Full cooperation applied to environmental improvements

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- **separate actions** - certificates are issued for each agent separately;
- **collusive actions** - there is one certificate that embraces two domains and each agent has full information about pollution levels in other's domain at any time $0 \leq t \leq 1$ but she/he can make improvements in her or his domain only;
- **fusion** - there is one certificate for two domains and, in practice, it could lead to **the free transfer of technologies**, because two agents act jointly as a one agent.

Model setting

$T = 1$ denotes time horizon.

$X_i(t)$ denotes the polution level at time t in the ith domain, $i = 1, 2$, $0 \leq t \leq T = 1$.

In the first case (separate actions) ith agent minimizes

$$\mathbf{E}\left(X_i^2(1) + \text{cost of improvements}\right).$$

In the second and the third case (collusive actions or fusion) agents minimize

$$\mathbf{E}\left(\left(X_1(1) + X_2(1)\right)^2 + \text{cost of improvements}\right).$$

Model setting - separate actions

We model pollution levels X_1, X_2 as independent geometric Brownian motions with improvements $u_1,\ u_2$ respectively:

$$\mathrm{d}X_i(t) = \alpha X_i(t)\mathrm{d}t + \beta X_i(t)\mathrm{d}W_i(t) - Au_i(t)\mathrm{d}t, \tag{1}$$

where $i = 1, 2$.

- $A, \alpha, \beta > 0$ are model parameters,
- W_1, W_2 are Brownian motions,
- $u_1, u_2 : [0; 1] \to [0; +\infty)$ are **(adapted) improvements**,
- $X_1(0) = x_1 > 0, X_2(0) = x_2 > 0$ are pollution levels at the moment 0.

Separate actions - costs of improvements

In any case we assume **quadratic costs of improvements**.

In the first case of separate actions ith agent minimizes

$$\mathbf{E}\left(X_i^2(1) + \int_0^1 u_i^2(s)\mathrm{d}s\right).$$

To solve the optimisation problem the simplified version of BSDE approach will be used.

Separate actions - optimization problem

We will show that there exists a deterministic function $Y : [0; +\infty) \to [0; +\infty)$ (in the general BSDE approach it is a process) such that

$$Y(1) = 1$$

and an adapted process u_i^* such that

$$Z_i(t) := Y(t)X_i^2(t) + \int_0^t u_i^{*2}(s)\mathrm{d}s$$

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and an adapted process u_i^* such that

$$Z_i(t) := Y(t)X_i^2(t) + \int_0^t u_i^{*2}(s)\mathrm{d}s$$

is a martingale and for any other adapted process u_i and $t \geq 0$ we have

$$\mathbf{E}\left(Y(t)X_i^2(t) + \int_0^t u_i^{*2}(s)\mathrm{d}s\right) \leq \mathbf{E}\left(Y(t)X_i^2(t) + \int_0^t u_i^2(s)\mathrm{d}s\right).$$

Thus u_i^* solves the optimization problem.

Separate actions - optimization problem, cont.

Assuming that $\mathrm{d}Y(t) = y(t)\mathrm{d}t$ we calculate easily the drift part of Z_i, which reads as

$$\underbrace{Y(t)X_i^2(t)(2\alpha + \beta^2) + X_i^2(t)y(t)}_{\text{independent of } u_i^*} + \underbrace{u_i^{*2}(t) - 2A \cdot Y(t)X_i(t)u_i^*(t)}_{\text{depends on } u_i^*} \quad (2)$$

and is minimized for

$$u_i^*(t) = A \cdot Y(t)X_i(t). \quad (3)$$

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and is minimized for

$$u_i^*(t) = A\cdot Y(t)X_i(t). \quad (3)$$

Substituting (3) into (2) we get that the drift term equals 0 (i.e. Z_i is a martingale) iff

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = y(t) = A^2Y^2(t) - (2\alpha+\beta^2)Y(t). \quad (4)$$

Separate actions - optimization problem, cont.

Equation (4) has unique solution in $[0;1]$ such that $Y(1) = 1$, namely

$$Y(t) = \frac{1}{\frac{A^2}{B} + \left(1 - \frac{A^2}{B}\right) e^{B(t-1)}} > 0$$

where $B = 2\alpha + \beta^2$.

Separate actions - optimization problem, cont.

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$$Y(t) = \frac{1}{\frac{A^2}{B} + \left(1 - \frac{A^2}{B}\right) e^{B(t-1)}} > 0$$

where $B = 2\alpha + \beta^2$. We also have

$$u_i^*(t) > 0,$$

for $t \in [0;1]$ since $u_i^*(t) = A \cdot Y(t) X_i(t)$ and substituting this into (1), we get that

$$\mathrm{d}X_i(t) = (\alpha - A \cdot Y(t))\, X_i(t)\mathrm{d}t + \beta X_i(t)\mathrm{d}W_i(t),$$

thus X_i is a gBm and $u_i^*(t) > 0$.

