Arbitrages arising with honest times

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based on a joint work with Monique Jeanblanc and Shiqi Song (University of Évry)

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Private information and arbitrage profits

- Let the publicly available information be represented by a filtration $\mathbb{F}$.
- Let $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be a filtration with $\mathcal{G}_t \supseteq \mathcal{F}_t$ for all $t \geq 0$.

  $\Rightarrow$ from a financial point of view, the filtration $\mathbb{G}$ represents the point of view of a better informed agent (*insider trader*).

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The basic example

- Let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}^W, P)$;
- let $S = (S_t)_{t \geq 0}$ represent the discounted price of a risky asset, with:

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  Define the *progressively enlarged filtration* $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ as:

$$\mathcal{G}_t := \bigcap_{s > t} \mathcal{G}^0_s \text{ with } \mathcal{G}^0_t := \mathcal{F}^W_t \vee \sigma(\tau \wedge t), \text{ for all } t \geq 0$$

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- An **arbitrage strategy** in the enlarged filtration $\mathbb{G}$:

buy at $t = 0$ and sell at $t = \tau$

Discussion and motivation

- In the previous example, the random time τ is an **honest time**:
for every $t > 0$, there exists an $\mathcal{F}_t^W$-measurable random variable ζ_t such that
$$\tau = \zeta_t \text{ on } \{\tau < t\}.$$
Indeed, we can take $\zeta_t := \sup\left\{u \in [0,t] : S_u = \sup_{r\in[0,t]} S_r\right\} \in \mathcal{F}_t^W$.

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What happens in general?

- continuous semimartingale setting;
- do arbitrage profits exist *before* τ?
- do arbitrage profits exist *at* τ?
- do arbitrage profits exist *after* τ?
- which is the appropriate notion of “arbitrage profit”?

Setting and preliminaries

- Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ be a given filtered probability space;
- let the $\mathbb{R}^d$-valued continuous semimartingale $S = (S_t)_{t \geq 0}$ represent the discounted price of d risky assets;
- let $\tau : \Omega \to [0, \infty]$ be a P-a.s. finite *honest time*;
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Definition

Let $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}\}$. For $a \in \mathbb{R}_+$, an element $\theta \in L^{\mathbb{H}}(S)$ is said to be an *a-admissible $\mathbb{H}$-strategy* if $(\theta \cdot S)_\infty$ exists and $(\theta \cdot S)_t \geq -a$ P-a.s. for all $t \geq 0$. We denote by $\mathcal{A}^{\mathbb{H}}_a$ the set of all a-admissible $\mathbb{H}$-strategies. We say that an element $\theta \in L^{\mathbb{H}}(S)$ is an *admissible $\mathbb{H}$-strategy* if $\theta \in \mathcal{A}^{\mathbb{H}} := \bigcup_{a \in \mathbb{R}_+} \mathcal{A}^{\mathbb{H}}_a$.

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Definition

- The *restricted financial market* is the tuple $\mathcal{M}^{\mathbb{F}} := (\Omega, \mathcal{F}, \mathbb{F}, P; S, \mathcal{A}^{\mathbb{F}})$;
- the *enlarged financial market* is the tuple $\mathcal{M}^{\mathbb{G}} := (\Omega, \mathcal{F}, \mathbb{G}, P; S, \mathcal{A}^{\mathbb{G}})$.

Two notions of arbitrage

For a strategy $\theta \in \mathcal{A}^{\mathbb{G}}$ and $x \in \mathbb{R}_+$, we denote by $V(x, \theta) = x + \int \theta dS$ the corresponding wealth process (self-financing trading).

Definition

- An element $\theta \in \mathcal{A}^{\mathbb{G}}$ is said to be an *arbitrage opportunity in* $\mathbb{G}$ if $V(0, \theta)_\infty \geq 0$ P-a.s. and $P(V(0, \theta)_\infty > 0) > 0$.
  The financial market $\mathcal{M}^{\mathbb{G}}$ satisfies **NA** if no such $\theta \in \mathcal{A}^{\mathbb{G}}$ exists.
- A non-negative random variable ξ with $P(\xi > 0) > 0$ is said to be an *arbitrage of the first kind in* $\mathbb{G}$ if for all $x > 0$ there exists an element $\theta^x \in \mathcal{A}_x^{\mathbb{G}}$ such that $V(x, \theta^x)_\infty := x + (\theta^x \cdot S)_\infty \geq \xi$ P-a.s.
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Remark: NA1 is the minimal condition for the solvability of expected utility maximization problems (see Karatzas & Kardaras, 2007).

