Optimal Investment with Illiquid Assets

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Sascha Desmettre and Frank Thomas Seifried

Fraunhofer ITWM

Agenda

1. Financial Market Model
2. Optimal Investment Problem with Illiquid Assets
   - Explain the differences to a classical investment problem
3. Solution by Duality Methods
4. Application: Investment with fixed Deposits

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Financial Market Model

- Money market account $B = \{B_t\}$ with

$$dB_t = B_t r_t dt \tag{1}$$

with an $\mathfrak{F}$-progressively measurable interest rate process $r = \{r_t\}$

- Risky asset $P = \{P_t\}$, a stock or stock index with

$$dP_t = P_t[(r_t + \eta_t)dt + \sigma_t dW_t] \tag{2}$$

with $\mathfrak{F}$-progressively measurable excess return and volatility processes $\eta = \{\eta_t\}$ and $\sigma = \{\sigma_t\}$, W Wiener process.

- The financial market is then $\mathfrak{F}_T$-complete.
- We denote the corresponding state-price deflator by $Z = \{Z_t\}$,

$$Z_t \triangleq \exp\left\{-\int_0^t \theta_s dW_s - \int_0^t (r_s + \tfrac{1}{2}\theta_s^2) ds\right\} \text{ for } t \in [0, T] \tag{3}$$

where $\theta = \{\theta_t\}$, $\theta_t \triangleq \frac{\eta_t}{\sigma_t}$ is the market price of risk process.

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Illiquid Asset

Additionally there is an illiquid asset which offers

- a random payoff F_T at time T and
- a continuous coupon payment at a rate $\delta = \{\delta_t\}$.

We suppose that F_T admits the representation

$$F_T(\omega) = F(M(\omega), N(\omega)) \text{ with } \begin{cases} M \ \mathfrak{F}_T\text{-measurable}, \\ N \text{ independent of } \mathfrak{F}_T. \end{cases}$$

→ M and N represent, respectively, the marketed (observable) and non-marketed (unobservable) components of the risk inherent in F_T.

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- We assume that δ is $\mathfrak{F}$-progressively measurable.
- The illiquid asset is traded at a price of F_0 at time 0.
- The investor obtains a payment ψF_T at time T and receives a continuous coupon $\psi\delta_t dt$ between time t and time $t + dt$, if she decides to buy ψ illiquid assets at time 0.
- The money market account and the stock are liquidly traded.
- But it is not possible to buy or sell the illiquid asset after time 0.
  - Agreed contractually as in the case of a fixed deposit.
  - Inherent illiquidity and intermediate usage of the illiquid investment (for instance, housing).
- Short positions in the illiquid asset are prohibited.

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Wealth Dynamics

- x_0 : initial wealth
- ψF_0 : time-0 value of the illiquid investment
- Investor can borrow - a the interest rate r - against a fraction $\alpha \in [0, 1)$ of the face value of her illquid wealth and against the total value of the outstanding coupon payments $\psi \mathbb{E}_t[\int_t^T Z_s \delta_s ds]$ at time t.
- In general, $\alpha < 1$ and the divident stream is sold completely.
- $\pi = \{\pi_t\}$: fraction of liquid wealth invested into the stock at time t and c_t: investor's consumption rate. Investor's liquid wealth $\{X_t^{\psi,\pi,c}\}$ is

$$dX_t^{\psi,\pi,c} = X_t^{\psi,\pi,c}[(r_t + \pi_t \eta_t)dt + \pi_t \sigma_t dW_t] - c_t dt,$$

$$X_0^{\psi,\pi,c} = x_0 - \psi F_0 + \alpha \psi F_0 + \psi \mathbb{E}\left[\int_0^T Z_t \delta_t dt\right] = x_0 - \psi[(1 - \alpha)F_0 - \bar{\delta}],$$

  and $\bar{\delta} \triangleq \mathbb{E}[\int_0^T Z_t \delta_t dt]$.
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Optimal Portfolio Problem

$\mathcal{A}(x_0)$: Class of admissible strategies (ψ, π, c) for initial wealth $x_0 > 0$. In particular, we must have $\psi \leq \psi_{max} \triangleq \frac{x_0}{(1-\alpha)F_0 - \bar{\delta}}$ $(\Rightarrow X_0^{\psi,\pi,c} \geq 0)$.

