Dynamic Limit Growth Indices

(based on a joint work with T. R. Bielecki and I. Cialenco)

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Outline

Introduction

Framework and notation.

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{T}}, P)$ - standard filtered probability space with $\mathbb{T} = \mathbb{N}_0$ and $\mathcal{F}_0 = \{\Omega, \emptyset\}$.
- $L^p := L^p(\Omega, \mathcal{F}, P)$, $L^p_t := L^p(\Omega, \mathcal{F}_t, P)$ for $p \in \{0, 1, \infty\}$.
- $V = \{V_t\}_{t\in\mathbb{N}_0}$ – process such that $V_t \in L^1_t$ and $V_t > 0$ for all $t \in \mathbb{T}$ (e.g. value of an asset, fund or portfolio). Let $\mathbb{V}$ – the space of all such processes. $\widetilde{\mathbb{V}} := \{V \in \mathbb{V} : \forall_{t\in\mathbb{T}} \quad \ln V_t \in L^1\}$.

Introduction

Dynamic Assessment Index

Definition (for processes)

A *dynamic assessment index for processes* is a family $\{\varphi_t\}_{t\in\mathbb{T}}$ of mappings $\varphi_t : \mathbb{V} \to \bar{L}^0_t$ which satisfies the following properties for every $t \in \mathbb{T}$ and $V, V' \in \mathbb{V}$:

(a) *Locality.* $\mathbf{1}_A\varphi_t(V) = \mathbf{1}_A\varphi_t(\mathbf{1}_A \cdot_t V)$, for $A \in \mathcal{F}_t$;

(b) *Monotonicity.* $V \leq V' \Rightarrow \varphi_t(V) \leq \varphi_t(V')$;

(c) *Quasi-concavity.* $\varphi_t(\lambda \cdot_t V + (1-\lambda) \cdot_t V') \geq \min[\varphi_t(V), \varphi_t(V')]$, for $\lambda \in L^\infty_t, 0 \leq \lambda \leq 1$.

Comments

- Operation $(\cdot_t)$ is defined by $m \cdot_t V := (V_1 \ldots, V_{t-1}, mV_t, mV_{t+1}, \ldots)$. In discrete case - it is the same as with the optional σ-algebra.

Introduction

Representatives of DAI

- Negatives of dynamic (convex) risk measures.
- Dynamic acceptability indices (performance measures).
- Subclassess of dynamic certainty equivalents (e.g. dynamic mean value principles).

Risk Sensitive Criterion

Definition

A *risk sensitive criterion* is a mapping $\varphi : \widetilde{\mathbb{V}} \to \bar{\mathbb{R}}$ indexed by $\gamma \leq 1$ and defined by

$$\varphi^{\gamma}(V) = \begin{cases} \liminf_{T\to\infty} \frac{1}{T}\frac{1}{\gamma} \ln E[V_T^{\gamma}] & \text{if } \gamma \leq 1, \gamma \neq 0 \\ \liminf_{T\to\infty} \frac{1}{T} E[\ln V_T] & \text{if } \gamma = 0 \end{cases}$$

Remarks

- Designed to measure the long-term growth rate of a process. We will focus on Infinite time horizon (discrete case but almost everything translates to continuous case).

Risk Sensitive Criterion - Motivation

Motivation

- We want the growth rate of our process to be approximately e^{aT} (for some $a \in \mathbb{R}$).
- We will use certainty equivalent (mean value principle), i.e. $U^{-1}E[U(X)]$ with given U as utility measure.
- We will use $U(x) = \frac{1}{\gamma}x^{\gamma}$ with some parameter γ (the power utility).

For a process (V_T) It is easy to check that:

$$U^{-1}(E[U(V_T)]) \sim e^{aT}$$

$$\ln U^{-1}(E[U(V_T)]) \sim aT$$

$$\frac{1}{T}\ln \gamma E[\frac{1}{\gamma}V_T^{\gamma}]^{\frac{1}{\gamma}} \sim a$$

$$\frac{1}{T}\frac{1}{\gamma}E[V_T^{\gamma}] \sim a$$

Risk Sensitive Criterion - Motivation

Other representation of Risk Sensitive Criterion

Representation using (negative of) entropic risk measure. We could write

$$\varphi^{\gamma}(V) = \liminf_{T\to\infty} \frac{-\rho^{\gamma}(\ln V_T)}{T}$$

or equivalently

$$\varphi^{\gamma}(V) = \liminf_{T\to\infty} \frac{-\rho^{\gamma}(\ln \frac{V_T}{V_0})}{T}$$

- We have cumulative log-return of the process (at time T).
- We use some utility measure (in fact an assessment index) to see how it performs.
- We normalize it with T.
- We take the limit in T.

