \geq \frac{x-1}{x} \qquad \text{and}$$

$$\ln(x) \left\{ \begin{array}{ll} < 2\,\frac{x-1}{x+1} & \text{if } x < 1 \\ > 2\,\frac{x-1}{x+1} & \text{if } x > 1 \end{array} \right\}$$

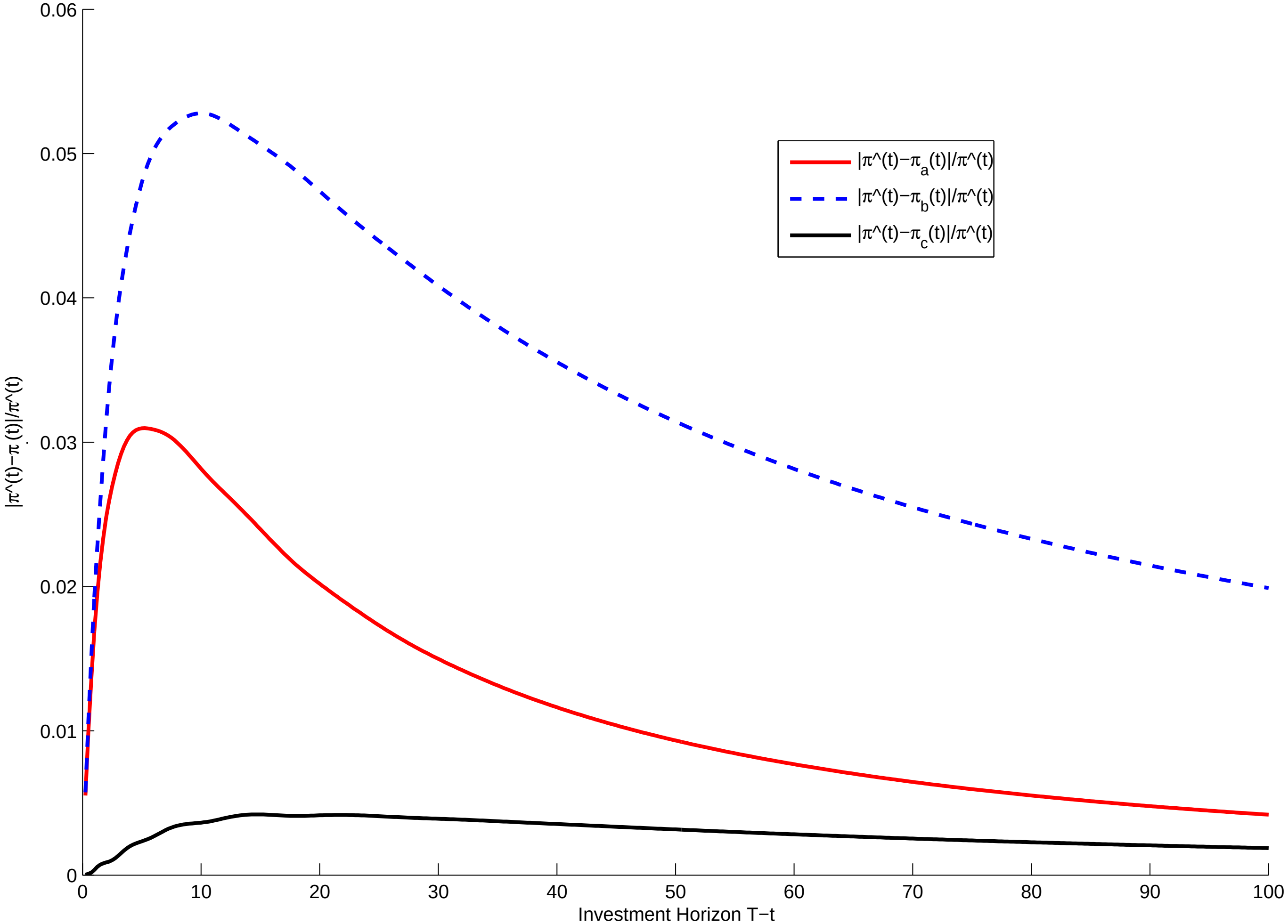
The condition $x < 1$ corresponds to the condition $\pi_0^* < \frac{1}{k^*}$ if $\hat{\pi}(t) > 0$. (If $\hat{\pi}(t) < 0$, the condition would be $\pi_0^* > \frac{1}{k^*}$). $\square$

Comparing Approximations for $\hat{\pi}(t)$ in the Case of $\pi_0^* < \frac{1}{k^*}$



This Figure is plotted with $\pi_0^* = 1.25$, $\sigma_0 = 0.25$, $r = 0.05$, $k^* = 0.25$, and $T = 100$. This implies that $\Psi_0 \approx 0.0988$, $\frac{1}{k^*} = 4$. The bounds $\frac{\pi_0^*}{2k^*\pi_0^*-1} \approx -3.3333$ and $\frac{\pi_0^*}{2-k^*\pi_0^*} \approx 0.7407$ should be obliged by $\tilde{\pi}_a$ and $\tilde{\pi}_b$, respectively. Of these, only the latter is relevant.

