Lévy Information and the Aggregation of Risk Aversion

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Advances in Mathematics of Finance
Sixth General AMaMeF Conference, Warsaw, 10-15 June 2013

This presentation is based on the following papers:

1. Brody, D.C., Hughston, L.P. & Mackie, E.
"General theory of geometric Lévy models for dynamic asset pricing"
Proceedings of the Royal Society London A**468**, 1778-1798 (2012)

2. Brody, D.C., Hughston, L.P. & Yang, X.
"Signal processing with Lévy information"
Proceedings of the Royal Society London A**469**, 2012.0433 (2013)

3. Brody, D.C. & Hughston, L.P.
"Lévy information and the aggregation of risk aversion"
Proceedings of the Royal Society London A**469**, 2013.0024 (2013)

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1. Questions to be addressed

- How are heterogeneous views on the level of risk aversion aggregated to form a single price in the context of Brownian-motion based models for asset prices?

- How are investors rewarded in exchange for the accommodation of jump risk?

- How are diverse views on jump risk aggregated?

- How can one estimate the excess rate of return on an asset from knowledge of the current price of the asset?

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2. Pricing kernel method in the Brownian context

We begin by consideration of the case of a single risky asset in the standard geometric Brownian motion (GBM) family of models.

For simplicity we present the case of a single such asset, and we assume that no dividends are paid over the time horizon considered.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the price we write

$$S_t = S_0\, \mathrm{e}^{(r+\sigma\lambda)t} \mathrm{e}^{\sigma B_t - \frac{1}{2}\sigma^2 t}. \tag{1}$$

Here S_0 is the initial price, $\{B_t\}_{t\geq 0}$ is a standard $\mathbb{P}$-Brownian motion, $r > 0$ is the interest rate, $\sigma > 0$ is the volatility, and $\lambda > 0$ is the risk aversion.

The term $\sigma\lambda$ is called the "risk premium" or "excess rate of return".

To complete the specification of the model we need a pricing kernel $\{\pi_t\}_{t>0}$, which in the standard GBM model takes the form

$$\pi_t = \mathrm{e}^{-rt} \mathrm{e}^{-\lambda B_t - \frac{1}{2}\lambda^2 t}. \tag{2}$$

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The pricing kernel in an arbitrage-free model has the property that its product with the price of any asset gives a martingale under the physical measure $\mathbb{P}$.

In the present situation we have

$$\pi_t S_t = S_0 \, \mathrm{e}^{(\sigma-\lambda)B_t - \frac{1}{2}(\sigma-\lambda)^2 t}, \tag{3}$$

which indeed is a martingale.

For pricing, if H_T represents a random cash flow determined by the trajectory of $\{S_t\}_{0\leq t\leq T}$, then the random value H_t of the asset at any time $t<T$ is

$$H_t = \frac{1}{\pi_t} \mathbb{E}_t[\pi_T H_T] \, . \tag{4}$$

For example, if $H_T = \max(S_T - K, \, 0)$, then a calculation shows that H_t is given by the familiar Black-Scholes formula.

The pricing kernel methodology gives this result rather directly, without the involvement of hedging arguments, replication portfolios, market completeness, risk neutrality, change of measure, and so on.

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3. Random risk aversion and market heterogeneity

Now suppose that the risk aversion factor (excess rate of return per unit of risk) is not directly observable, and that there is uncertainty in the market as to its value.

Let us write X for the unknown (random) value of the risk aversion factor.

Then in our model for the typical asset price we have

$$S_t = S_0 \, \mathrm{e}^{(r+\sigma X)t} \mathrm{e}^{\sigma B_t - \frac{1}{2}\sigma^2 t}. \tag{5}$$

Thus if we introduce a so-called "information process" $\{\xi_t\}_{t\geq 0}$ defined by

$$\xi_t = B_t + Xt, \tag{6}$$

we can write the price in the form

$$S_t = S_0 \, \mathrm{e}^{rt} \mathrm{e}^{\sigma \xi_t - \frac{1}{2}\sigma^2 t}. \tag{7}$$

Note that the filtration generated by the asset price is the same as the filtration generated by the information process.

It is rather natural to let the latter be the market filtration $\{\mathcal{F}_t\}$.

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Then the "true" value of the market risk aversion factor X remains hidden, and at best can only be estimated by observations of the asset price.

