On an Integral Equation for the Free-Boundary of Stochastic, Irreversible Investment Problems

Giorgio Ferrari

Center for Mathematical Economics, Bielefeld University
and
Hausdorff Research Institute for Mathematics, Bonn

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In this talk we discuss the link between

free-boundary of continuous time, stochastic irreversible investment problems

and

the optional solution of a representation problem for optional processes studied by Peter Bank and Nicole El Karoui (2004).

Such a link enables us

- to obtain a new integral equation for the free-boundary which does not require smooth-fit property or a priori continuity of the free-boundary to be derived.

A (Very Brief) Introduction to 'Monotone Follower' Problems

Irreversible investment problems may be mathematically modeled as singular stochastic control problems of the **monotone follower** type,

- a stochastic control problem in which a *given diffusion is controlled additively by a nondecreasing process* (the monotone follower).

A probabilistic treatment of *monotone follower problems* and their application to Economics started with the early papers by

- Karatzas (1981), Karatzas and Shreve (1984), El Karoui and Karatzas (1991), among others.

By purely probabilistic arguments, Karatzas and Shreve in 1984 show that

- the problem of *optimally* tracking a Brownian motion by a nondecreasing process is equivalent to a suitable optimal stopping problem.

A (Very Brief) Introduction to 'Monotone Follower' Problems

Then, the optimal control policy may be characterized as follows:

- The state space (t, x) is divided in two regions.
- In the 'waiting region' $\mathcal{C}$: optimal to do nothing.
- In the 'action region' $\mathcal{S}$: profitable to exercise immediately the investment option.
- A monotone boundary splits these two regions. That is the *free-boundary*.
- The optimal control acts like the *local time* of the (optimally controlled) process at the boundary. It is the least effort to keep the controlled diffusion in the closure of $\mathcal{C}$.

A (Very Brief) Introduction to 'Monotone Follower' Problems

Later on, the **equivalence** between singular stochastic control problems and optimal stopping has been established **also for more complicated dynamics of the controlled diffusion**

- Baldursson and Karatzas (1997), Boetius and Kohlmann (1998) and Benth (2004), among others,

and **irreversible investment problems** have been widely studied in the economic and mathematical literature

- Kobila (1993), Dixit and Pindyck (1994), Øksendal (2000), Chiarolla and Haussmann (2005 and 2009) and Pham (2006), among others.

In the last decade, many papers addressed **optimal irreversible investment or consumption problems by a stochastic Kuhn-Tucker approach** and application of the Bank-El Karoui representation problem

- Bank and Riedel (2001), Bank (2005), Riedel and Su (2011), Steg (2012), Chiarolla, F. (2012) or Chiarolla, F. and Riedel (2012).

The Optimal Investment Problem

A firm represents the productive sector of a stochastic, continuous time economy over an infinite time horizon.

- complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- exogenous Brownian motion $\{W(t), t \geq 0\}$;
- $\{\mathcal{F}_t, t \geq 0\}$, the filtration generated by W and augmented by $\mathbb{P}$-null sets.

The uncertain status of the economy is modeled as a **one-dimensional, time-homogeneous, diffusion** $\{X^x(t), t \geq 0\}$ with state space $\mathcal{I} \subseteq \mathbb{R}$, unique strong solution of the SDE

$$\begin{cases} dX^x(t) = \mu(X^x(t))dt + \sigma(X^x(t))dW(t) \\ X(0) = x, \end{cases}$$

for some Borel functions $\mu : \mathcal{I} \mapsto \mathbb{R}$ and $\sigma : \mathcal{I} \mapsto (0, +\infty)$ such that for every $x \in \text{int}(\mathcal{I})$

$$\int_{x-\epsilon}^{x+\epsilon} \frac{1 + |\mu(y)|}{\sigma^2(y)}\, dy < +\infty, \quad \text{for some } \epsilon > 0. \tag{1}$$

Under (1), X^x is **regular** and its scale and speed measures are well defined.

The Optimal Investment Problem

The firm's manager aims to increase the production capacity

$$C^{y,\nu}(t) = y + \nu(t), \qquad C^{y,\nu}(0) = y \geq 0,$$

by optimally choosing an irreversible investment plan $\nu \in \mathcal{S}_o$, where

$$\mathcal{S}_o := \{\nu : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}_+, \text{ nondecreasing, left-continuous,} \\ \text{adapted s.t. } \nu(0) = 0, \ \mathbb{P} - \text{a.s.}\}$$

is the non empty, convex set of irreversible investment processes.

