Existence of an Equivalent Martingale Measure in the Dalang–Morton–Willinger Theorem, which Preserves the Dependence Structure

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The Dalang–Morton–Willinger theorem (1990)

Theorem (DMW, Fundamental Theorem of Asset Pricing)

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,\dots,T}, \mathbb{P})$ be a (general) filtered probability space and let $S = \{S_t\}_{t\in\{0,\dots,T\}}$ be an adapted, $\mathbb{R}^d$-valued stochastic process describing the discounted prices of $d \in \mathbb{N}$ financial assets. Then the following properties are equivalent:

(a) *The financial market model is free of arbitrage.*

(b) *There exists a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that*
- *$\mathbb{P}^*$ is equivalent to $\mathbb{P}$ and the Radon–Nikodým density $\varrho := d\mathbb{P}^*/d\mathbb{P}$ is* **bounded**, *i.e. in $L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$,*
- *Integrability: $S_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P}^*)$ for all $t \in \{0, \dots, T\}$,*
- *Martingale property w.r.t. $\mathbb{P}^*$:*

$$\mathbb{E}_{\mathbb{P}^*}[S_t \,|\, \mathcal{F}_{t-1}] \overset{a.s.}{=} S_{t-1} \quad \text{for all } t \in \{1, \dots, T\}.$$

Remarks on the proof of the Dalang–Morton–Willinger theorem

- No integrability assumption on $S_0, \dots, S_T$ w.r.t. $\mathbb{P}$.
- Proving (a) $\Longrightarrow$ (b), i.e. the existence of the equivalent martingale measure $\mathbb{P}^*$, is the hard part.
- The proof uses the Hahn–Banach theorem (and therefore the axiom of choice), hence it is not constructive.
- Boundedness of the density ϱ originates from the fact that $L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$ "is" the topological dual of $L^1(\Omega, \mathcal{F}_T, \mathbb{P})$.
- For proving (b) $\Longrightarrow$ (a), i.e. absence of arbitrage, the boundedness of the density ϱ is not needed. (This works nicely with conditional expectations for σ-integrable random variables.)

Application of the Dalang–Morton–Willinger theorem to the pricing of contingent claims

- Add a discounted, contingent claim $C \in L^1(\Omega, \mathcal{F}_T, \mathbb{P})$ to an arbitrage-free financial market model.
- By DMW, (a) $\Longrightarrow$ (b), there exists an equivalent martingale measure $\mathbb{P}^*$ with **bounded density** ϱ, hence

$$\mathbb{E}_{\mathbb{P}^*}\big[|C|\big] = \mathbb{E}_{\mathbb{P}}\big[\varrho|C|\big] \le \|\varrho\|_\infty \, \mathbb{E}_{\mathbb{P}}\big[|C|\big] < \infty.$$

- Therefore, the additional discounted price process

$$S_t^{(d+1)} := \mathbb{E}_{\mathbb{P}^*}[C \,|\, \mathcal{F}_t], \quad t \in \{0, \ldots, T\},$$

is (also) a well-defined $\mathbb{P}^*$-martingale.
- By DMW, (b) $\Longrightarrow$ (a), the model $(S_t^{(1)}, \ldots, S_t^{(d)}, S_t^{(d+1)})$, $t \in \{0, \ldots, T\}$, with the additional discounted price process is free of arbitrage.

Example for gaining "nice" dependence properties

Binomial model:

- On $\Omega = \{-1, 1\}^T$ let $X_1, \dots, X_T$ denote the $\{-1, 1\}$-valued projections.
- Define $S_t = X_1 + \cdots + X_t$ and $\mathcal{F}_t = \sigma(S_0, \dots, S_t)$ for all $t \in \{0, \dots, T\}$.
- Under $\mathbb{P}^* = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes T}$ the projections $X_1, \dots, X_T$ are i.i.d. with symmetric distribution and $(S_0, \dots, S_T)$ is a martingale.
- $\mathbb{P}^*$ with this martingale property is unique.
- By DMW, (b) $\Longrightarrow$ (a), every measure $\mathbb{P}$ with full support gives an arbitrage-free model, hence DMW, (a) $\Longrightarrow$ (b), turns every $\mathbb{P}$ with full support into $\mathbb{P}^*$.