The conditional distribution of X given the relevant market information up to time t is given by the measure

$$p_t(\mathrm{d}x) = \frac{\exp\left[x\xi_t - \frac{1}{2}x^2 t\right] p(\mathrm{d}x)}{\int_0^\infty \exp\left[z\xi_t - \frac{1}{2}z^2 t\right] p(\mathrm{d}z)}, \tag{8}$$

where $p(\mathrm{d}x)$ is the unconditional distribution of X.

It follows that the conditional mean is

$$\begin{aligned} \mathbb{E}\left[X \mid \mathcal{F}_t\right] &= \int_0^\infty x\, p_t(\mathrm{d}x) \\ &= \frac{\int_0^\infty x \exp\left[x\xi_t - \frac{1}{2}x^2 t\right] p(\mathrm{d}x)}{\int_0^\infty \exp\left[z\xi_t - \frac{1}{2}z^2 t\right] p(\mathrm{d}z)} . \end{aligned} \tag{9}$$

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The statement “X is unknown” can be interpreted in several ways.

• One is that there is a “secret” value of X which none of the market participants know.

• Another is that there is variability of opinion in the market about the rate of return that ought to be expected for a given level of risk, and that the distribution of X represents this spread of opinion.

We know from experience that even a single individual can, depending on mood and circumstance, exhibit significantly variable attitudes towards risk.

It seems therefore both necessary and reasonable to suppose that an equilibrium can be established in a market where investors have widely differing attitudes towards risk, and that market prices are obtained by averaging in some sense over all these different attitudes.

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4. Modelling the pricing kernel

With these thoughts in mind, we need to consider how to model the pricing kernel in a situation where heterogeneous attitudes towards risk prevail.

One might be inclined simply to replace the parameter λ with the random variable X:

$$\pi_t \overset{?}{=} \mathrm{e}^{-rt}\mathrm{e}^{-XB_t-\frac{1}{2}X^2t} \tag{10}$$

Unfortunately, this will not quite work, since once we introduce ξ_t we obtain

$$\pi_t \overset{?}{=} \mathrm{e}^{-rt}\mathrm{e}^{-X\xi_t+\frac{1}{2}X^2t}, \tag{11}$$

which is clearly not $\mathcal{F}_t$-measurable.

However, if we take its conditional expectation, this gives a better candidate:

$$\begin{aligned}\pi_t &= \mathbb{E}_t\left[\exp\left(-rt - X\xi_t + \tfrac{1}{2}X^2t\right)\right] \\ &= \int_0^\infty \exp\left(-rt - x\xi_t + \tfrac{1}{2}x^2t\right)\, p_t(\mathrm{d}x), \end{aligned} \tag{12}$$

which has the virtue of being $\mathcal{F}_t$-measurable.

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After the insertion of expression (8) for $p_t(\mathrm{d}x)$ in the above, we obtain

$$\pi_t = \frac{1}{\int_0^\infty \exp\left(rt + x\xi_t - \frac{1}{2}x^2 t\right) p(\mathrm{d}x)}. \tag{13}$$

This formula is perhaps most easily understood as follows.

The process $\{n_t\}$ defined in terms of the pricing kernel by $n_t = 1/\pi_t$ is the so-called "natural numeraire" or "growth optimal portfolio".

In the present context we have:

$$n_t = \int_0^\infty \exp\left(rt + x\xi_t - \tfrac{1}{2}x^2 t\right) p(\mathrm{d}x). \tag{14}$$

Therefore, in the case of an unknown risk aversion factor we form a weighted portfolio of the numeraire assets obtained for various specific values of the risk aversion, and then invert this to obtain the pricing kernel $\pi_t = 1/n_t$.

Thus: *The pricing kernel associated with random risk aversion is given by the harmonic mean of the pricing kernels arising for various specific values of the risk aversion.*

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5. Information-based estimation of market risk aversion

Is it possible to formulate the price dynamics in such a way that representations for $\{S_t\}$ and $\{\pi_t\}$ are expressible in the language of stochastic differential equations, without reference to the "hidden" risk aversion variable X?

To see that the answer is affirmative, we proceed as follows.

As in the previous discussion, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and introduce a Brownian motion $\{B_t\}$ and an independent random variable X.

We introduce the information process $\{\xi_t\}$ defined by $\xi_t = B_t + Xt$.

The dynamical equation satisfied by the asset price then takes the form

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}\xi_t. \tag{15}$$

Our goal is to write the dynamics in a way that brings out more explicitly the fact that the price movements are being driven by Brownian motion.