Absolute Relative Difference in the Case of $\pi_0^* < \frac{1}{k^*}$



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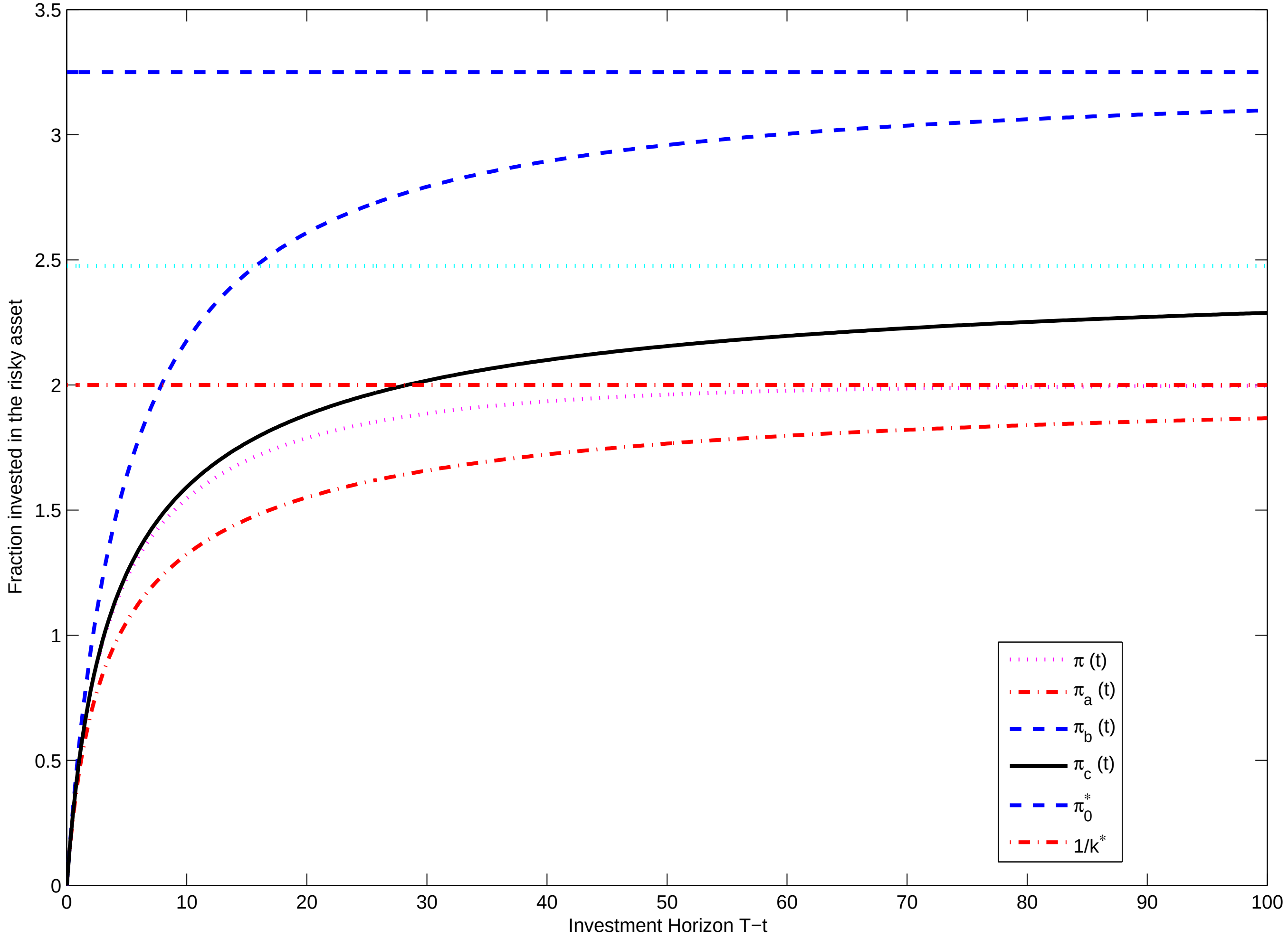
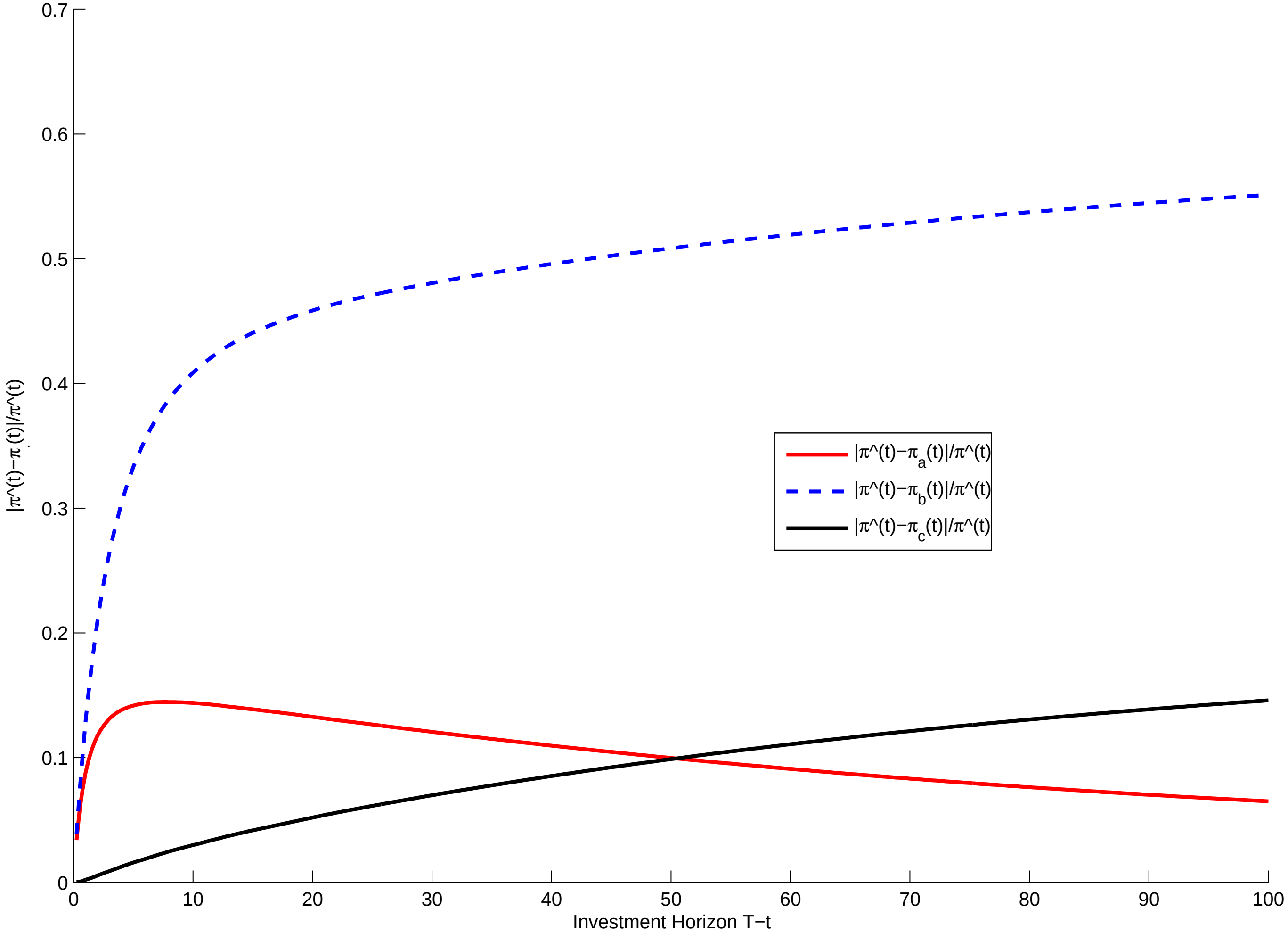
Comparing Approximations for $\hat{\pi}(t)$ in the Case of $\pi_0^* > \frac{1}{k^*}$



This Figure is plotted with $\pi_0^* = 3.25$, $\sigma_0 = 0.25$, $r = 0.05$, $k^* = 0.5$, and $T = 100$. This implies that $\Psi_0 \approx 0.3801$, $\frac{1}{k^*} = 2$. The bounds $\frac{\pi_0^*}{2k^*\pi_0^*-1} \approx 1.4444$ and $\frac{\pi_0^*}{2-k^*\pi_0^*} \approx 8.6667$ should be obliged by $\tilde{\pi}_a$ and $\tilde{\pi}_b$, respectively. Of these, only the first is relevant.

Absolute Relative Difference in the Case of $\pi_0^* > \frac{1}{k^*}$



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5 Approximating the Value Function

Corollary 5.1
In the special case of $\Psi_0 = \Psi_1$ *and* $\pi_0^* = \frac{1}{k^*}$ *(that is* $\pi_0^* \geq 1$*), the value function is given by*

$$\begin{aligned} \mathcal{J}_0\left(t, x, \widehat{\pi}(t)\right) = \widehat{\nu}(t,x) &= \nu_1\left(t, x\left[1 - \widehat{\pi}(t) k^*\right]\right) \\ &= \ln(x) + \Psi_0\left(T-t\right) - \frac{1}{2}\ln\left(\sigma_0^2\left[\pi_0^*\right]^2\left[T-t\right]+1\right). \end{aligned}$$

Note that either $\mathcal{J}_0$ or ν_1 can be used to calculate the value function $\widehat{\nu}$ and both give the same result given in the Corollary.

Proposition 5.2

In the case of $\Psi_0 = \Psi_1$ and $\pi_0^* \neq \frac{1}{k^*}$, the following performance functions can be used as an approximation for the value function $\widehat{\nu}(t,x)$:

(i)
$$\nu_1\left(t, x\left[1-\tilde{\pi}_a(t)k^*\right]\right) = \ln(x) + \Psi_0\left(T-t\right) + \ln\left(-\frac{k^*\kappa}{2} - \frac{k^*}{\Theta\left(t\right)} + \sqrt{\left(k^*\eta + \frac{k^*}{\Theta\left(t\right)}\right)^2 - k^*\pi_0^*}\right).$$

(ii)
$$\mathcal{J}_0^b\left(t, x, \tilde{\pi}_b(t)\right) = \ln(x) + \Psi_0\left(T-t\right) - \frac{\pi_0^* k^* \Theta\left(t\right)}{\pi_0^*\Theta\left(t\right) + 2k^*}.$$

(iii)
$$\nu_1\left(t, x\left[1-\tilde{\pi}_b(t)k^*\right]\right) = \ln(x) + \Psi_0\left(T-t\right) + \ln\left(1 - \frac{k^*\pi_0^*\Theta\left(t\right)}{2k^* + \Theta\left(t\right)}\right).$$

(iv)
$$\begin{aligned}\mathcal{J}_0^c\left(t, x, \tilde{\pi}_c(t)\right) = {} & \ln(x) + \Psi_0\left(T-t\right) + \frac{\pi_0^*\kappa}{2\eta^2} + \frac{\left[\pi_0^*\right]^2}{\eta^3 k^*}\ln\left(8\pi_0^*\right) + \\ & -\frac{\pi_0^*\kappa}{16\eta^2}\left[2+\eta\Theta(t)\right]\frac{\kappa\Theta(t) + \sqrt{\Theta^2(t)\kappa^2 + 16\eta\Theta(t) + 16}}{1+\eta\Theta(t)} + \\ & -\frac{\left[\pi_0^*\right]^2}{2\eta^3 k^*}\left\{\ln\left(8\eta + \kappa^2\Theta(t) + \kappa\sqrt{\kappa^2\Theta^2(t) + 16 + 16\eta\Theta(t)}\right) + \right. \\ & \left. + \ln\left(8\eta + \left[8\eta^2 - \kappa^2\right]\Theta(t) + \kappa\sqrt{\kappa^2\Theta^2(t) + 16\eta\Theta(t) + 16}\right)\right\}.\end{aligned}$$