The investor's total terminal wealth is then given by

$$X_T^{\psi,\pi,c} + \psi F_T - \alpha\psi F_0 e^{\int_0^T r_s ds} = X_T^{\psi,\pi,c} + \psi(F_T - \alpha F_0 e^{\int_0^T r_s ds}).$$

Optimal Portfolio Problem with Illiquid Assets

$$\max_{(\psi,\pi,c)\in\mathcal{A}(x_0)} \mathbb{E}\left[\int_0^T u_t(c_t)dt + u\left(X_T^{\psi,\pi,c} + \psi(F_T - \alpha F_0 e^{\int_0^T r_s ds})\right)\right] \quad (P)$$

Note that by choosing $u_t = 0$ for all $t \in [0, T]$ we obtain the corresponding optimal terminal wealth problem.

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Optimal Investment with a Given Fixed Deposit

In the following, we fix the investment $\psi \in [0, \psi_{\max}]$ into illiquid assets. The portfolio problem (P) rewrites as

$$\max_{(\pi,c)} \mathbb{E}\Big[\int_0^T u_t(c_t)dt + \bar{u}(\bar{X}_T^{\pi,c})\Big] \text{ with } (\psi, \pi, c) \in \mathcal{A}(x_0). \qquad (P_\psi)$$

Here the random utility function $\bar{u}_\omega : (0, \infty) \to \mathbb{R}$ is given by

$$\bar{u}_\omega(\bar{x}) \triangleq u\big(\bar{x} + \psi(F_T(\omega) - \alpha F_0 e^{\int_0^T r_s ds})\big) \text{ for } \bar{x} \in (0, \infty),$$

and the liquid wealth $\bar{X}^{\pi,c} = \{\bar{X}_t^{\pi,c}\}$ satisfies

$$\begin{aligned} d\bar{X}_t^{\pi,c} &= \bar{X}_t^{\pi,c}[(r_t + \pi_t\eta_t)dt + \pi_t\sigma_t dW_t] - c_t dt, \\ \bar{X}_0^{\pi,c} &= x_0 - \psi[(1-\alpha)F_0 - \bar{\delta}],\ \bar{\delta} \triangleq \mathbb{E}\Big[\int_0^T Z_t\delta_t dt\Big]. \end{aligned} \qquad (4)$$

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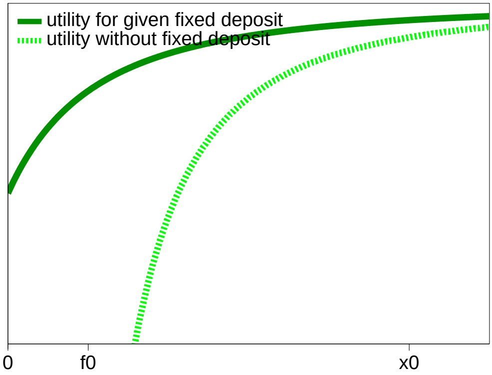

Figure: (Power) utility function $\bar{u}$ for given investment into illiquid assets.

- In general, the mapping $\omega \mapsto \bar{u}_\omega$ need not be $\mathfrak{F}_T$-measurable (recall $F_T(\omega) = F(M(\omega), N(\omega))$).
- Hence conditioning on $\mathfrak{F}_T$ we rewrite the criterion of problem (P_ψ) as

$$\mathbb{E}\Big[\int_0^T u_t(c_t)dt + \bar{u}(\bar{X}_T^\pi)\Big] = \mathbb{E}\Big[\int_0^T u_t(c_t)dt + \hat{u}(\bar{X}_T^\pi)\Big]$$

with an $\mathfrak{F}_T$-measurable random utility function $\hat{u}_\omega : (0, \infty) \to \mathbb{R}$ with

$$\begin{aligned}\hat{u}_\omega(x) &\triangleq \mathbb{E}[\bar{u}(x) \,|\, \mathfrak{F}_T]_\omega = \mathbb{E}\big[u\big(x + \psi(F(M,N) - \alpha F_0 e^{\int_0^T r_s ds})\big) \,|\, \mathfrak{F}_T\big]_\omega \\ &= \mathbb{E}\big[u\big(x + \psi(F(m,N) - \alpha F_0 e^{\int_0^T \varrho_s ds})\big)\big] \restriction_{m=M(\omega),\ \varrho_t = r_t(\omega),\, t\in[0,T]},\end{aligned}$$

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Theorem (Optimal Investment with Illiquid Assets I)

The portfolio problem with illiquid assets (P) has a solution $(\psi^\star, \pi^\star, c^\star)$. With $\gamma^\star = \gamma^\star(\psi^\star)$ the optimal terminal wealth and the optimal consumption rate are given by

$$\bar{X}_T^\star = \hat{\imath}(\gamma^\star Z_T), \quad c_t^\star = \iota_t(\gamma^\star Z_t),\ t \in [0, T].$$