Martingale measures and deflators

Definition

- A probability measure $Q \sim P$ is said to be an *Equivalent Local Martingale Measure in* $\mathbb{G}$ (ELMM$_{\mathbb{G}}$) if S is a $\mathbb{G}$-local martingale under Q.
- A strictly positive $\mathbb{G}$-local martingale $L = (L_t)_{t \geq 0}$ with $L_0 = 1$ and $L_\infty > 0$ P-a.s. is said to be a *local martingale deflator in* $\mathbb{G}$ if LS is a $\mathbb{G}$-local martingale;

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- Let Q be an ELMM$_{\mathbb{G}}$ and denote by Z^Q its density process, i.e., $Z_t^Q = dQ/dP|_{\mathcal{G}_t}$, for $t \geq 0$.
  $\Rightarrow$ Then Z^Q is a local martingale deflator in $\mathbb{G}$.
- Let L be a local martingale deflator in $\mathbb{G}$.
  $\Rightarrow$ Then we can define an ELMM$_{\mathbb{G}}$ Q by letting $dQ/dP := L_\infty$ *if and only if* $E[L_\infty] = 1$.

Fundamental theorem of asset pricing

Theorem (Delbaen-Schachermayer, 1994-1998; Kardaras, 2007-2012)

- *NA1 holds in the financial market $\mathcal{M}^{\mathbb{G}}$ if and only if there exists a local martingale deflator in $\mathbb{G}$;*
- *NFLVR holds in the financial market $\mathcal{M}^{\mathbb{G}}$ if and only if there exists an ELMM$_{\mathbb{G}}$;*
- *NFLVR holds in the financial market $\mathcal{M}^{\mathbb{G}}$ if and only if both NA1 and NA hold.*

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What happens in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$?

Two technical results

Lemma (Nikeghbali-Yor, 2006)

There exists a continuous non-negative $\mathbb{F}$*-local martingale* $N = (N_t)_{t\geq 0}$ *with* $N_0 = 1$ *and* $\lim_{t\to\infty} N_t = 0$ P*-a.s. such that the following holds, for all* $t \geq 0$*:*

$$Z_t := P(\tau > t|\mathcal{F}_t) = N_t/N_t^*$$

where $N_t^* := \sup_{u\leq t} N_u$*. Furthermore:*

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Let $X = (X_t)_{t \geq 0}$ be an $\mathbb{F}$-local martingale. Then X has the following canonical decomposition as a semimartingale in the filtration $\mathbb{G}$:

$$X_t = \widetilde{X}_t + \int_0^{t \wedge \tau} \frac{d\langle X, N \rangle_s}{N_s} - \int_\tau^{t \vee \tau} \frac{d\langle X, N \rangle_s}{N_\infty^* - N_s}$$

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- by MRP in the $\mathbb{F}$-filtration, we can write $N = 1 + \varphi \cdot S$, with $\varphi \in L^{\mathbb{F}}(S)$.

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Proposition

The enlarged financial market $\mathcal{M}^{\mathbb{G}}$ does not satisfy NA on $[0, \tau]$.

Proof: it suffices to consider the arbitrage strategy $\bar{\varphi} := \mathbf{1}_{[\![0,\tau]\!]}\varphi \in \mathcal{A}_1^{\mathbb{G}}$.

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Does the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ satisfy NA1 on $[0,\tau]$?

Arbitrages on the time horizon $[0, \tau]$

Existence of a local martingale deflator in $\mathbb{G}$ on $[0, \tau]$

The stopped process S^τ admits the following $\mathbb{G}$-canonical decomposition:

$$S^\tau = \widetilde{S}^\tau + \frac{1}{N} \cdot \langle S^\tau, N \rangle$$

where $\widetilde{S} = (\widetilde{S}_t)_{t \geq 0}$ is a continuous $\mathbb{G}$-local martingale with $\widetilde{S}_0 = S_0 = s$.