(v)
$$\begin{aligned}\nu_1\left(t, x\left[1-\tilde{\pi}_c(t)k^*\right]\right) &= \ln(x) + \Psi_0\left(T-t\right) + \\ &+ \ln\left(1-\pi_0^* k^*\left[1-\frac{\frac{1}{4}\Theta(t)\kappa + \sqrt{\left(1+\frac{1}{4}\Theta(t)\kappa\right)^2 + \frac{\Theta(t)}{k^*}}}{1+\eta\Theta(t)}\right]\right).\end{aligned}$$

Moreover, one has that

(a)
$$\mathcal{J}_0^a\left(t, x, \tilde{\pi}_a(t)\right) \le \hat{\nu}(t,x) \le \nu_1\left(t, x\left[1-\tilde{\pi}_a(t)k^*\right]\right)$$

(b)
$$\mathcal{J}_0^b\left(t, x, \tilde{\pi}_b(t)\right) \ge \hat{\nu}(t,x) \ge \nu_1\left(t, x\left[1-\tilde{\pi}_b(t)k^*\right]\right)$$

(c)
$$\mathcal{J}_0^c\left(t, x, \tilde{\pi}_c(t)\right)\left\{\begin{array}{c}\ge\\ \le\end{array}\right\}\hat{\nu}(t,x)\left\{\begin{array}{c}\ge\\ \le\end{array}\right\}\nu_1\left(t, x\left[1-\tilde{\pi}_c(t)k^*\right]\right) \quad \text{if} \left\{\begin{array}{c}\pi_0^* > \frac{1}{k^*}\\ \pi_0^* < \frac{1}{k^*}\end{array}\right\}$$

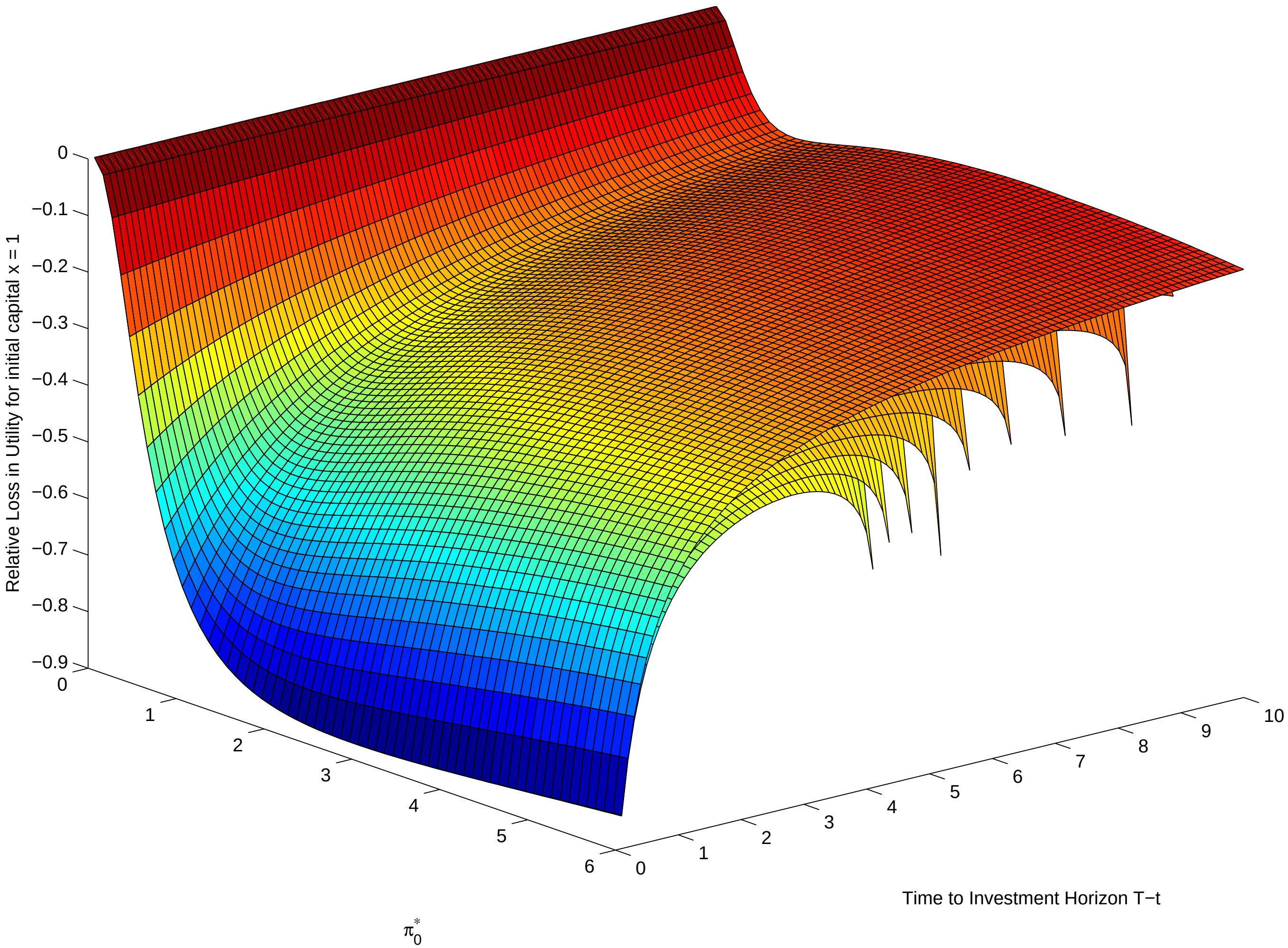
with strict inequality applying for $t < T$.

6 Costs and Benefits

Let us define the **relative loss of utility** by

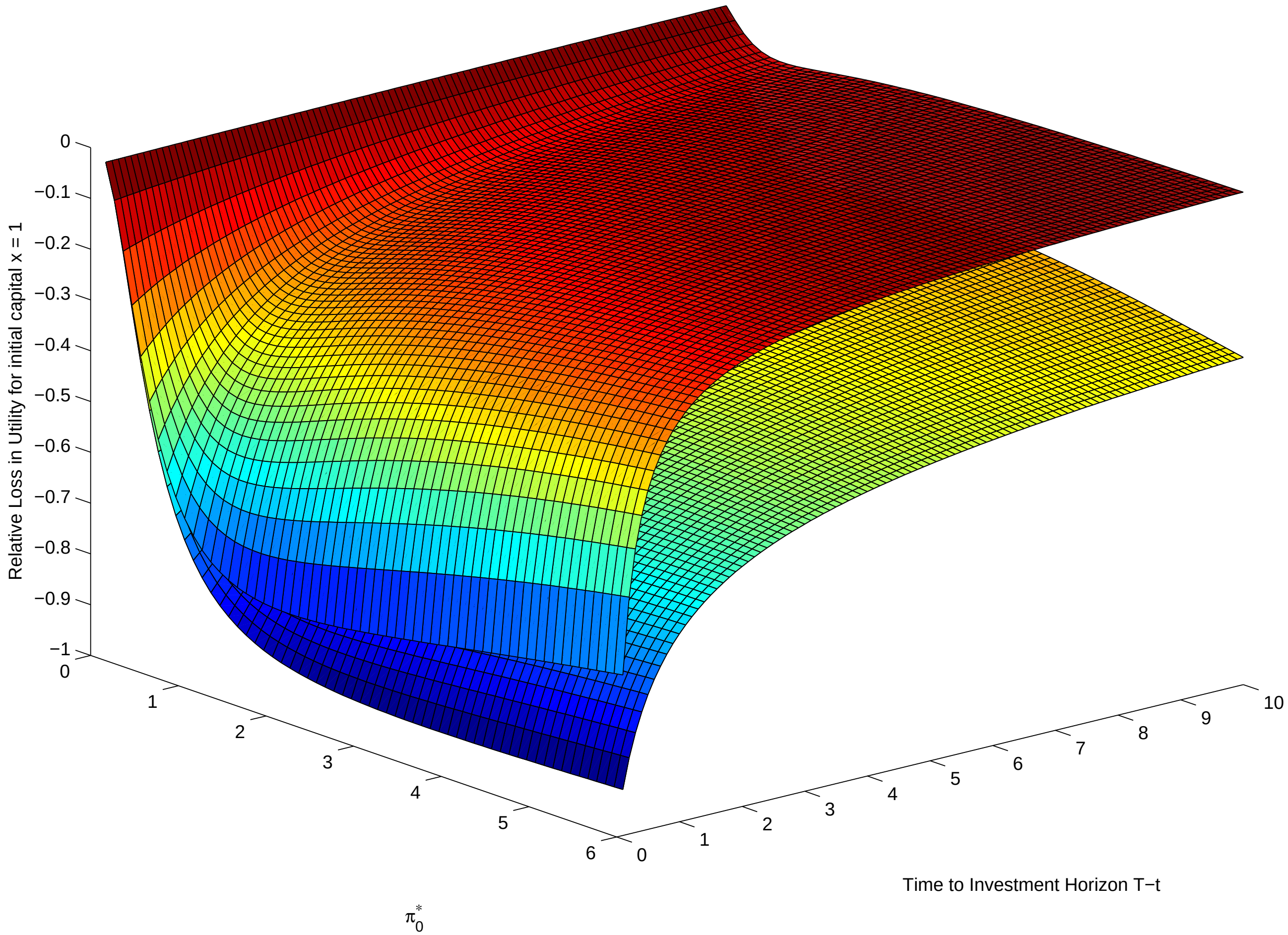
$$\frac{\mathcal{J}_0^c\left(t,x,\tilde{\pi}_c(t)\right)-\nu_0(t,x)}{\nu_0(t,x)},.$$

