Testing for non-correlation between price and volatility jumps and ramifications

Claudia Klüppelberg
Technische Universität München
cklu@ma.tum.de
www-m4.ma.tum.de

Joint work with
Jean Jacod, Gernot Müller, Carsten Chong and Anita Behme

June 2013

Outline

Jacod, J. and Protter, P. (2011)
Discretization of Processes.
Springer, Berlin.

Stochastic volatility models for financial data

- **Presence of jumps in the price and the volatility process**

  Merton (1976)

  ⋮

  Lee and Mykland (2008)
  Aït-Sahalia and Jacod (2009)

- **Do price and volatility jump together?**

  Jacod and Todorov (2010)

- **Are common jumps in price and volatility correlated?**

  Jacod, Klüppelberg and Müller (2012a,b)

Prominent continuous-time models

Consider any model for (log) price X and squared volatility $V = \sigma^2$.
All prominent models satisfy a relationship between their jump sizes:

$$f(X_{t-}, X_t) = \gamma\, g(V_{t-}, V_t)$$

for known functions f, g and one fixed parameter $\gamma \in \mathbb{R}$.

Prominent continuous-time models

- Linear models: CARMA (the OU process is a CAR(1) model):

$$f_{\text{CARMA}}(x,y) = g_{\text{CARMA}}(x,y) = y - x \qquad \Delta X_t = \gamma \Delta V_t$$

(in these models, joint jumps of X and V are always positive).

- COGARCH models:

$$f_{\text{COG}}(x,y) = (y-x)^2, \qquad g_{\text{COG}}(x,y) = y - x \qquad (\Delta X_t)^2 = \gamma \Delta V_t$$

- ECOGARCH models:

$$f_{\text{ECOG}}(x,y) = y - x, \qquad g_{\text{ECOG}}(x,y) = \sqrt{x}\,(\log y - \log x)$$

Such relationships seem too strong, but we can ask:
are jump sizes in price and squared volatility correlated?

Semimartingale framework

X (log-)price process, observed on a discrete time grid with grid size $\Delta_n \to 0$

$V = \sigma^2$ squared volatility process (càdlàg), unobserved

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s \, dW_s + \int_0^t \int_E \delta(s,z)\,(\mu - \nu)(ds, dz)$$

$$V_t = V_0 + \int_0^t \widetilde{b}_s ds + \int_0^t \widetilde{\sigma}_s dW_s + \int_0^t \widehat{\sigma}_s dW'_s + \int_0^t \int_E \widetilde{\delta}(s,z)\,(\mu - \nu)(ds, dz)$$

Assumptions

- jumps of X have finite activity (otherwise rates change)
- all moments of V are bounded in t, those of X are finite
- the processes $b, \widetilde{b}, \widetilde{\sigma}, \ldots$ are bounded
- some ergodicity property for jumps
- …

The Goal

- **Goal:** Tests based on data within $[0, T]$

  for non-correlation of $f(\Delta X)$ and $g(\Delta V)$

  using observations from a discrete time grid with $\Delta_n \to 0$

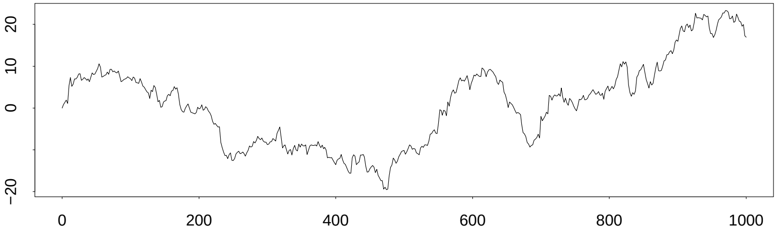
- Define for joint jump times S_m

$$U(f,g)_{S_m} \ := \ \mathbb{E}[f(\Delta X_{S_m})\, g(\Delta V_{S_m}) \mid \mathcal{F}_{S_m-}]$$

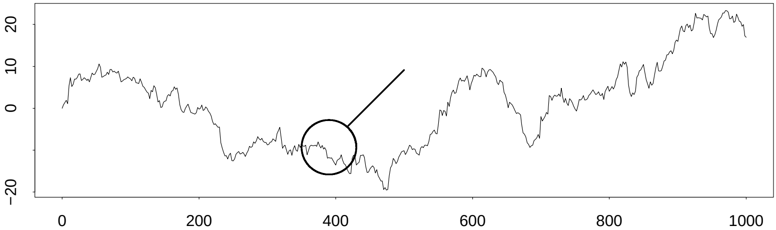
- **Null hypothesis:** jump sizes are uncorrelated, i.e.

$$\mathrm{H}_0 : \ U(f,g)_{S_m} = U(f,1)_{S_m}\, U(1,g)_{S_m} \text{ for all } m$$