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To reach this goal we define a process $\{W_t\}$ by

$$W_t = \xi_t - \int_0^t \mathbb{E}[X \,|\, \mathcal{F}_s] \mathrm{d}s. \tag{16}$$

One can show that $\{W_t\}$ is an $\{\mathcal{F}_t\}$-Brownian motion.

Next we define a process $\{\lambda_t\}$ by setting

$$\lambda_t = \mathbb{E}[X \,|\, \mathcal{F}_t]. \tag{17}$$

By virtue of (9), (16), and (17), one sees that the information process $\{\xi_t\}$ satisfies a stochastic differential equation of the form

$$\mathrm{d}\xi_t = \lambda_t \mathrm{d}t + \mathrm{d}W_t, \tag{18}$$

where

$$\lambda_t = \frac{\int_0^\infty x \exp\left(x\xi_t - \frac{1}{2}x^2 t\right) p(\mathrm{d}x)}{\int_0^\infty \exp\left(x\xi_t - \frac{1}{2}x^2 t\right) p(\mathrm{d}x)}. \tag{19}$$

It follows that the dynamical equation for the price is given by

$$\mathrm{d}S_t = (r + \lambda_t \,\sigma) S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t. \tag{20}$$

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An analogous calculation shows that

$$d\pi_t = -r\,\pi_t\,dt - \lambda_t\pi_t\,dW_t. \tag{21}$$

We therefore find that *the observable drift of an asset is determined not by the "actual" risk premium, but rather by the market best estimate for the risk premium*.

In particular, given the price S_t of the asset at time t, the best estimate of the market price of risk is given by:

$$\lambda_t = \frac{\int_0^\infty x(S_t/S_0)^{x/\sigma}\exp\left[-\frac{1}{2}x^2t + (\frac{1}{2}\sigma - r/\sigma)xt\right]p(dx)}{\int_0^\infty (S_t/S_0)^{x/\sigma}\exp\left[-\frac{1}{2}x^2t + (\frac{1}{2}\sigma - r/\sigma)xt\right]p(dx)}. \tag{22}$$

This formula shows in explicit terms how the investor is able to update the *a priori* estimate for the market price of risk given the current price level of the asset.

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6. Geometric Lévy models

Remarkably, the considerations that we have presented in connection with GBM models generalise very naturally to geometric Lévy models (GLMs).

To set the notation we begin with a few definitions.

A Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a process $\{X_t\}$ such that $X_0 = 0$, $X_t - X_s$ is independent of $\mathcal{F}_s$ for $t \geq s$ (independent increments), and

$$\mathbb{P}(X_t - X_s \leq y) = \mathbb{P}(X_{t+h} - X_{s+h} \leq y) \tag{23}$$

(stationary increments), where $\{\mathcal{F}_t\}$ denotes the augmented filtration generated by $\{X_t\}$.

In order for $\{X_t\}$ to give rise to a geometric Lévy model, we require that it should admit exponential moments. In other words,

$$\mathbb{E}[\mathrm{e}^{\alpha X_t}] < \infty \tag{24}$$

for all $t \geq 0$, for some connected real interval $\alpha \in A$ containing the origin.

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It follows from the stationary and independent increments property that there exists a function $\psi(\alpha)$, the Lévy exponent, such that

$$\mathbb{E}[\mathrm{e}^{\alpha X_t}] = \mathrm{e}^{t\psi(\alpha)} \tag{25}$$

for $\alpha \in \{w \in \mathbb{C} : \mathrm{Re}(w) \in A\}$, and one can check that the process defined by

$$M_t = \mathrm{e}^{\alpha X_t - t\psi(\alpha)} \tag{26}$$

is an $\{\mathcal{F}_t\}$-martingale.

With these definitions at hand, we are able to construct the pricing kernel $\{\pi_t\}_{t\geq 0}$ in the case of a GLM:

$$\pi_t = \mathrm{e}^{-rt}\mathrm{e}^{-\lambda X_t - t\psi(-\lambda)}. \tag{27}$$

We require that the product of the pricing kernel and the asset price should be a martingale:

$$\pi_t S_t = S_0 \mathrm{e}^{\beta X_t - t\psi(\beta)} \tag{28}$$

for some $\beta \in A$, and we deduce as a consequence that

$$S_t = S_0\, \mathrm{e}^{rt}\, \mathrm{e}^{\sigma X_t + t\psi(-\lambda) - t\psi(\sigma-\lambda)}, \tag{29}$$

where $\sigma = \beta + \lambda$ (we assume that $\sigma > 0$).