Arbitrages on the time horizon $[0, \tau]$

Existence of a local martingale deflator in $\mathbb{G}$ on $[0, \tau]$

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Proposition

The process $1/N^\tau$ is a local martingale deflator in $\mathbb{G}$ on the time horizon $[0, \tau]$.
Furthermore, the process $1/N^\tau$ is not a u.i. $\mathbb{G}$-martingale.

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Proposition

The process $1/N^\tau$ is a local martingale deflator in $\mathbb{G}$ on the time horizon $[0, \tau]$. Furthermore, the process $1/N^\tau$ is not a u.i. $\mathbb{G}$-martingale.

Proof: First argue that $N_\tau > 0$ P-a.s. Then, by Itô's formula:

$$\frac{1}{N^\tau} = 1 - \frac{1}{(N^\tau)^2} \cdot N^\tau + \frac{1}{(N^\tau)^3} \cdot \langle N \rangle^\tau = 1 - \frac{\varphi}{(N^\tau)^2} \cdot S^\tau + \frac{\varphi}{(N^\tau)^3} \cdot \langle S^\tau, N \rangle = 1 - \frac{\varphi}{(N^\tau)^2} \cdot \widetilde{S}^\tau$$

Hence, $1/N^\tau$ is a P-a.s. strictly positive $\mathbb{G}$-local martingale with $1/N_0 = 1$ and $1/N_\tau > 0$ P-a.s. The product rule shows that S^τ / N^τ is a $\mathbb{G}$-local martingale and, hence, $1/N^\tau$ is a local martingale deflator in $\mathbb{G}$ on $[0, \tau]$. Finally, observe that $E[1/N^\tau_\infty] = E[1/N_\tau] < 1 = E[1/N_0]$.

Arbitrages on the time horizon $[0, \tau]$

Validity of NA1 in $\mathcal{M}^{\mathbb{G}}$ on $[0, \tau]$

As an immediate consequence, we get the following theorem:

Theorem

In the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ the following hold:

(i) *NA1 holds on the time horizon $[0, \tau]$;*

(ii) *NA and NFLVR fail on the time horizon $[0, \tau]$.*

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A characterization of local martingale deflators in $\mathcal{M}^{\mathbb{G}}$ on $[0, \tau]$:

Lemma

Let $L = (L_t)_{t \geq 0}$ be a local martingale deflator in $\mathbb{G}$ on the time horizon $[0, \tau]$. Then L admits the following representation:

$$L^{\tau} = \frac{1}{N^{\tau}} \exp\left(-\frac{k}{N^*} \cdot N^*\right) \left(1 + k_{\tau} \mathbf{1}_{[\![\tau,\infty[\![} + \eta \mathbf{1}_{[\![\tau,\infty[\![}\right)$$

where $k = (k_t)_{t \geq 0}$ is an $\mathbb{F}$-predictable process such that $k_{\tau} > -1$ P-a.s. and η is a $\mathcal{G}_{\tau}$-measurable random variable such that $E[\eta | \mathcal{G}_{\tau-}] = 0$.

Lemma

Let σ be an $\mathbb{F}$-stopping time and $L = (L_t)_{t \geq 0}$ a local martingale deflator in $\mathbb{G}$ on the time horizon $[0, \sigma \wedge \tau]$. Then the following holds:

$$E\left[L_{\sigma \wedge \tau}\right] = E\left[1 - \exp\left(-\int_0^{\tau} \frac{1+k_s}{N_s^*}\, dN_s^*\right) \mathbf{1}_{\{\nu \leq \sigma\}}\right] \leq 1$$

where $\nu := \inf\{t \geq 0 : N_t = 0\}$. Furthermore, we have $\int_0^{\tau} \frac{1+k_s}{N_s^*}\, dN_s^* > 0$ P-a.s.

Arbitrages before time τ

Impossibility of arbitrages before time τ

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Corollary

Let σ be an $\mathbb{F}$-stopping time. The following are equivalent:

- *NFLVR holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on $[0, \sigma \wedge \tau]$;*
- $P(\sigma \geq \nu) = 0$.

In particular, NFLVR holds in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[0, \varrho]$ for every $\mathbb{G}$-stopping time ϱ with $\varrho < \tau$ P-a.s.