The optimal investment into the illiquid asset is given by

$$\psi^\star = \operatorname*{arg\,max}_{\psi \in [0, \psi_{\max}]} v(\psi)$$

where v is given by

$$v(\psi) \triangleq \mathbb{E}\Big[\int_0^T u_t\big(\iota_t(\gamma^\star(\psi) Z_t)\big) dt + u\Big(\hat{\imath}(\gamma^\star(\psi) Z_T)\Big)\Big].$$

Remarks

- v depends on ψ only via the Lagrange multiplier $\gamma^\star(\psi)$ which was determinded via the Budget constraint!
- The solvency requirement $\bar{X}_t^\pi \geq 0$, $t \in [0, T]$, a.s. forces us to to set the inverse marginal utility $\hat{\iota}_\omega(\lambda) = 0$ if $\lambda > \hat{u}'_\omega(0)$ since the marginal utility funtion $\hat{u}'$ is defined on the range $(0, \infty)$ with image $(0, \hat{u}'(0))$ and we have to ensure that the optimal final wealth $\bar{X}_T^\star = \hat{\iota}(\gamma^\star Z_T)$ stays non-negative.
- Problem: how to compute $\bar{X}_T^\star = \hat{\iota}(\gamma^\star Z_T)$, i.e. $\hat{\iota}_\omega(\lambda)$
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- In general, $\hat{u}$ and $\hat{\iota}$ must be computed numerically.
- In the special case $F_T = F(M)$, i.e. F_T is $\mathfrak{F}_T$-measurable, we have $\hat{u} = \bar{u}$ and the quantities of interest can be computed explicitly.
- Since we have that

$$\bar{u}'(x) = u'(x + \bar{F}) \text{ with } \bar{F} \triangleq \psi(F_T - \alpha F_0 e^{\int_0^T r_s ds}),$$

- it follows that $\bar{\iota}$ associated to $\bar{u}$ is given as

$$\bar{\iota}: \ (0,\infty) \to [0,\infty), \quad \bar{\iota}(\lambda) = (\iota(\lambda) - \bar{F})^+$$

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- In general, $\hat{u}$ and $\hat{\iota}$ must be computed numerically.
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- Since we have that

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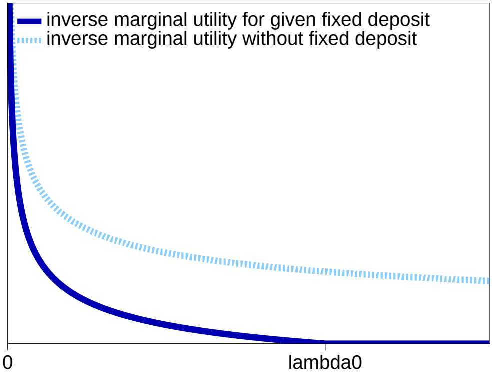

Figure: Inverse marginal (power) utility $\bar{\iota}$ corresponding to $\bar{u}$.

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Theorem (Optimal Investment with Illiquid Assets II)

Suppose integrability conditions as before. Then the optimal portfolio problem with illiquid assets (P) has a solution $(\psi^\star, \pi^\star, c^\star)$. The optimal investment into the illiquid asset is given by

$$\psi^\star = \underset{\psi \in [0, \psi_{\max}]}{\arg\max}\ v(\psi) \text{ with}$$

$$v(\psi) \triangleq \mathbb{E}\Big[\int_0^T u_t\big(\iota_t(\gamma^\star(\psi) Z_t)\big)dt + u\Big(\iota(\gamma^\star(\psi) Z_T) \vee \bar{F}\Big)\Big].$$

The optimal terminal wealth and the optimal consumption rate are

$$\bar{X}^\star_\psi \triangleq \bar{\iota}(\gamma^\star(\psi) Z_T) \text{ and } c^\star_t \triangleq \iota_t(\gamma^\star(\psi) Z_t),\ t \in [0, T]$$

where the Lagrange multiplier $\gamma^\star(\psi) \in (0, \infty)$ is determined by the BC

$$\mathbb{E}\big[\int_0^T Z_t \iota_t(\gamma^\star(\psi) Z_t) dt + Z_T \bar{\iota}(\gamma^\star(\psi) Z_T)\big] = x_0 - \psi[(1-\alpha)F_0 - \bar{\delta}].$$