The firm

- makes profit at rate $\pi(x, c)$, when its own capacity is c and the status of economy is x;
- discounts revenues and costs at constant rate $r \geq 0$.

The Optimal Investment Problem

The operating profit function $\pi : \mathbb{R} \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is such that

Assumptions on π

The mapping $c \mapsto \pi(x, c)$ is strictly increasing and strictly concave with continuous derivative $\pi_c(x, c) := \frac{\partial}{\partial c}\pi(x, c)$, satisfying the Inada conditions

$$\lim_{c \to 0} \pi_c(x, c) = \infty, \qquad \lim_{c \to \infty} \pi_c(x, c) = 0.$$

The Optimal Investment Problem

The firm's optimal investment problem is then

$$V(x,y) := \sup_{\nu \in \mathcal{S}_o} \mathcal{J}_{x,y}(\nu), \tag{2}$$

where the net profit functional $\mathcal{J}_{x,y}(\nu)$ is defined as

$$\mathcal{J}_{x,y}(\nu) = \mathbb{E}\left\{ \int_0^\infty e^{-rt} \pi(X^x(t), C^{y,\nu}(t))dt - \int_0^\infty e^{-rt} d\nu(t) \right\}.$$

That is a singular stochastic control problem!

Under further minor assumptions on π and X, existence and uniqueness of a solution is a well known result (cf. Riedel and Su (2011)).

The First-Order Conditions for Optimality

We take care of the irreversible investment problem (2) by means of a **stochastic first order conditions approach**.

Let $\mathcal{T}$ denote the set of all $\mathcal{F}_t$-stopping times $\tau \geq 0$ a.s.

We may associate to $\mathcal{J}_{x,y}(\nu)$ its supergradient as the unique optional process defined by

$$\nabla \mathcal{J}_{x,y}(\nu)(\tau) := \mathbb{E}\left\{ \int_{\tau}^{\infty} e^{-rs} \pi_c(X^x(s), C^{y,\nu}(s)) ds \middle| \mathcal{F}_\tau \right\} - e^{-r\tau},$$

for any $\tau \in \mathcal{T}$.

The First-Order Conditions for Optimality

Theorem

Under the Assumptions on π, a process $\nu^* \in \mathcal{S}_o$ is the unique optimal investment strategy for problem (2) if and only if the following first order conditions for optimality

$$\begin{cases} \nabla \mathcal{J}_{x,y}(\nu^*)(\tau) \leq 0, \quad \textit{a.s. for any } \tau \in \mathcal{T}, \\ \mathbb{E}\left\{ \int_0^\infty \nabla \mathcal{J}_{x,y}(\nu^*)(t) d\nu^*(t) \right\} = 0, \end{cases} \tag{3}$$

hold true.

- The first order conditions are not binding at any time and so they cannot be directly applied to determine the optimal control ν^*.
- Nevertheless, ν^* may be obtained in terms of the solution of a suitable Bank-El Karoui's representation problem directly related to the FOCs.

The Bank-El Karoui Representation Theorem

The **Bank-El Karoui Representation Theorem** (2004) states that, given

- an optional process $X = \{X(t), t \in [0, T]\}$ of class (D), lower-semicontinuous in expectation with $X(T) = 0$,
- a nonnegative optional random Borel measure $\mu(\omega, dt)$,
- $f(\omega, t, x) : \Omega \times [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ such that $f(\omega, t, \cdot) : \mathbb{R} \mapsto \mathbb{R}$ is continuous, strictly decreasing from $+\infty$ to $-\infty$, and the stochastic process $f(\cdot, \cdot, x) : \Omega \times [0, T] \mapsto \mathbb{R}$ is progressively measurable and integrable with respect to $d\mathbb{P} \otimes \mu(\omega, dt)$,

then there exists an optional process $\xi = \{\xi(t), t \in [0, T]\}$ taking values in $\mathbb{R} \cup \{-\infty\}$ such that for all $\tau \in \mathcal{T}$,

$$f(t, \sup_{\tau \leq u < t} \xi(u)) \mathbb{1}_{(\tau, T]}(t) \in L^1 \left(d\mathbb{P} \otimes \mu(\omega, dt)\right)$$

and

$$\mathbb{E}\left\{ \int_{(\tau, T]} f(s, \sup_{\tau \leq u < s} \xi(u)) \, \mu(ds) \,\Big|\, \mathcal{F}_\tau \right\} = X(\tau).$$