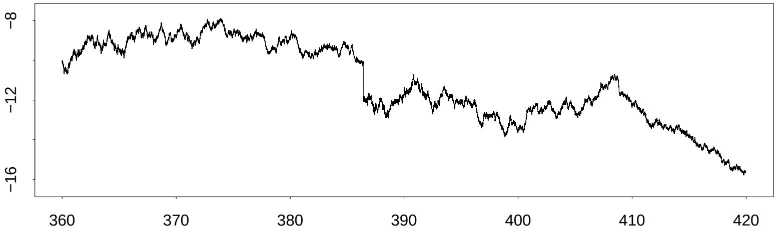
Log-price process



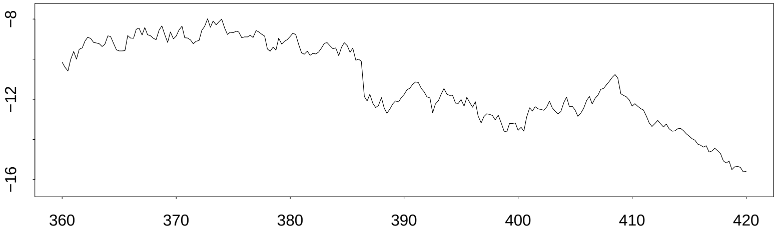
Log-price process



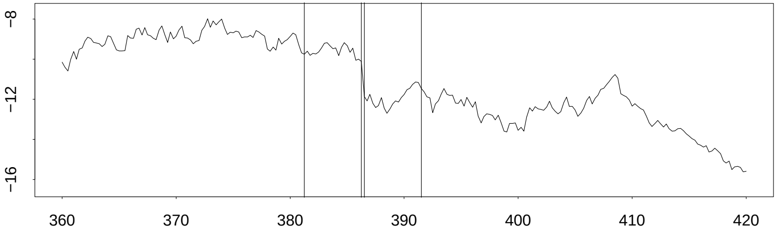
Log-price process: continuously observed



Log-price process: $\Delta_n = 0.25$



Local volatility estimation



Local volatility estimation

Local volatility estimates [Mancini (2001)]:

$$\widehat{V}_i^n = \frac{1}{k_n \Delta_n} \sum_{j=1}^{k_n} |\Delta_{i+j}^n X|^2 \, 1_{\{|\Delta_{i+j}^n X| \leq u_n\}}$$

u_n: threshold used to identify the jumps of X

k_n: number of observations used for volatility estimation

The test statistic (1)

Recall:

$$U(f,g)_{S_m} \;:=\; \mathbb{E}[f(\Delta X_{S_m})\, g(\Delta V_{S_m}) \mid \mathcal{F}_{S_m-}]$$

Define

$$\widehat{U}(f,g)_t^n \;=\; \sum_{i=k_n+1}^{[t/\Delta_n]-k_n} f(\Delta_i^n X)\, g(\widehat{V}_i^n - \widehat{V}_{i-k_n-1}^n)\, 1_{\{|\Delta_i^n X|>u_n\}}$$

The test statistic (2)

Set $\widehat{\Gamma}_n = \widehat{U}(1,1)^n_{T_n}\,\widehat{U}(f,g)^n_{T_n} - \widehat{U}(f,1)^n_{T_n}\,\widehat{U}(1,g)^n_{T_n}$.

As test statistic take

$$\widehat{\Psi}_n = \frac{\widehat{\Gamma}_n}{\sqrt{\widehat{\Phi}_n / \widehat{U}(1,1)^n_{T_n}}}$$

where

$$\begin{aligned}\Phi_n = {} & (U(1,1)^n_{T_n})^3 U(f^2,g^2)^n_{T_n} \\ & + U(1,1)^n_{T_n}\,(U(f,1)^n_{T_n})^2\,U(1,g^2)^n_{T_n} \\ & + U(1,1)^n_{T_n}\,(U(1,g)^n_{T_n})^2\,U(f^2,1)^n_{T_n} \\ & + 4U(1,1)^n_{T_n}\,U(1,g)^n_{T_n}\,U(f,1)^n_{T_n}\,U(f,g)^n_{T_n} \\ & - 2U(1,1)^n_{T_n}\,U(f,1)_t\,U(f,g^2)^n_{T_n} \\ & - 2U(1,1)^n_{T_n}\,U(1,g)^n_{T_n}\,U(f^2,g)^n_{T_n} \\ & - 3(U(f,1)^n_{T_n})^2\,(U(1,g)^n_{T_n})^2\end{aligned}$$

Theorem 1

Let

- $T_n \to \infty$ and $\Delta_n \to 0$ s.t. $T_n \Delta_n^{1/2-\eta} \to 0$ for some $\eta \in (0, \frac{1}{2})$,
- $u_n \to 0$ more slowly than $\Delta_n^{1/2}$, and
- $k_n \to \infty$ more slowly than $\Delta_n^{-1/2}$.