Fusion - optimization problem

In the fusion case each agent may act in both domains (in fact they act as a one agent) and the dynamics of X_1, X_2 is given by

$$\begin{aligned} \mathrm{d}X_1(t) &= \alpha X_1(t)\mathrm{d}t + \beta X_1(t)\mathrm{d}W_1(t) - Au_1(t)\mathrm{d}t - Au_2(t)\mathrm{d}t, \\ \mathrm{d}X_2(t) &= \alpha X_2(t)\mathrm{d}t + \beta X_2(t)\mathrm{d}W_2(t) - Au_1(t)\mathrm{d}t - Au_2(t)\mathrm{d}t, \end{aligned}$$

and the agents minimize

$$\mathbf{E}\left((X_1(1) + X_2(1))^2 + \int_0^1 (u_1(s) + u_2(s))^2 \,\mathrm{d}s \right). \tag{5}$$

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and the agents minimize

$$\mathbf{E}\left((X_1(1) + X_2(1))^2 + \int_0^1 (u_1(s) + u_2(s))^2 \, ds\right). \quad (5)$$

So we have the free transfer of technologies, and the costs of improvements correspond rather to the cost of the development of technologies no to the costs of implementing them on a specific area (domain) - this could be proportional to the area.

Fusion - optimization problem, cont.

To solve this optimization problem we use similar techniques as before. Setting $u = u_1 + u_2$ and

$$Z(t) = X_1^2(t)Y(t) + X_2^2(t)Y(t) + 2X_1(t)X_2(t)\tilde{Y}(t) + \int_0^t u^2(s)\mathrm{d}s \quad (6)$$

we get that the drift term of Z is minimized for

$$u^*(t) = A\left(X_1(t) + X_2(t)\right)\left(Y(t) + \tilde{Y}(t)\right).$$

Fusion - optimization problem, cont.

Substituting u^* into the equation defining Z, (6), and assuming that

$$\mathrm{d}Y(t) = f'(t)\mathrm{d}t,\ \mathrm{d}\tilde{Y}(t) = g'(t)\mathrm{d}t,$$

we get that Z is a martingale when f and g satisfy the system of equations

$$\begin{cases} f' = A^2 (f+g)^2 - (2\alpha + \beta^2) f, \\ g' = A^2 (f+g)^2 - 2\alpha g. \end{cases} \tag{7}$$

To solve the optimization problem (5), we set the border conditions $f(1) = g(1) = 1$.

Fusion - optimization problem, cont.

Now, the problem will be solved if we show that the system (7) (with the border conditions $f(1) = g(1) = 1$) has a solution such that

$$u^*(t) = A\left(X_1(t) + X_2(t)\right)\left(f(t) + g(t)\right) \geq 0.$$

(We can not make negative improvements...)

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(We can not make negative improvements...) But we have

Lemma

The solution of system (7) with the border conditions $f(1) = g(1) = 1$ exists for $t \in [0;1]$ and for such t we have

$$f(t) + g(t) > 0.$$

and by the fact that u^* minimizes (5) we have that $X_1 + X_2 \geq 0$.

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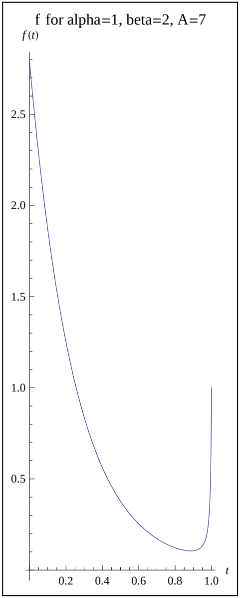
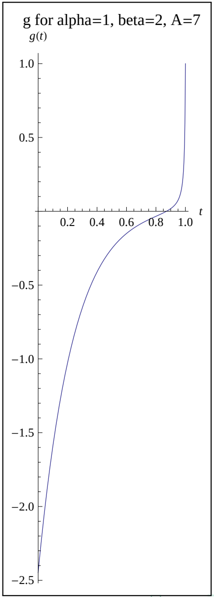
Remark

Let (f, g) be a solution of (7) in $[0;1]$ with the border conditions $f(1) = g(1) = 1$ then f is strictly positive but g may be negative.

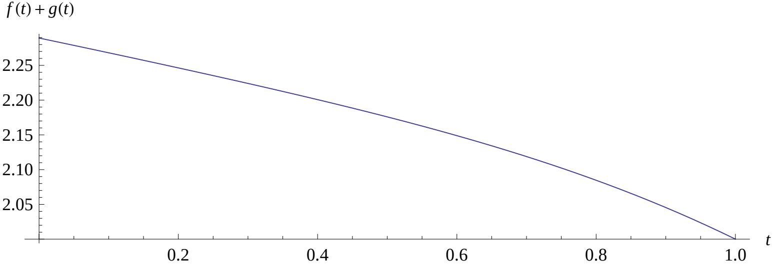
Functions f and g





Function $f + g$

f+g for alpha=1, beta=2, A=7



Collusive actions

The dynamics of X_i, $i = 1, 2$, is given by

$$\mathrm{d}X_i(t) = \alpha X_i(t)\mathrm{d}t + \beta X_i(t)\mathrm{d}W_i(t) - Au_i(t)\mathrm{d}t, \tag{8}$$