Relative Utility Loss using $\mathcal{J}_0^c(t, x, \tilde{\pi}_c(t))$



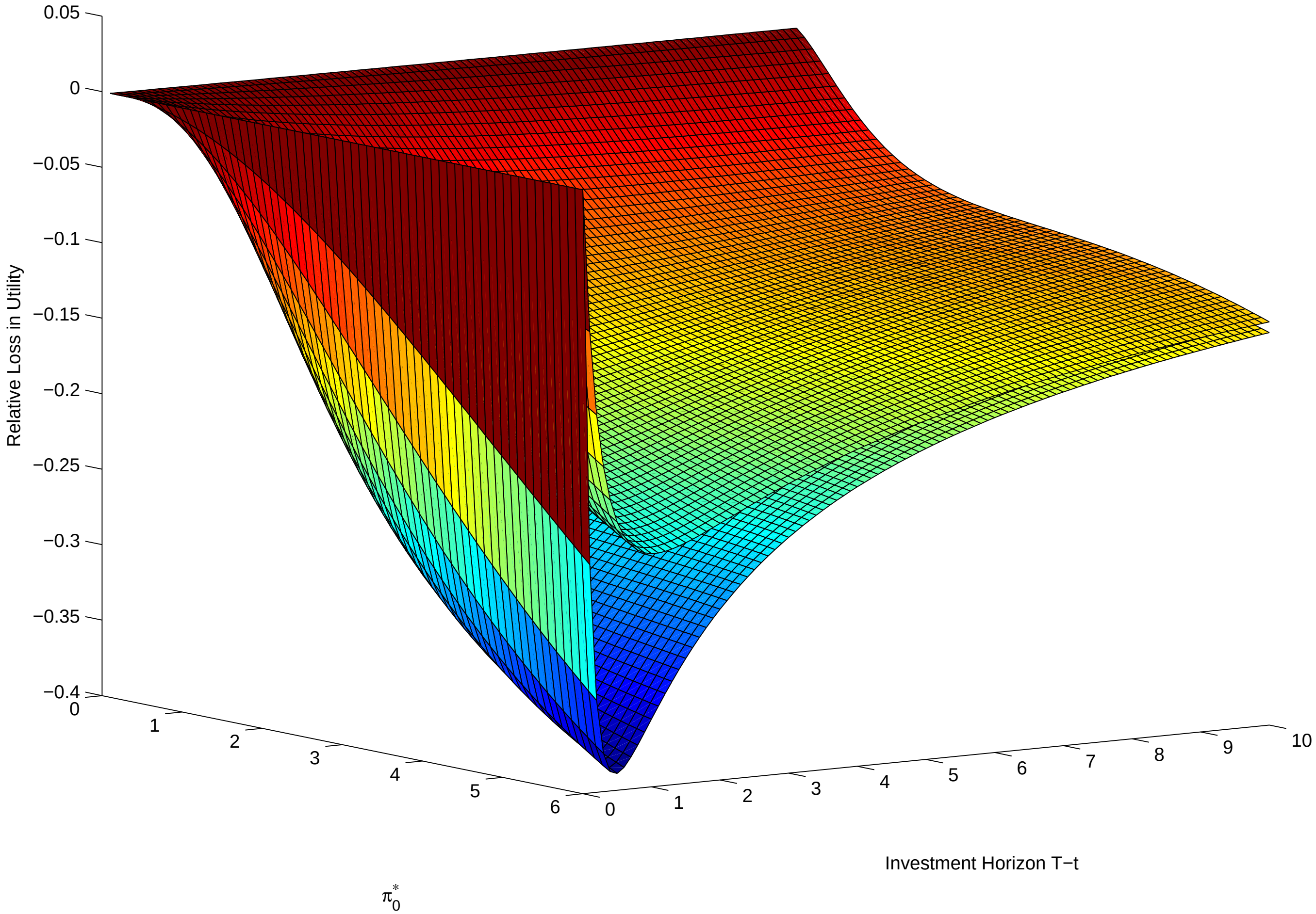
This Figure is plotted with $\sigma_0 = 0.25$, $r_0 = 0.03$, $k^* = 0.25$, $T = 10$, and an initial capital of $x = 1$. This implies that $\frac{1}{k^*} = 4$.

Relative Utility Loss using $\mathcal{J}_{\cap}^{c}(t, x, \tilde{\pi}_c(t))$



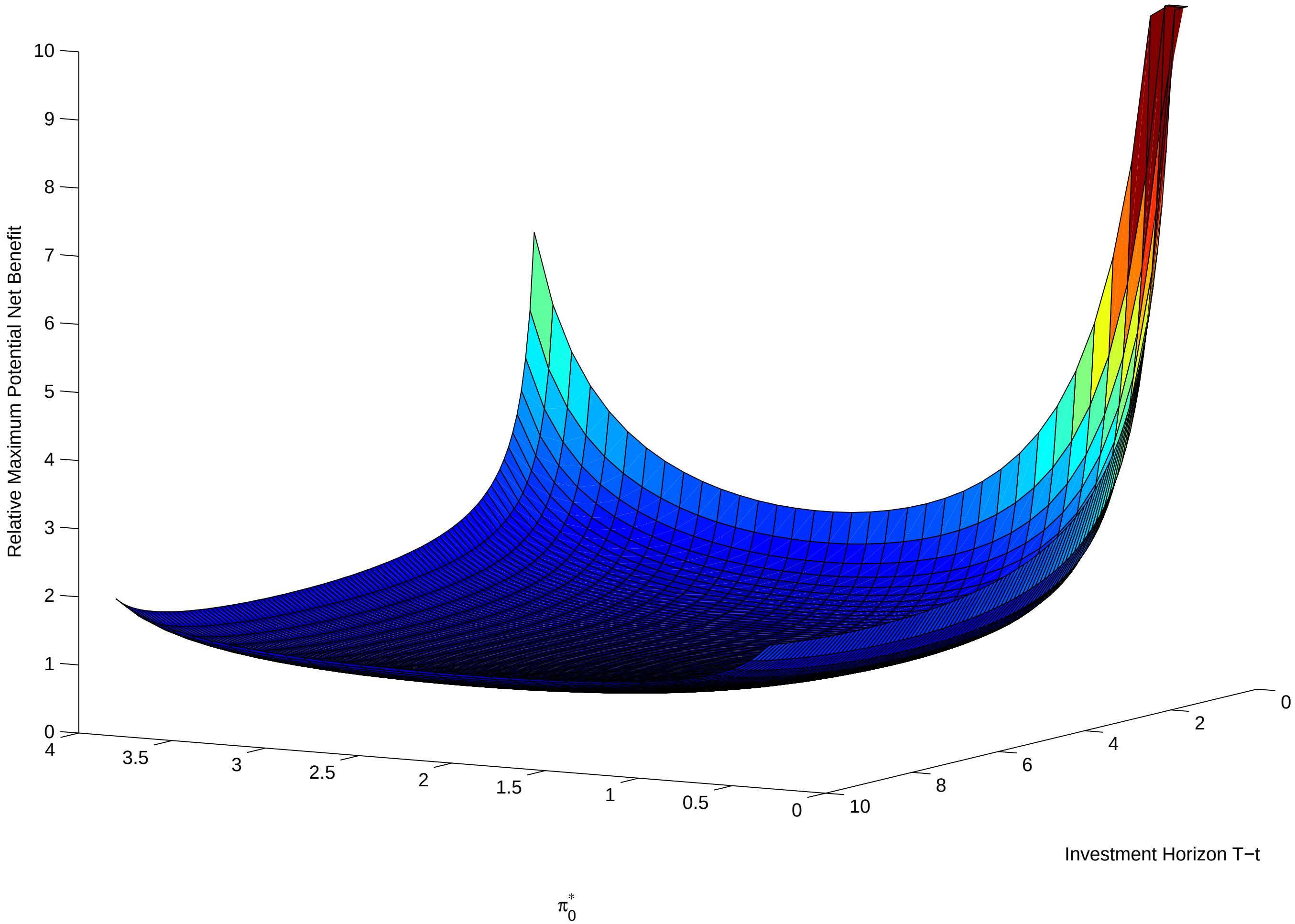
This Figure is plotted with $\sigma_0 = 0.25$, $r_0 = 0.03$, $T = 10$, and an initial capital of $x = 1$. The upper surface is the case $k^* = 0.1$ and the lower surface is $k^* = 0.5$.