Central question concerning the measure change

If the discounted asset price process $S = \{S_t\}_{t\in\{0,\dots,T\}}$ has a "nice" dependence property under the original measure $\mathbb{P}$, does there exist an equivalent martingale measure $\mathbb{P}^*$ as in DMW (with bounded density) such that S has the same dependence property under $\mathbb{P}^*$?

Examples:

- Martingale property (trivial, holds for every $\mathbb{P}^*$)
- If S is an integrable process which is increasing in the convex order w.r.t. $\mathbb{P}$ (i.e, S is a peacock), then S keeps this property under every $\mathbb{P}^*$, because every martingale is a peacock (apply the conditional Jensen inequality).

Further dependence properties

Does there exist at least one $\mathbb{P}^*$ from the DMV theorem to preserve the following dependence properties:

- independence of additive (or multiplicative) increments of S,
- independence and identical distribution of additive (or multiplicative) increments of S,
- Markov property,
- k-multiple Markov property,
- exchangeablility,
- . . . ?

Result for independent additive increments

Theorem (Gülüm/S.)

Assume that the adapted process $S = \{S_t\}_{t\in\{0,\dots,T\}}$ of the discounted asset prices does not offer arbitrage possibilities.

- *Assume that the additive increment $X_t := S_t - S_{t-1}$ is independent of $\mathcal{F}_{t-1}$ under $\mathbb{P}$ for every $t \in \{1, \dots, T\}$. Then there exists an equivalent martingale measure $\mathbb{P}^*$ for S with bounded density such that the increments $\{X_t\}_{t\in\{1,\dots,T\}}$ of S keep this independence under $\mathbb{P}^*$.*
- *If, in addition, $X_1, \dots, X_T$ have the same distribution under $\mathbb{P}$, then $\mathbb{P}^*$ can be chosen to preserve this property, too.*

Preservation of the Markov property (history)

- Stanley Pliska shows in his book “Introduction to Mathematical Finance, Discrete Time Models” (1998) that, if the discounted asset price process S is a Markov chain under $\mathbb{P}$ with **finite state space**, then an equivalent martingale measure $\mathbb{P}^*$ can be chosen such that S also has the Markov property under $\mathbb{P}^*$. Due to the finite state space, the boundedness of the density ϱ is trivial.
- This result was generalized by Freddy Delbaen (lectures on discrete-time mathematical finance at ETH Zürich) to Markov chains with a general state space, but **boundedness of the density ϱ was left open**.
- To my (and Freddy Delbaen’s) knowledge, this problem remained open for the last decade.
- While searching for a counterexample to close a matter, we instead found a proof, even for k-multiple Markov chains.

Definition of k-multiple Markov chains

Slightly more general setting:

Let $I \subseteq \mathbb{Z}$ be a discrete interval, $(E, \mathcal{E})$ be a measurable space, $S = \{S_t\}_{t\in I}$ a E-valued stochastic process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in I}, \mathbb{P})$, and $k \in \mathbb{N}$. Define $I_k = \{t \in I \mid t - k \in I\}$.

Definition

We say that S is a k-multiple Markov chain with respect to $\mathbb{P}$ and $\{\mathcal{F}_t\}_{t\in I}$, if S is adapted and if

$$\mathbb{P}(S_t \in B \,|\, \mathcal{F}_{t-1}) \overset{\text{a.s.}}{=} \mathbb{P}(S_t \in B \,|\, S_{t-1}, S_{t-2}, \ldots, S_{t-k})$$

for every $B \in \mathcal{E}$ and for every $t \in I_k$.

Remark (no enlargement of the state space E to E^k)

The usual reduction to the Markov chain $S'_t := (S_t, \ldots, S_{t-k+1})$, $t \in I_{k-1}$, doesn't fit nicely with the martingale property for $E = \mathbb{R}^d$.