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It follows that the price can be expressed by the formula

$$S_t = S_0 \, \mathrm{e}^{rt} \, \mathrm{e}^{R(\lambda,\sigma)t} \, \mathrm{e}^{\sigma X_t - t\psi(\sigma)}, \tag{30}$$

where

$$R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda). \tag{31}$$

It is a remarkable fact that the excess rate of return function $R(\lambda, \sigma)$ thus arising is positive and is increasing with respect to both of its arguments.

7. On the aggregation of jump-risk aversion

With the help of the theory of nonlinear filtering associated with Lévy processes developed in Brody, Hughston & Yang (2013), we are able to extend the analysis presented in the first part of the talk to identify how investors are rewarded for taking on jump risk in a heterogeneous market.

First we observe that the price process (30) can be expressed in the form

$$S_t = S_0 \mathrm{e}^{rt} \mathrm{e}^{\sigma X_t - t\phi(\sigma)}, \tag{32}$$

where

$$\phi(\sigma) = \psi(\sigma - \lambda) - \psi(-\lambda). \tag{33}$$

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Thus we regard the "fiducial" exponent $\phi(\alpha)$ as given, and define the exponent $\psi(\alpha)$ of the Lévy process $\{X_t\}$ in terms of $\phi(\alpha)$ and λ by setting

$$\psi(\alpha) = \phi(\alpha + \lambda) - \phi(\lambda). \tag{34}$$

Then in the case of a heterogeneous market we let the asset price process be given by

$$S_t = S_0 \mathrm{e}^{rt} \mathrm{e}^{\sigma \xi_t - t\phi(\sigma)}, \tag{35}$$

where $\{\xi_t\}$ is a so-called Lévy information process with information X and fiducial exponent $\phi(\alpha)$.

By a Lévy information process we mean a process such that conditional on X, $\{\xi_t\}$ is a Lévy process with exponent

$$\psi_X(\alpha) = \phi(\alpha + X) - \phi(X). \tag{36}$$

for $\alpha \in \{w \in \mathbb{C} : \mathrm{Re}(w) = 0\}$.

For the pricing kernel we obtain

$$\pi_t = \frac{1}{\int_0^\infty \exp\left(rt + x\xi_t - \phi(x)t\right) p(\mathrm{d}x)}. \tag{37}$$

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The natural numeraire in the case of a Lévy model with random risk aversion is thus given by

$$n_t = \int_0^\infty \exp\left(rt + x\xi_t - \phi(x)t\right) p(\mathrm{d}x). \tag{38}$$

For the jump-risk aversion factor we have

$$\lambda_t = \frac{\int_0^\infty x \exp\left(x\xi_t - \phi(x)t\right) p(\mathrm{d}x)}{\int_0^\infty \exp\left(x\xi_t - \phi(x)t\right) p(\mathrm{d}x)}. \tag{39}$$

Substitution of this in

$$R(\lambda_t, \sigma) = \psi(\sigma) + \psi(-\lambda_t) - \psi(\sigma - \lambda_t) \tag{40}$$

gives us the excess rate of return arising from the accommodation of jump risk in a geometric Lévy model.

In particular, in a geometric Lévy model, the best estimate at time t for the risk aversion factor λ_t, given the price S_t at that time, is given by

$$\lambda_t = \frac{\int_0^\infty x(S_t/S_0)^{x/\sigma} \exp\left[-\phi(x)t + (\phi(\sigma) - r)xt/\sigma\right] p(\mathrm{d}x)}{\int_0^\infty (S_t/S_0)^{x/\sigma} \exp\left[-\phi(x)t + (\phi(\sigma) - r)xt/\sigma\right] p(\mathrm{d}x)}. \tag{41}$$

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8. Conclusions

• We are able to construct arbitrage-free models for the prices of assets in heterogeneous markets where investors have diverse attitudes towards risk.

• Jump risk can be accommodated consistently in such models by use of the notion of a so-called Lévy information process, leading to explicit expressions for the market excess rate of return with jump risk, and allowing for heterogeneous attitudes towards jump risk.

• The pricing kernel in a heterogeneous market is obtained by taking the harmonic mean of the pricing kernels associated with the individual investors in such a market.

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