The Bank-El Karoui Representation Theorem

Moreover, any progressively measurable, upper right-continuous solution ξ to the representation problem, i.e. such that

$$\xi(t) = \limsup_{s \downarrow t} \xi(s), \quad t \in [0, T),$$

is uniquely determined up to optional sections on $[0, T)$ in the sense that

$$\xi(\tau) = \operatorname*{ess\,inf}_{\tau < \sigma \leq T} \Xi_{\tau,\sigma}, \qquad \tau \in [0, T),$$

where $\Xi_{\tau,\sigma}$ is the unique (up to a $\mathbb{P}$-null set) $\mathcal{F}_\tau$-measurable random variable satisfying

$$\mathbb{E}\{X(\tau) - X(\sigma)|\mathcal{F}_\tau\} = \mathbb{E}\left\{ \int_{(\tau,\sigma]} f(t, \Xi_{\tau,\sigma})\, \mu(dt) \Big| \mathcal{F}_\tau \right\}.$$

The Bank-El Karoui Representation Theorem

In Bank-Föllmer (2003) it is shown that the representation problem is closely linked to the solution of

- stochastic optimization problems as Gittins Index Problems (e.g., cf. El Karoui and Karatzas (1994));
- optimal consumption choice problems with Hindy-Huang-Kreps utility functional (cf. Bank and Riedel (2001));
- parameter-dependent optimal stopping problems (see also Bank-Baumgarten (2010)).

Moreover, it is also related to the solution of irreversible investment problems as in

- Bank (2005), Riedel and Su (2011), Steg (2012), Chiarolla, F. (2012) or Chiarolla, F. and Riedel (2012).

The Optimal Solution

Suitably applying the Bank-El Karoui's Representation Theorem we can show that

Theorem

Under the Assumptions on π, the unique optimal irreversible investment process is

$$\nu^*(t) = \Big(\sup_{0 \le s \le t} l^*(s) - y \Big) \vee 0, \tag{4}$$

with $l^*(t)$ the unique optional, positive, upper-right continuous solution to

$$\mathbb{E}\left\{ \int_\tau^\infty e^{-rs} \pi_c(X^x(s), \sup_{\tau \le u \le s} l^*(u)) ds \Big| \mathcal{F}_\tau \right\} = e^{-r\tau}, \quad \tau \in \mathcal{T}.$$

Proof

ν^* of (4) satisfies the FOCs.

Base Capacity and Free-Boundary

So far

- The optimal policy is given in terms of the *base capacity process* l^*, a desirable value of capacity.
- If the current capacity at time t is below $l^*(t)$, then it is optimal to reach immediately that value; otherwise, no investment is optimal.
- This optimal control strategy acts like as that of the original monotone follower problems.

A link between l^* and the free-boundary b of an associated optimal stopping problem should exist!

Base Capacity and Free-Boundary

We show that

Theorem

Under our Assumptions on π, one has

$$l^*(t) = \sup\{y > 0 \,:\, v(X^x(t), y) = 1\},$$

where

$$v(x, y) := \inf_{\tau \geq 0} \mathbb{E}\left\{ \int_0^\tau e^{-rs} \pi_c(X^x(s), y) ds + e^{-r\tau} \right\}$$

is the value function of the optimal stopping problem naturally associated to the investment one.

v is the *optimal cost of not investing*.

Base Capacity and Free-Boundary

Since

Proposition

Under the Assumptions on π, *the mapping* $y \mapsto v(x,y)$ *is decreasing for any* $x \in \mathcal{I}$,

then

$$b(x) := \sup\{y > 0 \,:\, v(x,y) = 1\}, \qquad x \in \mathcal{I},$$

is the boundary between the continuation region

$$\mathcal{C} := \{(x,y) \in \mathcal{I} \times (0,\infty) : v(x,y) < 1\}$$

and the stopping region

$$\mathcal{S} := \{(x,y) \in \mathcal{I} \times (0,\infty) : v(x,y) = 1\}.$$

Base Capacity and Free-Boundary

It is now clear that

Theorem

Under our Assumptions on π, one has

$$I^*(t) = b(X^x(t)).$$

- Such result clarifies why in the literature one usually refers to I^* as a *desirable value of capacity* that the controller aims to maintain in a 'minimal way'.
- The optimal investment ν^* at time t is indeed the least effort needed to reflect the production capacity at the moving (random) boundary $I^*(t) = b(X^x(t))$; that is,

$$\nu^*(t) = \sup_{0 \leq s \leq t} \left(b(X^x(s)) - y \right) \vee 0.$$

Two intermediate (simple) results

Assume now that

- $x \mapsto \pi_c(x, c)$ is nondecreasing (if π is C^2, then π is supermodular)

Proposition

Under our Assumptions on π, $x \mapsto v(x, y)$ is nondecreasing for any $y > 0$.