The free transfer of technologies does not occur! ;-(

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The free transfer of technologies does not occur! ;-(and the agents minimize

$$\mathbf{E}\left((X_1(1) + X_2(1))^2 + \int_0^1 u_1^2(s)\mathrm{d}s + \int_0^1 u_2^2(s)\mathrm{d}s\right). \tag{9}$$

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Again, we proceed as in previous cases but now we work with

$$Z = X_1^2 Y + X_2^2(t)Y + 2X_1X_2\tilde{Y} + \int_0^{\cdot} u_1^2(s)\mathrm{d}s + \int_0^{\cdot} u_2^2(s)\mathrm{d}s. \tag{10}$$

Collusive actions - optimization

The drift is minimimal for $u_1 = u_1^* = A\left(X_1 Y + X_2 \tilde{Y}\right)$ and $u_2 = u_2^* = A\left(X_2 Y + X_1 \tilde{Y}\right)$, and, assuming that

$$\mathrm{d}Y(t) = \varphi'(t)\mathrm{d}t,\ \mathrm{d}\tilde{Y}(t) = \psi'(t)\mathrm{d}t,$$

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we get that the drift term vanishes (i.e. Z is a martingale) iff φ and ψ satisfy the system of two equations

$$\begin{cases} \varphi' = A^2\left(\varphi^2 + \psi^2\right) - \left(2\alpha + \beta^2\right)\varphi, \\ \psi' = 2A^2\varphi\psi - 2\alpha\psi. \end{cases} \tag{11}$$

To solve the optimization problem (9), we set the border conditions $\varphi(1) = \psi(1) = 1$.

Collusive actions - optimization problem, cont.

Now, the problem will be solved (recall that we can not make negative improvements) if we show that the system (11) (with the border conditions $\varphi(1) = \psi(1) = 1$) has a solution such that $\varphi \geq 0$ and $\psi \geq 0$ for $t \in [0; 1]$ and then

$$u_1^* = A\left(X_1\varphi + X_2\psi\right) \geq 0,$$

$$u_2^* = A\left(X_2\varphi + X_1\psi\right) \geq 0.$$

(By (8) X_1 and X_2 are gBm.)

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But we have

Lemma

The solution of system (11) with the border conditions $\varphi(1) = \psi(1) = 1$ exists for $t \in [0; 1]$ and for such t we have $\varphi > 0$ and $\psi > 0$.

Optimizaton results

- Separate actions. In this case, due to the martingale property of Z_i, $\mathbf{E}\left(X_i^2(1)+\int_0^1 u_i^2(s)\,\mathrm{d}s\right)=y(0)\,X_i^2(0)=y(0)\,x_i^2$

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$$\mathbf{E}\left(X_1^2(1)+\int_0^1 u_1^2(s)\,\mathrm{d}s + X_1^2(1)+\int_0^1 u_1^2(s)\,\mathrm{d}s\right) = y(0)\left(x_1^2+x_2^2\right).$$

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$$\mathbf{E}\left(X_1^2(1)+\int_0^1 u_1^2(s)\,\mathrm{d}s+X_1^2(1)+\int_0^1 u_1^2(s)\,\mathrm{d}s\right)=y(0)\left(x_1^2+x_2^2\right).$$
- Fusion. In this case
$$\mathbf{E}\left((X_1(1)+X_2(1))^2+\int_0^1 (u_1(s)+u_2(s))^2\,\mathrm{d}s\right)$$
$$=f(0)(x_1+x_2)^2+2g(0)\,x_1x_2.$$

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- Collusive actions. In this case
$$\mathbf{E}\left((X_1(1)+X_2(1))^2+\int_0^1 u_1^2(s)\,\mathrm{d}s+\int_0^1 u_1^2(s)\,\mathrm{d}s\right)$$
$$=\varphi(0)(x_1+x_2)^2+2\psi(0)\,x_1x_2.$$

Optimizaton results - numerical examples

Recall the optimal values.

- Separate actions.

$$y(0)\,x_i^2.$$

- Fusion.

$$f(0)\,(x_1+x_2)^2+2g(0)\,x_1x_2.$$

- Collusive actions.

$$\varphi(0)\,(x_1+x_2)^2+2\psi(0)\,x_1x_2.$$

Numerical examples

Model parameters	Quantity	$\alpha=1,\beta=0.7$ $A=0.7$	$\alpha=1,\beta=2$ $A=2$
Separate actions	$y(0)$	3.80	1.5
Fussion	$f(0)\,,\,g(0)$	2.16, 0.19	16.7, -13.81
Collusive optima	$\varphi(0)\,,\,\psi(0)$	3.05, 1.10	1.49, 0.00

Thank you!