Relative Utility Loss using $\mathcal{J}_0^c(t, x, \tilde{\pi}_c(t))$



This Figure is plotted with $\sigma_0 = 0.25$, $r_0 = 0.03$, $T = 10$, and $k^* = 0.25$. The upper surface is the case $x = 10$ and the lower surface is $x = 2$.

Relative Maximum Potential Net Benefit using $\mathcal{J}_0^c(t,x,\tilde{\pi}_c(t))$



This Figure is plotted with $\sigma_0 = 0.25$, $r_0 = 0.03$, and $T = 10$.

7 Efficiency of Crash Hedging

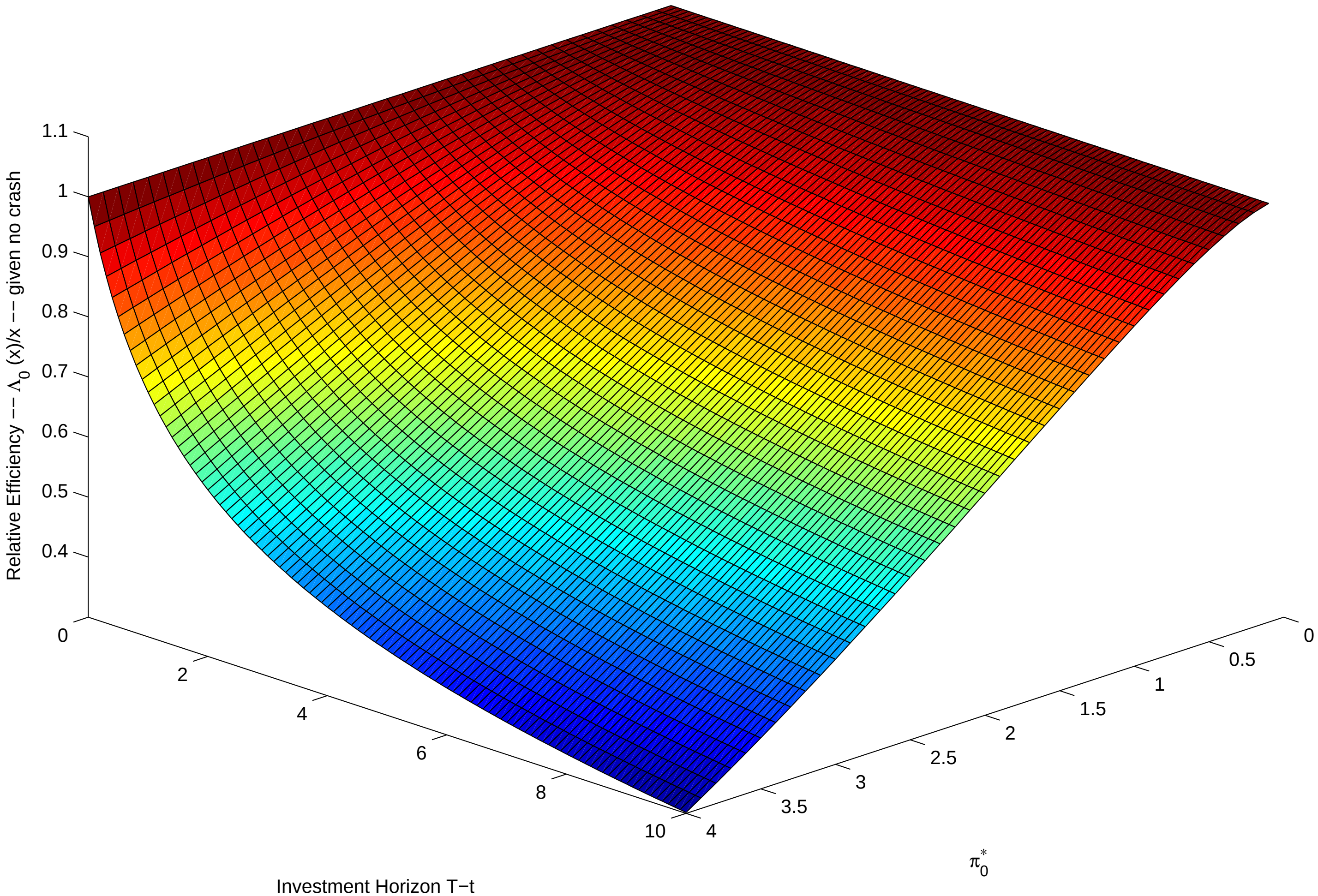
There is another possibility to compare the performance of the crash hedging strategy with the performance of the classical optimal Merton strategy. This is known as **efficiency** and is defined as follows (see e.g. Rogers (2013))

$$\widehat{\nu}(t,x) = \nu_0\left(t, \Lambda_0(x)\right) ,$$

where $\Lambda_0(x)$ is the efficiency of the optimal worst case portfolio strategy $\widehat{\pi}$ compared the the classical case of Merton with optimal portfolio strategy π_0^* in the initial market (assuming that no crash happens). The definition means that $\Lambda_0(x)$ is the amount of initial capital needed in the classical Merton case to ensure the same utility as in the considered worst case scenario approach with initial capital x. Since the worst case scenario approach can be considered as the classical Merton case with an additional constraint, it is clear that $0 \leq \Lambda_0(x) \leq 1$. This should be compared with the case that a crash of the worst possible size happens, that is