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Fraunhofer ITWM

Application: Optimal Investment with Fixed Deposits

- Illiquid asset: fixed deposit investment with initial price $F_0 = 1$ with terminal payoff $F_T = e^{\bar{r}T}$, $\bar{r} > r$, $\alpha = 0$ (no borrowing)
- $u_t = 0$ for all $t \in [0, T]$, i.e. terminal wealth problem only
- ψ is optimal amount of wealth invested into fixed deposits.
- In view of the preceeding discussion, the portfolio problem (P) reduces to the maximization of a continuous function on a compact interval, i.e. to

$$\text{maximize } v(\psi) \text{ over } \psi \in [0, \psi_{\max}], \tag{5}$$

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- Problem: Since $\mathbb{P}(\iota(\gamma^\star(\psi)Z_T) = \psi e^{\bar{r}T}) > 0$, it is not clear how to differentiate the function $v(\psi)$ w.r.t. ψ to obtain a FOC!
- $\rightarrow$ Compute the optimal fixed deposit investment $\psi^\star$ numerically.
- The qualitative shapes of $v(\psi)$ are robust so that the maximization can be performed efficiently with numerical methods.
  - The shape seems to be even concave for suitable values of r and $\bar{r}$.
  - $\rightarrow$ Show concavity of $v(\psi)$ in dependence of the parameters!
- Unless stated otherwise, we use the parameter specifications

$$\rho = 3,\ T = 5,\ r = 4\%,\ \bar{r} = 5\%,\ \eta = 8\%,\ \sigma = 20\%.$$

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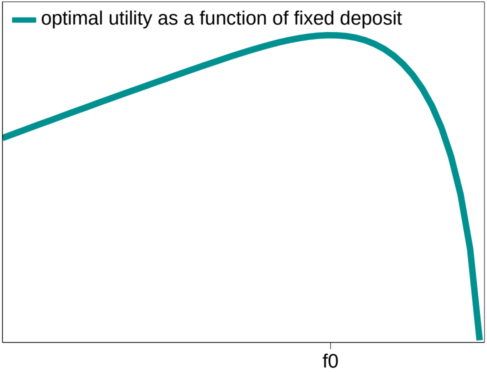

Figure: Optimal terminal utility $v(\psi)$ as a function of fixed deposit investment ψ.

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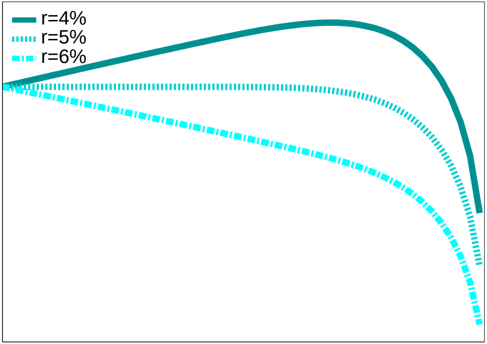

Figure: Optimal terminal utility v(ψ) as a function of fixed deposit investment ψ for different levels of the riskless interest rate r.

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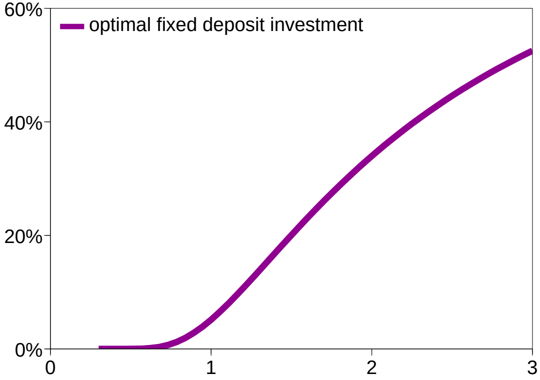

Figure: Optimal fixed deposit investment ψ as a function of risk aversion ρ.

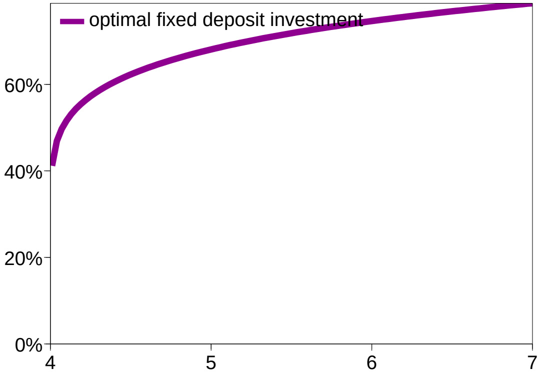

Figure: Optimal fixed deposit investment ψ as a function of the fixed interest rate $\bar{\mathrm{r}}$.

Thank You for Your Attention!

Questions/Remarks?