Under the assumptions for the stochastic volatility model and the test functions f and g, we have, as $n \to \infty$,

$$\text{under } \mathrm{H}_0 : \ \widehat{\Psi}_n \ \stackrel{d}{\to} \ \mathcal{N}(0,1)$$

$$\text{under } \mathrm{H}_1 : \ |\widehat{\Psi}_n| \ \stackrel{\mathbb{P}}{\longrightarrow} \ \infty.$$

Theorem 2

Under the assumptions of Theorem 1 the critical regions

$$C_n := \{|\psi_n| > z_\alpha\}$$

$(P(N(0,1) > z_\alpha) = \alpha)$
have the asymptotic size α for testing H_0 and are consistent for H_1.

Conclusions from an extended simulation study

- for a substantial number of jumps the test works very well
- the more jumps are considered, the bigger is the power of the test
- sensitivity on k_n is weak
- test works better for lower values of u_n

The data

- 1-minute data of the SPDR S&P 500 ETF (SPY) from 2005 to 2011 traded at NASDAQ
- use data between 9:30 am and 4:00 pm
  $\Rightarrow$ 390 observations per day
- days with periods of more than 60 consecutive seconds without trades deleted
- choice of parameters:

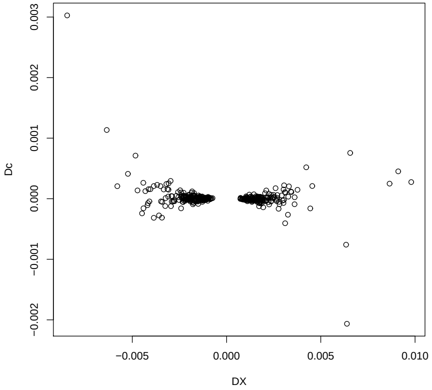
  length of volatility window:
  $k^* = 3.00$, leading to $k_n = 56$ [minutes]

  price jump detection:
  $u^* = 3.89$ (99.995% quantile of standard normal)

Selection of jumps

- the threshold u_n is locally adapted to the **current volatility level** calculated as a moving average over 20 days (10 before, 10 after)
- the threshold u_n is locally adapted to the **daily volatility smile** (taken from Mykland, Shephard and Sheppard (2012))
- **only isolated jumps:** for the test statistic we only use jumps, where within 56 minutes before and 56 minutes after no other jump(s) occurred
- **no overnight jumps for volatility estimation:** we account only for jumps between 10:26 am and 3:04 pm
- **thresholds for volatility jumps**: at least 10% upwards or 9% downwards from current volatility level
- **this way, 330 jumps are selected**

Price jumps ΔX versus volatility jumps $\Delta\sigma$



Results

	SPY data set $f(\Delta X)$ vs. $g(\Delta V)$		
	$f(x) = x$	$f(x) = \|x\|$	$f(x) = x^2$
$g(v) = v$	-1.4213	0.8859	1.1421
$g(v) = \|v\|$	-0.3561	**1.7766**	**2.0007**
$g(v) = v^2$	-0.6019	1.0949	1.2920

Are jump sizes in price and volatility correlated?

For the SPY data set:

- on a 10% level, the test rejects the null hypothesis of **no correlation** between price and volatility jump sizes, for 2 out of 9 choices of (f, g)

Superpositioned COGARCH model (supCOGARCH)

[Klüppelberg, Lindner, Maller (2004)]

Let L be a Lévy process with discrete quadratic variation $S = [L, L]^{\mathrm{d}}$.

The **COGARCH squared volatility** V^{φ} is the solution of the SDE

$$\mathrm{d}V_t^{\varphi} = (\beta - \eta V_t^{\varphi})\,\mathrm{d}t + V_{t-}^{\varphi}\varphi\,\mathrm{d}S_t \quad t \geq 0.$$

It admits the integral representation

$$V_t^{\varphi} = V_0^{\varphi} + \beta t - \eta \int_{(0,t]} V_s^{\varphi}\,\mathrm{d}s + \sum_{0<s\leq t} V_{s-}^{\varphi}\varphi\Delta S_s \quad t \geq 0.$$

The **integrated COGARCH price process** is then

$$X_t^{\varphi} = \int_0^t \sqrt{V_{s-}^{\varphi}}\,\mathrm{d}L_s, \quad t \geq 0.$$

Stationary solutions exist for all $\varphi \in \Phi = (0, \varphi_{\max})$ with $\varphi_{\max} < \infty$

supCOGARCH: [Behme, Chong and Klüppelberg (2013)]

Replace L by an independently scattered infinitely divisible random measure Λ such that $L_t := \Lambda((0,t] \times \Phi), t \geq 0$ and $\Lambda^S := [\Lambda, \Lambda]^{\mathrm{d}}$ is the jump part of the quadratic variation measure. Let Λ^S have characteristics $(0, 0, \mathrm{d}t\, \nu_S(\mathrm{d}y)\, \pi(\mathrm{d}\varphi))$, where π is a probability measure on $\Phi = (0, \varphi_{\max})$.