Proposition

Under our Assumptions on π, the free-boundary $b(\cdot)$ between the continuation region and the stopping region is nondecreasing for any $x \in \mathcal{I}$.

The Integral Equation for the Free-Boundary

We may now state our main result.

Theorem

Let the Assumptions on π hold. Denote by

- $\mathcal{G}$ *the infinitesimal generator associated to* X;
- $\psi_r(x)$ *the increasing solution to the ODE* $\mathcal{G}u = ru$;
- $m(dx)$ *and* $s(dx)$ *the speed measure and the scale function measure, respectively, associated to* X;
- $\underline{x}$ *and* $\overline{x}$ *the lower and upper endpoints of the state space* $\mathcal{I}$ *of* X.

Then, the free-boundary $b(\cdot)$ *is the unique nondecreasing positive solution to the integral equation*

$$\psi_r(x) \int_x^{\overline{x}} \left(\int_{\underline{x}}^{z} \pi_c(y, b(z)) \psi_r(y) m(dy) \right) \frac{s(dz)}{\psi_r^2(z)} = 1.$$

The Integral Equation for the Free-Boundary: the Proof

Since l^* uniquely solves a backward stochastic equation and $l^*(t) = b(X^x(t))$, then, for any $\tau \in \mathcal{T}$,

$$r = \mathbb{E}\left\{ \int_\tau^\infty re^{-r(s-\tau)} \pi_c(X^x(s), \sup_{\tau \le u \le s} b(X^x(u))) ds \Big| \mathcal{F}_\tau \right\}$$
$$= \mathbb{E}\left\{ \int_0^\infty re^{-rt} \pi_c(X^x(t+\tau), b(\sup_{0 \le u \le t} X^x(u+\tau))) dt \Big| \mathcal{F}_\tau \right\},$$

where in the second equality we have used the fact that $b(\cdot)$ is nondecreasing.

Now, by strong Markov property, the previous one amounts to find $b(\cdot)$ such that

$$\mathbb{E}_x\left\{ \int_0^\infty re^{-rt} \pi_c(X(t), b(\sup_{0 \le u \le t} X(u))) dt \right\} = r;$$

The Integral Equation for the Free-Boundary: the Proof

that is, such that

$$\mathbb{E}_x\Big\{\pi_c(X(\tau_r), b(M(\tau_r)))\Big\} = r,$$

where

- $M(t) := \sup_{0 \le s \le t} X(s)$
- τ_r independent exponentially distributed random time with parameter r.

But now for a one-dimensional regular diffusion X^x (cf. Csáki et al. (1987)) one has

$$\mathbb{P}_x(X(\tau_r) \in dy, M(\tau_r) \in dz) = r\frac{\psi_r(x)\psi_r(y)}{\psi_r^2(z)} m(dy)s(dz), \quad y \le z, \;\; x \le z,$$

and this concludes the proof.

A Remark

Our integral equation

- follows immediately from the backward equation for $I^*(t) = b(X^x(t))$;
- does not require smooth-fit property or a priori continuity of $b(\cdot)$ to be applied;
- may be analitically solved for some non-trivial diffusions X.

The case of the 3-dimensional Bessel Process

- $X^x(t)$ is a three-dimensional Bessel process; that is,

$$dX^x(t) = \frac{1}{X^x(t)}dt + dW(t), \qquad X^x(0) = x > 0.$$

- $s(dx) = x^{-2}dx, \qquad m(dx) = 2x^2 dx.$
- $\psi_r(x) = \frac{\sinh(\sqrt{2r}x)}{x}.$
- Operating profit of Cobb-Douglas type: $\pi(x,c) = \frac{x^\alpha c^\beta}{\alpha+\beta}$ for $\alpha, \beta \in (0,1)$.

The case of the 3-dimensional Bessel Process

Proposition

For any $x > 0$ one has

$$b(x) = \left[\left(\frac{\alpha + \beta}{2\beta} \right) x^2 \frac{\psi_r'(x)}{g(x)} \right]^{-\frac{1}{1-\beta}},$$

where ψ_r' denotes the first derivative of the increasing function $\psi_r(x) = \frac{\sinh(\sqrt{2r}x)}{x}$, and $g(x) := \int_0^x y^{\alpha+1} \sinh(\sqrt{2r}y) dy$.