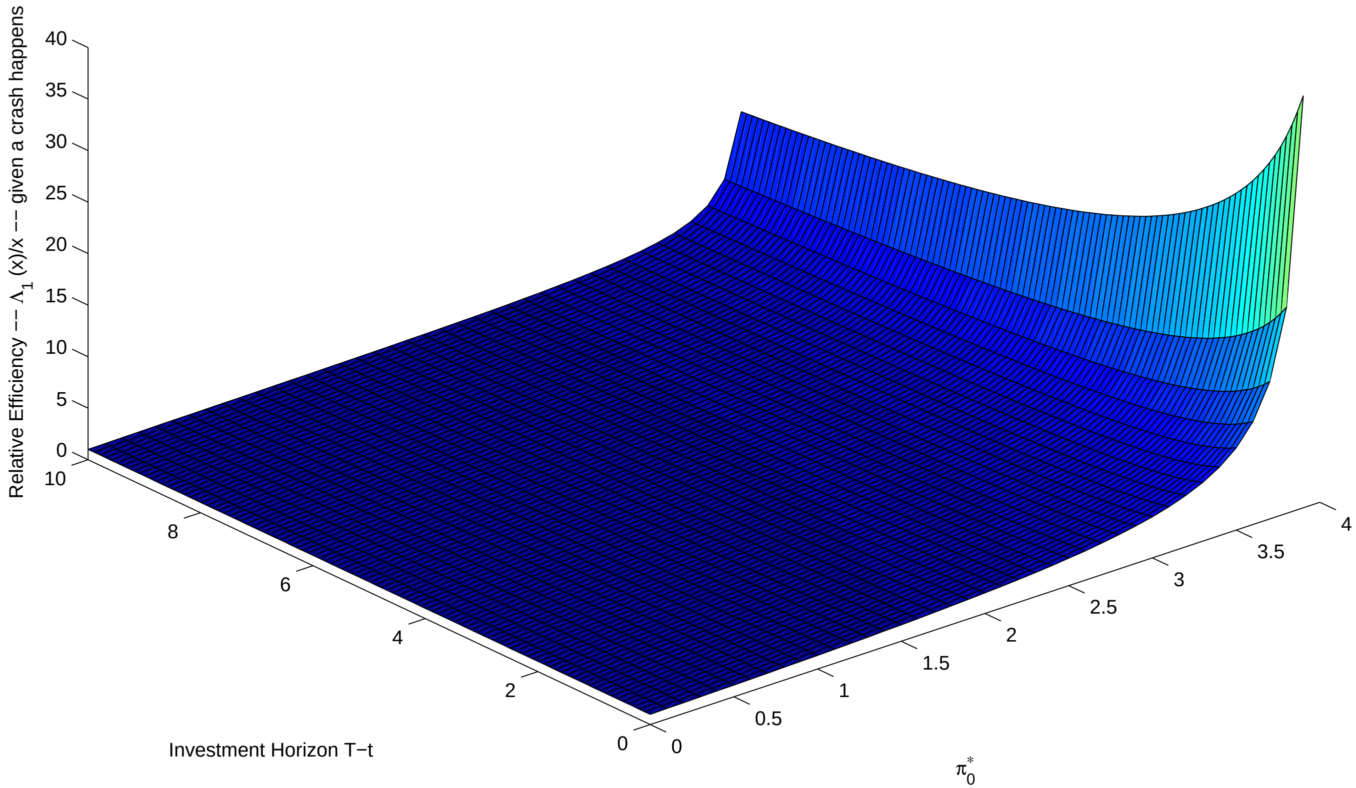
$$\widehat{\nu}(t,x) = \nu_1\left(t, \Lambda_1(x)\left[1 - \pi_0^* k^*\right]\right) .$$

Efficiency per Unit of Initial Capital $\frac{\Lambda_0^{0,c}(x)}{x}$ without Crash



This Figure is plotted assuming $\sigma_0 = 0.25$, $r_0 = 0.03$, $k^* = 0.25$, $T = 10$, and using the approximation $\mathcal{J}_0^c$.

Efficiency per Unit of Initial Capital $\frac{\Lambda_1^{0,c}(x)}{x}$ with Crash



This Figure is plotted assuming $\sigma_0 = 0.25$, $r_0 = 0.03$, $k^* = 0.25$, $T = 10$, and using the approximation $\mathcal{J}_0^c$.

8 Break Even Crash Sizes

Let us calculate the crash size $k(t) \in [0, k^*]$ with $t \in [s, T]$ for which

$$\nu_1\left(t, X_0^{\pi_0^*,s,x}(t)\left[1 - \pi_0^* k(t)\right]\right) = \nu_1\left(t, X_0^{\widehat{\pi},s,x}(t)\left[1 - \widehat{\pi}(t)k(t)\right]\right) \quad (13)$$

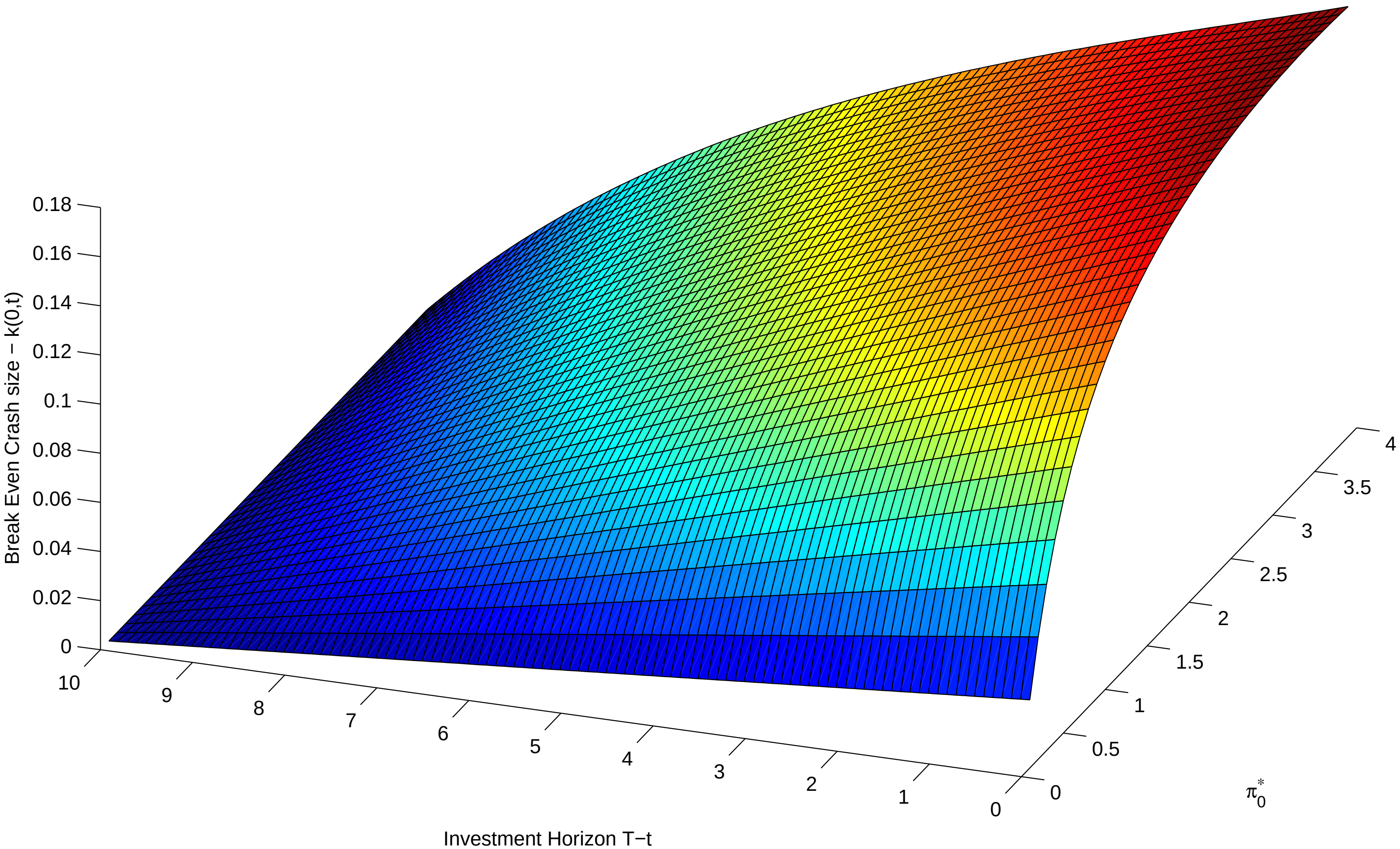
with initial endowment x at time s. Obviously, $k(t)$ is the crash size at time t which makes the investor indifferent between using the optimal worst case portfolio strategy $\widehat{\pi}$ or using the classical optimal Merton strategy π_0^*. Equation (13) can be simplified to

$$\begin{aligned} \mathbb{E}\left[\ln\left(X_0^{\pi_0^*,s,x}(t)\right)\right] + \ln\left(1 - \pi_0^* k(t)\right) & \\ = \ & \mathbb{E}\left[\ln\left(X_0^{\widehat{\pi},s,x}(t)\right)\right] + \ln\left(1 - \widehat{\pi}(t)k(t)\right) \\ \Longleftrightarrow \qquad k(t) = \ & \frac{\exp\left(\frac{\sigma_0^2}{2}\int_s^t \left(\widehat{\pi}(u) - \pi_0^*\right)^2 du\right) - 1}{\pi_0^* \exp\left(\frac{\sigma_0^2}{2}\int_s^t \left(\widehat{\pi}(u) - \pi_0^*\right)^2 du\right) - \widehat{\pi}(t)}, \end{aligned}$$

where $\mathcal{J}_0(s, t, x, \pi) := \mathbb{E}\left[\ln\left(X_0^{\pi,s,x}(t)\right)\right]$.