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Arbitrages on the global time horizon

Failure of NA1in $\mathcal{M}^{\mathbb{G}}$ after time τ

The process S admits the following $\mathbb{G}$-canonical decomposition:

$$S_t = \widetilde{S}_t + \int_0^{t\wedge\tau} \frac{d\langle S, N\rangle_s}{N_s} - \int_\tau^{t\vee\tau} \frac{d\langle S, N\rangle_s}{N^*_\infty - N_s} = \widetilde{S}_t + \int_0^t d\langle \widetilde{S}, \widetilde{S}\rangle_s \, \tilde{\alpha}_s$$

where $\widetilde{S} = (\widetilde{S}_t)_{t\geq 0}$ is a continuous $\mathbb{G}$-local martingale with $\widetilde{S}_0 = S_0 = s$ and the $\tilde{\alpha}_t := \mathbf{1}_{\{\tau\geq t\}} \frac{\varphi_t}{N_t} - \mathbf{1}_{\{\tau<t\}} \frac{\varphi_t}{N^*_\infty - N_t}$, for all $t \geq 0$.

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Proposition

The random variable $\xi := N_\tau - 1$ yields an arbitrage of the first kind in $\mathbb{G}$. As a consequence, NA1 fails in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ on $[0, \infty]$.

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Sketch of the proof:

Note that $\xi \geq 0$ P-a.s. and $P(\xi > 0) > 0$ and let $\hat{\varphi} := -\mathbf{1}_{(\!(\tau,\infty]\!]} \in \mathcal{A}_0^{\mathbb{G}}$.

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Note that $\xi \geq 0$ P-a.s. and $P(\xi > 0) > 0$ and let $\hat{\varphi} := -\mathbf{1}_{(\!(\tau,\infty]\!]} \in \mathcal{A}^{\mathbb{G}}_0$.

$\Rightarrow$ The enlarged financial market $\mathcal{M}^{\mathbb{G}}$ does not admit a local martingale deflator in $\mathbb{G}$ on $[0, \infty]$.

$\Rightarrow$ As a consequence, NFLVR fails in the enlarged financial market $\mathcal{M}^{\mathbb{G}}$.

Arbitrages strictly after τ

Validity of NA1 in $\mathcal{M}^{\mathbb{G}}$ on the time horizon $[\tau + \varepsilon, \infty]$

Proposition

For every $\varepsilon > 0$, the process ${}^{\varepsilon}L = ({}^{\varepsilon}L_t)_{t \geq 0}$ defined by

$$ {}^{\varepsilon}L_t = \frac{N_\tau - N_{\tau+\varepsilon}}{N_\tau - N_{t \vee (\tau+\varepsilon)}} = \frac{N^*_\infty - N_{\tau+\varepsilon}}{N^*_\infty - N_{t \vee (\tau+\varepsilon)}}, \qquad \text{for } t \geq 0, $$

is a local martingale deflator in $\mathbb{G}$ for the process ${}^{\tau+\varepsilon}S := S - S^{\tau+\varepsilon}$.
$\Rightarrow$ As a consequence, NA1 holds in $\mathcal{M}^{\mathbb{G}}$ on $[\tau + \varepsilon, \infty]$.

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$\Rightarrow$ As a consequence, NA1 holds in $\mathcal{M}^{\mathbb{G}}$ on $[\tau+\varepsilon,\infty]$.

However...

- For every $\varepsilon\in(0,\delta)$, the strategy $-\varphi/N^*_\infty$ belongs to $\mathcal{A}^{\mathbb{G}}_1$ for the process ${}^{\tau+\varepsilon}S=S-S^{\tau+\varepsilon}$ and realizes an arbitrage opportunity on $[\tau+\varepsilon,\infty]$.

  $\Rightarrow$ NA (and, hence, NFLVR as well) fails on $[\tau+\varepsilon,\infty]$.

Summing up

Theorem

In the enlarged financial market $\mathcal{M}^{\mathbb{G}}$ the following hold:

(i) *NA and NFLVR fail to hold on the time horizon $[0, \tau]$;*

(ii) *NA1 holds on the time horizon $[0, \tau]$;*

(iii) *NA1, NA and NFLVR fail to hold on the global time horizon $[0, \infty]$;*

(iv) *NA1 holds on the time horizon $[\tau + \varepsilon, \infty]$.*

$\Rightarrow$ Each of the above assertions can be proved:

- by means of explicit constructions of arbitrage strategies;
- by probabilistic arguments, using the multiplicative decomposition $P(\tau > t|\mathcal{F}_t) = N_t/N_t^*$.

Thank you for your attention