Dynamic Risk Sensitive Criterion

Dynamic analog of RSC

It is a natural thing to consider the dynamic analog of RSC, i.e. a family $\{\varphi_t^\gamma\}_{t\in\mathbb{T}}$ of mappings $\varphi_t^\gamma : \widetilde{\mathbb{V}} \to \bar{L}_t^0$ indexed by $\gamma \leq 1$ and defined by

$$\varphi_t^\gamma(V) := \begin{cases} \liminf_{T\to\infty} \frac{1}{T}\frac{1}{\gamma} \ln E[V_T^\gamma|\mathcal{F}_t] & \text{if } \gamma \leq 1, \gamma \neq 0 \\ \liminf_{T\to\infty} \frac{1}{T} E[\ln V_T|\mathcal{F}_t] & \text{if } \gamma = 0 \end{cases}$$

We could also define it with dynamic entropic risk measure:

$$\varphi_t^\gamma(V) = \liminf_{T\to\infty} \frac{-\rho_t^\gamma(\ln \frac{V_T}{V_t})}{T}$$

Some questions:

1. Is there any connection between dynamic assessment indices and dynamic risk sensitive criterion?
2. If yes, then can we use tools from the DAI theory to check time-consistency properties of DRSC?
3. Maybe some wider class of DAIs could be introduced?

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Some Answers (Question 1 - easy answer!)

$$\varphi_t^{\gamma}(V) := \begin{cases} \liminf_{T\to\infty} \frac{1}{T}\frac{1}{\gamma} \ln E[V_T^{\gamma}|\mathcal{F}_t] & \text{if } \gamma \leq 1, \gamma \neq 0 \\ \liminf_{T\to\infty} \frac{1}{T} E[\ln V_T|\mathcal{F}_t] & \text{if } \gamma = 0 \end{cases}$$

Theorem 1

For any $\gamma \leq 1$ the dynamic risk sensitive criterion is a dynamic assessment index for processes.

Remarks on the proof.

It is quite easy to proove it from the representation with dynamic entropic risk measure.

$$\varphi_t^{\gamma}(V) = \liminf_{T\to\infty} \frac{-\rho_t^{\gamma}(\ln V_T)}{T}$$

Question 2

Time consistency - remarks

- Unfortunatelly it is not strongly time consistent (basically speaking: it is not a "certainty equivalent"..).
- There are other forms of time-consistency [Acciaio, Penner (2010)]. The equivalent conditions and different definitions are introduced for RVs in L^{∞}. It is not easy to generalise it.
- There is a nice definition of time-consistency for Dynamic Acceptability Indices which easily could be used for dynamic assessment indices [Bielecki, Cialenco, Zhang (2011)].

Question 2

Definition (Time consistency - definition)

A dynamic assesment index $\{\varphi\}_{t\in\mathbb{T}}$ is *acceptance consistent* (resp. *rejection consistent*) if for all $s, t \geq 0$, $V \in \widetilde{\mathbb{V}}$ and $m_t \in \bar{L}_t^0$

$$\varphi_s(V) \geq m_t \quad (resp. \leq) \quad \Longrightarrow \quad \varphi_t(V) \geq m_t \quad (resp. \leq).$$

Remark

Note that it is not easy to compare different gain processes if we have this property. It is quite different from strong time consistency property.

Question 2

Theorem (time consistency - characterisation for DRSC)

1. *For risk-averse case* $\gamma < 0$*, DRSC is rejection consistent and not acceptance consistent.*
2. *For risk-seeking case* $\gamma \in (0, 1]$*, DRSC is acceptance consistent and not rejection consistent.*
3. *For risk-neutral case* $\gamma = 0$*, DRSC is both not rejection and not acceptance consistent.*

Question 2

Time consistency - risk-averse case ($\gamma < 0$).

$\varphi_s(V) \leq m_t \implies \varphi_t(V) \leq m_t$

- Intuition: If we reject something in the future, we reject it now.
- Acceptance consistency does not work. Counterexample: $\gamma = -1$, $([0,1], \mathcal{L}([0,1]), \{\mathcal{F}_t\}_{t\in\mathbb{N}_0}, P)$, P–Lebesgue measure, $\mathcal{F}_0 = \{\emptyset, [0,1]\}$, $\mathcal{F}_1 = \mathcal{L}([0,1])$ $(\mathcal{F}_1 = \mathcal{F}_2 = \ldots = \mathcal{F})$.

$$V_T(\omega) := \begin{cases} \frac{1}{T} & \omega \in [0, \frac{1}{T}] \\ e^T & \omega \in [\frac{1}{T}, 1] \end{cases}$$

The process on average is "small" $(\varphi_0^{(-1)}(V) = 0)$ but for every realisation it is "big" $(\varphi_1^{(-1)}(V) = 1)$.

Question 2

Time consistency - risk-seeking case $(-1 \leq \gamma < 0)$.