We define a **supCOGARCH squared volatility process** by

$$\bar{V}_t = \bar{V}_0 + \int_{(0,t]} (\beta - \eta \bar{V}_s)\, \mathrm{d}s + \int_{(0,t]} \int_\Phi \varphi V_{s-}^{\varphi}\, \Lambda^S(\mathrm{d}s, \mathrm{d}\varphi), \quad t \geq 0,$$

where $\bar{V}_0$ is independent of the restriction of Λ to $\mathbb{R}_+ \times \Phi$.

The stochastic integral: [Chong and Klüppelberg (2013)]

- We work in L_0 (space of all P-a.s. finite random variables X) with $\|X\|_0 = E[|X| \wedge 1]$, and
- we do not require independence of integrand and integrator;

Λ is an independently scattered infinitely divisible random measure and we have to combine this with an adaptedness concept;
cf. Bichteler and Jacod (1983);
i.e. Λ is a **filtration-based Lévy basis** on $\mathbb{R} \times \Phi$.
In particular, we want

$$\Lambda(A \times (s,t] \times \Phi) = \mathbf{1}_A(L_t - L_s) \quad s < t \text{ and } A \in \mathcal{F}_s.$$

Example: The two-factor supCOGARCH

Let $\pi = p_1\delta_{\varphi_1} + p_2\delta_{\varphi_2}$ with $p_1 + p_2 = 1$ and $\varphi_1, \varphi_2 \in \Phi = (0, \varphi_{\max})$.

The subordinator S drives two COGARCH processes V^{φ_1} and V^{φ_2}: when S jumps, a value is randomly chosen from $\{\varphi_1, \varphi_2\}$: φ takes the value φ_1 with prob. p_1 and the value φ_2 with prob. p_2.

The jump size of $\overline{V}$ is the jump size of the COGARCH with this φ. If $(T_i)_{i\in\mathbb{N}}$ denote the jump times of S, we have

$$\Delta\bar{V}_{T_i} = \Delta V^{\varphi_i}_{T_i} = \varphi_i V^{\varphi_i}_{T_i-}\Delta S_{T_i} \quad i \in \mathbb{N},$$

and $(\varphi_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence with distribution π, independent of S.

Example: $\pi = p_1\delta_{\varphi_1} + p_2\delta_{\varphi_2}$, Λ is CPRM

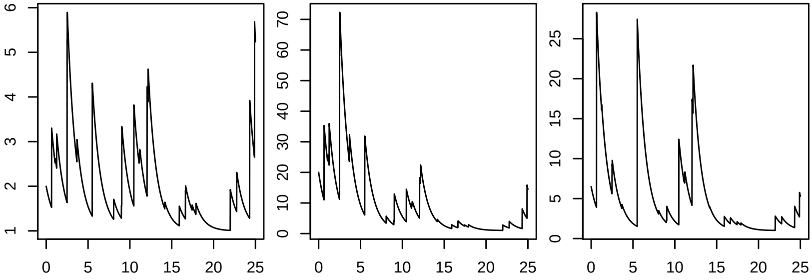


Figure: COGARCH V^{φ_1}, V^{φ_2} and supCOGARCH $\bar{V}$.
$\beta = 1, \eta = 1, \varphi_1 = 0.5, \pi_1 = 0.75, \varphi_2 = 0.95, \pi_2 = 0.25$,
Poisson rate $\lambda = 1$, jumps are $N(0, 1)$.

The supCOGARCH price process

We define the **integrated supCOGARCH price process** by

$$X_t := \int_{(0,t]} \sqrt{\bar{V}_{t-}}\, \mathrm{d}L_t \quad t \geq 0.$$

X has stationary increments and jumps at exactly the times as $\bar{V}$.

References

- Behme, A., Chong, C., and Klüppelberg, C. (2013)
  Superposition of COGARCH processes. Submitted.
- Chong, C. and Klüppelberg, C. (2013)
  Integrability conditions for space-time stochastic integrals: theory and applications. Submitted.
- Jacod, J., Klüppelberg, C. and Müller, G. (2012)
  Functional relationships between price and volatility jumps and its consequences for discretely observed data.
  J. Appl. Prob. **49**(4), 901-914.
- Jacod, J., Klüppelberg, C. and Müller, G. (2012)
  Testing for non-correlation between price and volatility jumps.
  Under revision.