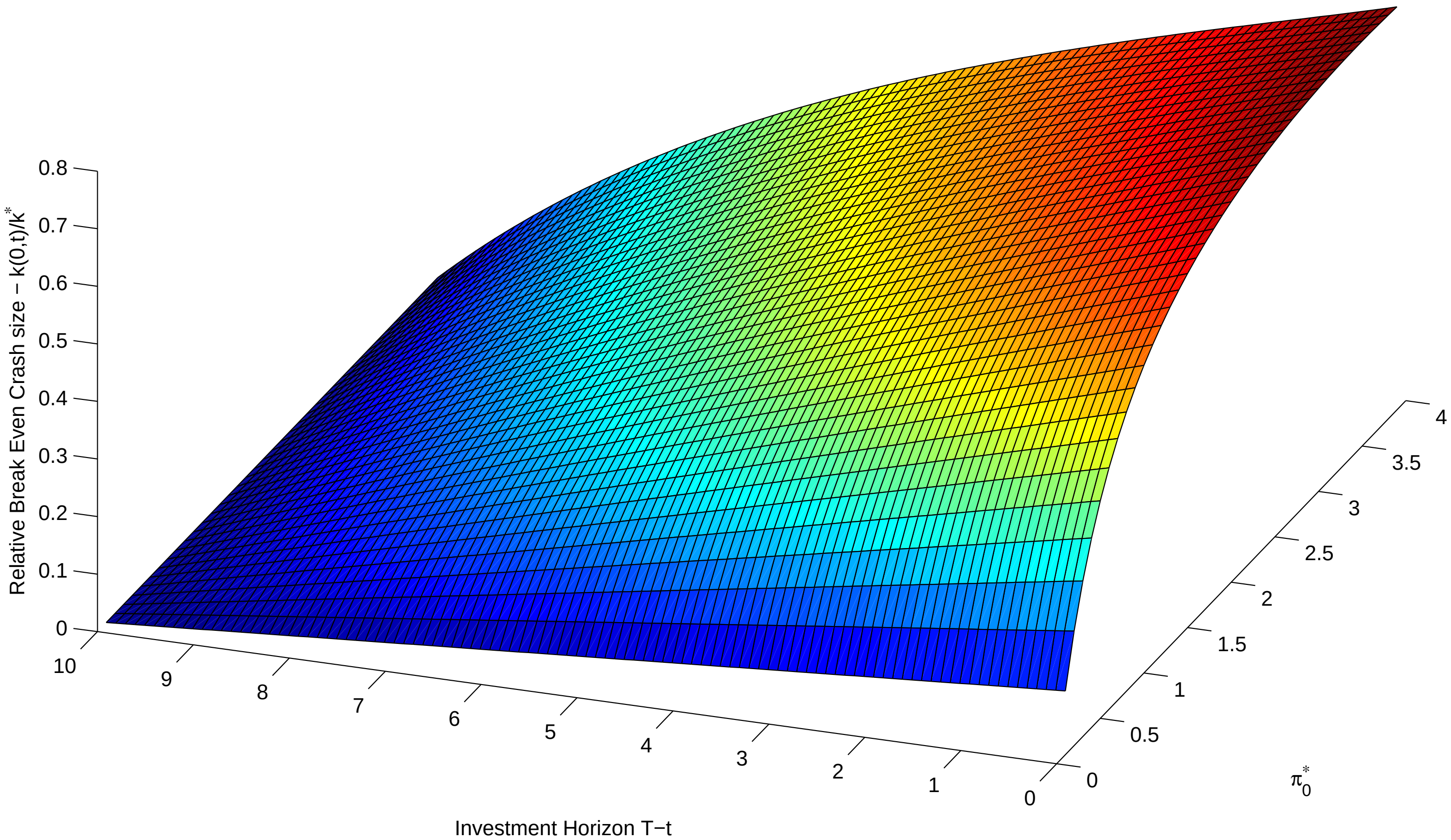
Since $\widehat{\pi}(t) \leq \pi_0^*$ for all $t \in [0, T]$ and $\nu_1(t, x[1 - \pi(t)k])$ is decreasing in k for any fixed t, k, $\pi(t)$, it follows that for crash sizes below $k(t)$, the Merton investor (using π_0^*) has a higher utility than the crash hedging investor (using $\widehat{\pi}(t)$). Correspondingly, for crash sizes above $k(t)$ the utility of the crash hedging investor will be higher than the utility of the Merton investor.

Break Even Crash Height $k(t)$



This Figure is plotted with $\sigma_0 = 0.25$, $r_0 = 0.03$, $k^* = 0.25$, $T = 10$, and using the approximation $\mathcal{J}_0^c$.

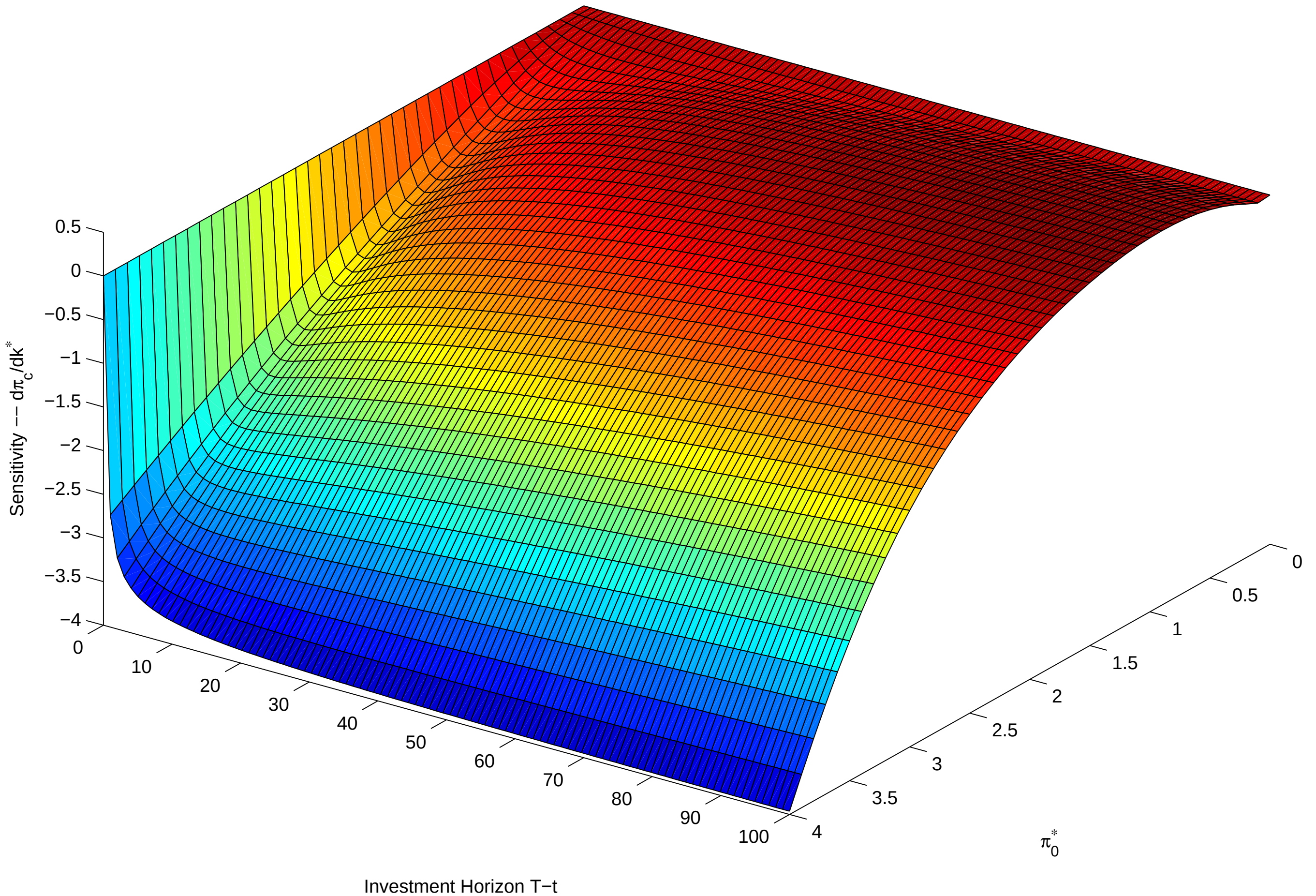
Relative Break Even Crash Height $\frac{k(t)}{k^*}$



This Figure is plotted with $\sigma_0 = 0.25$, $r_0 = 0.03$, $k^* = 0.25$, $T = 10$, and using the approximation $\mathcal{J}_0^c$.