The case of the 3-dimensional Bessel Process

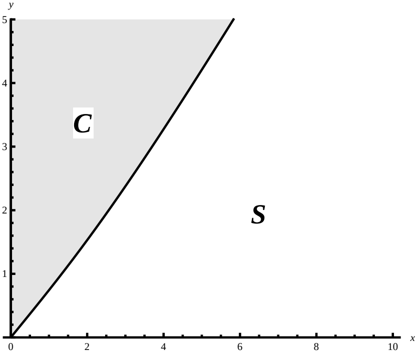

Figura: A computer drawing of the free-boundary for $\alpha = \beta = r = \frac{1}{2}$.

The case of the CEV Process

- $X^x(t)$ is a CEV process; that is,

$$dX^x(t) = rX^x(t)dt + \sigma(X^x)^{1-\gamma}(t)dW(t), \;\; X^x(0) = x > 0,$$

  for some $r > 0$, $\sigma > 0$ and $\gamma \in (0, \frac{1}{2}]$.

- $m(dx) = \frac{2}{\sigma^2 x^{2(1-\gamma)}} e^{\frac{r}{\gamma\sigma^2} x^{2\gamma}} dx, \qquad s(dx) = e^{-\frac{r}{\gamma\sigma^2} x^{2\gamma}} dx,$
- $\psi_r(x) = x$.
- Operating profit of Cobb-Douglas type: $\pi(x, c) = \frac{x^\alpha c^\beta}{\alpha+\beta}$ for $\alpha, \beta \in (0, 1)$.

The case of the CEV Process

Proposition

For any $x > 0$ one has

$$b(x) = \left[\frac{2\beta}{\sigma^2(\alpha+\beta)} g(x) e^{-\frac{r}{\gamma\sigma^2} x^{2\gamma}} \right]^{\frac{1}{1-\beta}}, \tag{5}$$

with $g(x) := \int_0^x y^{2\gamma+\alpha-1} e^{\frac{r}{\gamma\sigma^2} y^{2\gamma}} dy$.

The case of the CEV Process

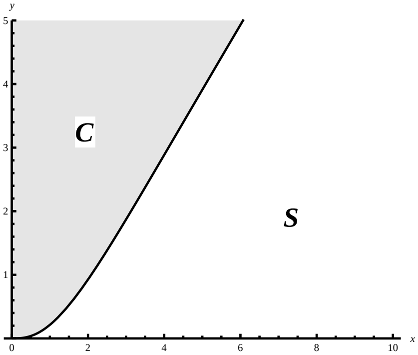

Figura: A computer drawing of the free-boundary for $\alpha = \beta = r = \frac{1}{2}$ and $\sigma = 1$.

The case of the CES Profit Function (non-separable)

Operating profit of CES type: $\pi(x,c) = \left(x^{\frac{1}{n}} + c^{\frac{1}{n}}\right)^n, \quad n \geq 2.$

The free boundary is the solution of an algebraic equation of order $n-1$.

- **Geometric Brownian motion**: $b(x) = C_n x$, with C_n the unique positive solution of

$$F_{2,1}(-(n-1), n\theta, n\theta+1, -C_n^{\frac{1}{n}}) = r, \quad r > 1.$$

- **Bessel 3D**: the free boundary $b(\cdot)$ is the unique positive, nondecreasing solution of

$$\sum_{k=1}^{n-1} \alpha_k(x) b^{-\frac{k}{n}}(x) = (r-1)\int_0^x y \sinh(\sqrt{2r}y)dy, \quad x > 0, \; r > 1.$$

- **CEV of parameter** $\gamma \in (0, \frac{1}{2}]$: the free boundary $b(\cdot)$ is the unique positive, nondecreasing solution of

$$\sum_{k=1}^{n-1} \beta_k(x) b^{-\frac{k}{n}}(x) = \frac{\sigma^2}{2r}[(r-1)e^{\frac{r}{\gamma\sigma^2}z^{2\gamma}} + 1], \quad x > 0, \; r > 1.$$

Conclusions and Current Research

- We have completely solved a general class of stochastic, continuous time irreversible investment problems over an infinite time horizon by means of a generalized Kuhn-Tucker approach.
- We have characterized their free-boundary in terms of the unique optional solution of a suitable representation problem à la Bank-El Karoui.
- Such identification enabled us to obtain a new handy integral equation for the free-boundary $b(\cdot)$.
- Such integral equation does not require smooth-fit or a priori continuity of $b(\cdot)$ to be applied.
- What if we consider a bounded variation control problem?
- Which is the connection between our integral equation and that one can be derived from local time-space calculus à la Peskir (2005)?

Grazie a tutti per l'attenzione

Essential References

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