Characterization of k-multiple Markov chains

Lemma

For an E-valued process S adapted to the filtration $\{\mathcal{F}_t\}_{t\in I}$, the following statements are equivalent:

1. S is a k-multiple Markov chain.
2. For every $\mathcal{E}$-measurable function $g\colon E \to \mathbb{R}$, which is bounded or nonnegative, and every $t \in I_k$,
$$\mathbb{E}\big[g(S_t)\,\big|\,\mathcal{F}_{t-1}\big] \overset{\text{a.s.}}{=} \mathbb{E}\big[g(S_t)\,\big|\,S_{t-1}, S_{t-2}, \dots, S_{t-k}\big].$$
3. For every $\mathcal{E}^{\otimes(k+1)}$-measurable function $h\colon E^{k+1} \to \mathbb{R}$, which is bounded or nonnegative, and every $t \in I_k$,
$$\begin{aligned}&\mathbb{E}\big[h(S_t, S_{t-1}, \dots, S_{t-k})\,\big|\,\mathcal{F}_{t-1}\big]\\ &\qquad \overset{\text{a.s.}}{=} \mathbb{E}\big[h(S_t, S_{t-1}, \dots, S_{t-k})\,\big|\,S_{t-1}, S_{t-2}, \dots, S_{t-k}\big].\end{aligned}$$

DMW theorem for k-multiple Markov chains

Theorem (Gülüm/S.)

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0,\dots,T}, \mathbb{P})$ be a (general) filtered probability space and let $S = \{S_t\}_{t\in\{0,\dots,T\}}$ be an adapted, $\mathbb{R}^d$-valued discounted asset price process, which has the k-multiple Markov property w.r.t. $\mathbb{P}$. Then the following properties are equivalent:

(a) *The financial market model is free of arbitrage.*

(b) *There exists a probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}, \mathbb{P})$ such that*

- $\mathbb{P}^* \sim \mathbb{P}$ *with* $\varrho := d\mathbb{P}^*/d\mathbb{P} \in L^\infty(\Omega, \mathcal{F}_T, \mathbb{P})$,
- *Integrability:* $S_t \in L^1(\Omega, \mathcal{F}_t, \mathbb{P}^*)$ *for all* $t \in \{0, \dots, T\}$,
- *Martingale property w.r.t.* $\mathbb{P}^*$*:*

$$\mathbb{E}_{\mathbb{P}^*}[S_t \,|\, \mathcal{F}_{t-1}] \overset{a.s.}{=} S_{t-1} \qquad \text{for all } t \in \{1, \dots, T\}.$$

- *k-multiple Markov property: For all* $B \in \mathcal{E}$ *and* $t \in \{k, \dots, T\}$

$$\mathbb{P}^*(S_t \in B \,|\, \mathcal{F}_{t-1}) \overset{a.s.}{=} \mathbb{P}^*(S_t \in B \,|\, S_{t-1}, S_{t-2}, \dots, S_{t-k}).$$

Idea of the proof: lemma for backward induction

Suppose we have shown the following result:

Lemma (Gülüm/S.)

Let $u \in \{1, \ldots, T\}$. Assume that there is a measure $\mathbb{Q}$ such that

- $S = \{S_t\}_{t \in \{0,\ldots,T\}}$ *is a k-multiple Markov chain under* $\mathbb{Q}$ *and*
- $\{S_t\}_{t \in \{u,u+1,\ldots,T\}}$ *is a* $\mathbb{Q}$*-martingale.*

Then there is a probability measure $\mathbb{Q}^*$ *on* $(\Omega, \mathcal{F})$ *such that:*

- $\mathbb{Q}^* \sim \mathbb{Q}$ *and the density* $d\mathbb{Q}^*/d\mathbb{Q}$ *is bounded and* $\mathcal{F}_u$*-measurable,*
- S_{u-1} *is* $\mathbb{Q}^*$*-integrable and* $\{S_t\}_{t \in \{u-1,u,\ldots,T\}}$ *is a* $\mathbb{Q}^*$*-martingale,*
- $S = \{S_t\}_{t \in \{0,\ldots,T\}}$ *is a k-multiple Markov chain under* $\mathbb{Q}^*$.