$\varphi_s(V) \geq m_t \Longrightarrow \varphi_t(V) \geq m_t$

- Intuition: If we accept something in the future, we accept it now.
- Rejection consistency does not work. Counterexample: $\gamma = 1$, $([0,1], \mathcal{L}([0,1]), \{\mathcal{F}_t\}_{t\in\mathbb{N}_0}, P)$, P–Lebesgue measure, $\mathcal{F}_0 = \{\emptyset, [0,1]\}$, $\mathcal{F}_1 = \mathcal{L}([0,1])$ $(\mathcal{F}_1 = \mathcal{F}_2 = \ldots = \mathcal{F})$.

$$V_T(\omega) := \begin{cases} Te^T & \omega \in [0, \frac{1}{T}] \\ 1 & \omega \in [\frac{1}{T}, 1] \end{cases}$$

The process on average is "big" $(\varphi_0^{(1)}(V) = 1)$ but for every realisation it is "small" $(\varphi_1^{(1)}(V) = 0)$.

Question 2

Time consistency - risk-neutral case ($\gamma = 0$).

$$V_T(\omega) := \begin{cases} e^{T^2} & \omega \in [0, \frac{1}{T}] \\ 1 & \omega \in [\frac{1}{T}, 1] \end{cases} \qquad V'_T(\omega) := \begin{cases} e^{-T^2} & \omega \in [0, \frac{1}{T}] \\ e^{T} & \omega \in [\frac{1}{T}, 1] \end{cases}$$

Important remark

This is a general result. For restricted class of dynamic the situation could be different.

1. Restrictions are often assumed in practise (for base intruments/underlyings).
2. Under restricted dynamics very often the optimal strategy could be found.
3. Usually we require some kind of Markov/Martingale property .

Question 3

Dynamic Limit Growth Indices

A *dynamic limit growth index* is a family $\{\varphi_t\}_{t\in\mathbb{T}}$ of mappings $\varphi_t : \widetilde{\mathbb{V}} \to \bar{L}_t^0$ such that

$$\varphi_t(V) := \liminf_{T\to\infty} \frac{\mu_t(\ln \frac{V_T}{V_t})}{T}$$

where $\{\mu_t\}_{t\in\mathbb{T}}$ is a dynamic assessment index and $T \in \mathbb{T}$.

Which families of $\{\mu_t\}_{t\in\mathbb{T}}$ are good to define DLGI?

1. Dynamic certainty equivalents (intuition similar to the first interpretation of RSC)
2. Negatives of dynamic convex risk measures (spectral risk measures with mass in 'high' quantiles, shortfall risk measures, entropic risk measure).

Some properties of Dynamic limit growth indices

$$\varphi_t(V) := \liminf_{T\to\infty} \frac{\mu_t(\ln \frac{V_T}{V_t})}{T}$$

Proposition 1

If $\{-\mu_t\}_{t\in\mathbb{T}}$ is a dynamic convex risk measure (for random variables) then corresponding $\{\varphi_t\}_{t\in\mathbb{T}}$ is a dynamic assessment index (for processes).

Proposition 2

Dynamic limit growth indices which could be expressed as

$$\varphi_t(V) = \liminf_{T\to\infty} \frac{\mu_t(\ln V_T)}{T}$$

fulfills all conditions of dynamic acceptability indices introduced in [Bielecki, Cialenco, Zhang, 2011] (except of positiveness and normalisation).

Some properties of Dynamic limit growth indices (cont.)

$$\varphi_t(V) := \liminf_{T\to\infty} \frac{\mu_t(\ln \frac{V_T}{V_t})}{T}$$

Proposition 3

For risk seeking case in DRSC ($\gamma \in (0,1]$) we know that

$$\left[\varphi_t^{\gamma}(V)\right]^+ = \liminf_{T\to\infty} \frac{-\rho_t^{\gamma}([\ln \frac{V_T}{V_t}]^+)}{T}.$$

Comment: It shows that this criterium could lead to very risky strategies. We don't care about the losses.

Proposition 4

Let $U \in \mathcal{C}^1$. If $a < U' < b$ for $a, b \in \mathbb{R}$ then DLGI defined with $\{\mu_t\}_{t\in\mathbb{T}}$, where

$$\mu_t(X) = U^{-1}(E[U(X)|F_t])$$

is a dynamic assessment index.

Comment: In general we need some form of a Lipschitz condition for $\{\mu_t\}_{t\in\mathbb{T}}$ if we want the corresponding $\{\varphi_t\}_{t\in\mathbb{T}}$ to be a dynamic assessment index.

Quick conclusions

1. Dynamic risk sensitive criterion is a dynamic assessment index.
   1.1 Risk-averse case - rejection consistency.
   1.2 Risk-seeking case - acceptance consistency.
2. New (promising?) class of maps: Dynamic limit growth indices

$$\varphi_t(V) := \liminf_{T\to\infty} \frac{\mu_t(\ln \frac{V_T}{V_t})}{T}$$

Thank you!