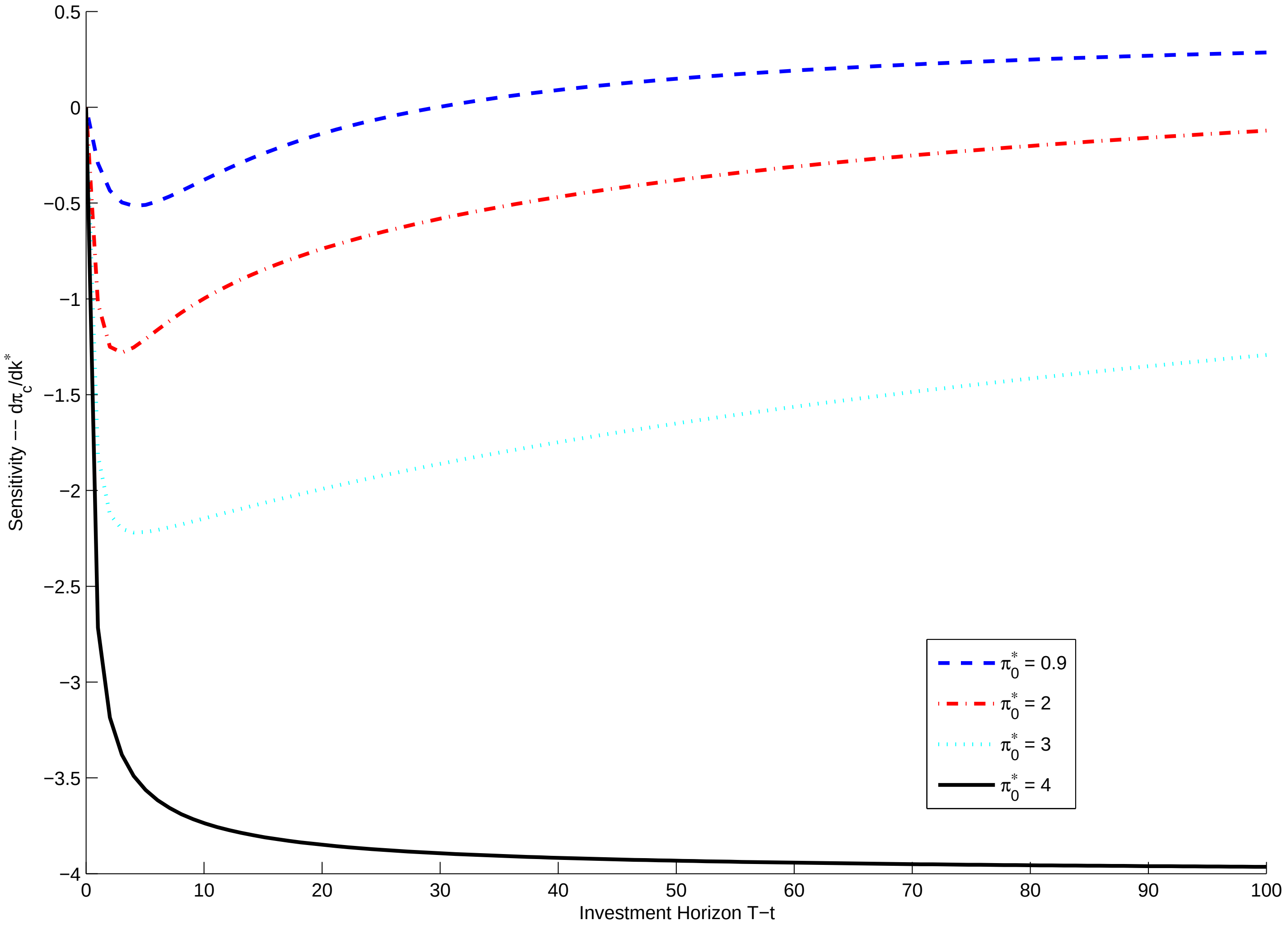
9 Sensitivities with Respect to k^*

Sensitivity of $\pi_c(t)$ with Respect to k^*



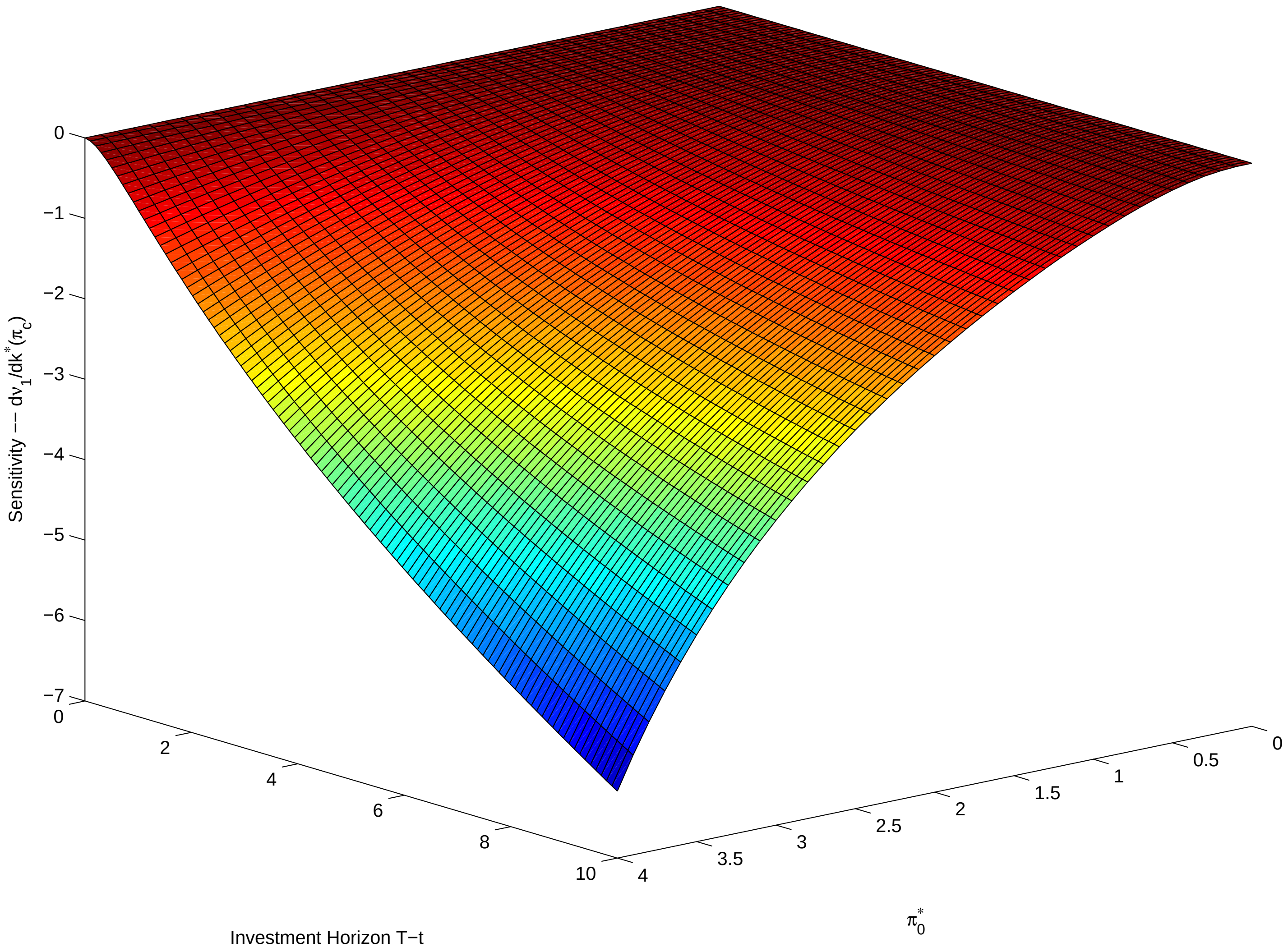
This Figure is plotted with $\sigma_0 = 0.25$, $r_0 = 0.03$, $k^* = 0.25$, and using the approximation $\pi_c(t)$.

Sensitivity of $\pi_c(t)$ with Respect to k^*



This Figure is plotted with $\sigma_0 = 0.25$, $r_0 = 0.03$, $k^* = 0.25$, and using the approximation $\pi_c(t)$.

Sensitivity of ν_1^c with Respect to k^*



This Figure is plotted with $\sigma_0 = 0.25$, $r_0 = 0.03$, $k^* = 0.25$, and using the approximation ν_1^c.