Idea of the proof: application of the lemma

After changing from $\mathbb{P}$ to a $\mathbb{Q}_T$ to make S_T integrable w.r.t. $\mathbb{Q}_T$, the model of the theorem satisfies all of the conditions of the lemma for $u = T$. So we get a probability measure $\mathbb{Q}_T^* \sim \mathbb{Q}_T$ with bounded density Z_T. The model $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\{0,\dots,T-1\}}, \mathbb{Q}_T^*)$ with asset prices $\{S_t\}_{t\in\{0,\dots,T-1\}}$ is arbitrage-free and satisfies all of the above conditions, hence we get a new probability measure $\mathbb{Q}_{T-1}^* \sim \mathbb{Q}_T^*$ with bounded, $\mathcal{F}_{T-1}$-measurable density Z_{T-1}. Doing this iteratively, we get probability measures $\mathbb{Q}_T^*, \mathbb{Q}_{T-1}^*, \dots, \mathbb{Q}_1^*$ such that $Z_u = \mathrm{d}\mathbb{Q}_u^*/\mathrm{d}\mathbb{Q}_{u+1}^* \in L^\infty$ for $u \in \{1, \dots, T-1\}$. Set $\mathbb{P}^* = \mathbb{Q}_1^*$. All that remains to show is that $\mathrm{d}\mathbb{P}^*/\mathrm{d}\mathbb{P}$ is bounded and strictly positive. This follows from

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{d}\mathbb{Q}_1^*}{\mathrm{d}\mathbb{Q}_2^*} \cdot \frac{\mathrm{d}\mathbb{Q}_2^*}{\mathrm{d}\mathbb{Q}_3^*} \cdot \dots \cdot \frac{\mathrm{d}\mathbb{Q}_T^*}{\mathrm{d}\mathbb{Q}_T} \cdot \frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{d}\mathbb{Q}_T}{\mathrm{d}\mathbb{P}} \prod_{u=1}^{T} Z_u \,.$$

Proof of the lemma (sketch)

- Since the model $\{S_t\}_{t\in\{0,\dots,u\}}$ is also arbitrage-free, there is by the DMW theorem an equivalent martingale measure $\mathbb{Q}'$ with bounded and $\mathcal{F}_u$-measurable density Z' with respect to $\mathbb{Q}$ such that $\{S_t\}_{t\in\{0,\dots,u\}}$ is a $\mathbb{Q}'$-martingale.
- If $u \in \{k,\dots,T\}$, then we define Z as the density of $\mathbb{Q}'$ with respect to $\mathbb{Q}$ on $\sigma(S_u, S_{u-1},\dots,S_{u-k})$, i.e.

$$Z = \mathbb{E}_{\mathbb{Q}}\big[Z' \,\big|\, S_u, S_{u-1},\dots,S_{u-k}\big].$$

  If $u \in \{1,\dots,k-1\}$, then we define $Z = Z'$.
- Z is almost surely bounded, strictly positive and $\mathbb{E}_{\mathbb{Q}}[Z] = 1$, so we can define a new probability measure $\mathbb{Q}^*$ on $\mathcal{F}$ via

$$\frac{d\mathbb{Q}^*}{d\mathbb{Q}} = Z.$$

- It follows that $\mathbb{Q}^* \sim \mathbb{Q}$, that Z is $\mathcal{F}_u$-measurable, and by calculation that $\mathbb{Q}^*$ has the desired properties.

Announcements

18th ÖMG Congress and Annual DMV Meeting

- Monday–Friday, 23.–27. September 2013
- University of Innsbruck, Austria
- Minisymposium on Financial and Actuarial Mathematics
- http://math-oemg-dmv-2013.uibk.ac.at/

Portfolio Risk Management Day 2013

- Friday, 27. September 2013, 9:00 – 18:00
- TU Vienna, no registration fee
  (but no free lunch, just free coffee)
- http://www.fam.tuwien.ac.at/prisma2013/

2nd European Actuarial Journal Conference 2014

- Wednesday–Friday, 10.–12. September 2014
- TU Vienna
- http://www.fam.tuwien.ac.at/eaj2014/