Generalized Good-Deal Bounds and Robust Hedging under Model Uncertainty

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Motivation and objective

In an arbitrage-free financial market.

Question: How to price and hedge a financial risk X?

- Complete Market: unique no-arbitrage price obtained by perfect hedging (replication)... $\rightsquigarrow$ no problem.
- Incomplete Market: infinitely many pricing measures $\rightsquigarrow$ interval of no-arbitrage prices:

$$\mathcal{I} = \Big(\underbrace{\inf_{Q \in \mathcal{M}^e} E^Q[X]}_{\text{buyer's price}}, \underbrace{\sup_{Q \in \mathcal{M}^e} E^Q[X]}_{\text{seller's price}} \Big).$$

... $\rightsquigarrow$ super-hedging/super-replication.

Inconvenience: Price interval $\mathcal{I}$ typically too large for practical use

Aim: Obtain tighter interval of prices by ruling out not only arbitrage opportunities but also "good deals". But HOW?

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Good-deal pricing and hedging idea

Pricing idea:

- Price using only a subset $\mathcal{Q}^{\mathrm{ngd}}$ of the set $\mathcal{M}^e$ of equivalent local martingale measures (ELMMs) with financial meaning.
- For a financial risk X (derivatives, contingent claim, ... etc), the upper and lower good-deal bounds are

$$\pi_t^l(X) := \operatorname*{essinf}_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X] \quad \text{and} \quad \pi_t^u(X) := \operatorname*{esssup}_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X].$$

Hedging idea: minimize over all trading strategies a suitable dynamic risk measure (of no-good-deal type) such that at every time, the minimal capital requirement to make the position acceptable coincides with the good-deal bound.

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Outline

1. No-Good-Deal Restriction
2. Good-Deal Valuation and Hedging
3. Explicit Results
4. Model Ambiguity

Setup and assumptions

The setup is the following:

- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\overline{\mathcal{F}^W_t}^P)_{t \leq T}$ and W an n-dimensional P-Brownian motion.
- Financial market with interest rate $r = 0$ and d risky assets with prices $S = (S^i)_{i=1}^d$.
- S^i are non-negative locally bounded (càdlàg) semimartingales.
- Assume $(d < n)$ and $\mathcal{M}^e(S) \neq \emptyset \rightsquigarrow$ arbitrage-free and incomplete market.

Set $\mathcal{Q}^{\mathrm{ngd}}$ of no-good-deal measures

- Choose $\mathcal{Q}^{\mathrm{ngd}} \subset \mathcal{M}^e(S)$ s.t. $Q \in \mathcal{Q}^{\mathrm{ngd}}$ with $Z^Q := \frac{dQ}{dP}$ satisfies

$$E_\tau^P\left[-\log\frac{Z_\sigma^Q}{Z_\tau^Q}\right] \le E_\tau^P\left[\frac{1}{2}\int_\tau^\sigma h_s^2 ds\right], \quad \tau \le \sigma \le T, \qquad (1)$$

  where $h > 0$ is predictable and τ, σ are stopping times.

- Using convex duality techniques one obtains from (1) that for $Q \in \mathcal{Q}^{\mathrm{ngd}}$ and for any Q-local martingale $N > 0$,

$$E_\tau^P\left[\log\frac{N_\sigma}{N_\tau}\right] \le E_\tau^P\left[-\log\frac{Z_\sigma^Q}{Z_\tau^Q}\right] \le E_\tau^P\left[\frac{1}{2}\int_\tau^\sigma h_s^2 ds\right], \ \forall \tau \le \sigma.$$

Interpretation of a good-deal

- So for $Q \in \mathcal{Q}^{\text{ngd}}$ and any Q-local martingale $N > 0$, we have

$$E_\tau^P\left[\log \frac{N_\sigma}{N_\tau}\right] \le E_\tau^P\left[-\log \frac{Z_\sigma^Q}{Z_\tau^Q}\right] \le E_\tau^P\left[\frac{1}{2}\int_\tau^\sigma h_s^2 ds\right], \ \forall \tau \le \sigma$$

- ... $\rightsquigarrow$ no-good-deal constraint is a bound on the conditional expected growth rate of log-returns on any fair investment in the whole financial market.
- more specifically, a good-deal is an investment for which the expected growth rate of returns exceeds $\frac{1}{2}h^2$.

No-good-deal restriction on the Girsanov kernels

- Our no-good-deal restriction to $\mathcal{Q}^{\mathrm{ngd}}$ is actually equivalent to a bound on the Girsanov kernels of measures in $\mathcal{M}^e(S)$:

$$Q \in \mathcal{Q}^{\mathrm{ngd}} \text{ iff } Q \in \mathcal{M}^e \text{ with } Z^Q = \mathcal{E}\left(\lambda^Q \cdot W\right), \text{ and } |\lambda^Q| \leq h$$

- ... $\rightsquigarrow$ Girsanov kernels λ^Q for $Q \in \mathcal{Q}^{\mathrm{ngd}}$ are selections of the correspondence (multivalued mapping) $C : [0,T] \times \Omega \to 2^{\mathbb{R}^n}$ defined by $C(t,\omega) = \overline{B_0(h_t(\omega))}$, $\forall (t,\omega)$.

Note:

- We will consider more general correspondences C, yielding more alternatives to no-good-deal constraints.
- The values of C could be e.g. ellipsoids, polytopes, ...etc

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Generalized good-deal bounds via correspondences

Fix an arbitrary compact- and convex-valued, predictable correspondence $C : [0, T] \times \Omega \to 2^{\mathbb{R}^n}$ with $0 \in C$.

- Definition:

$$\mathcal{Q}^{\mathrm{ngd}} := \left\{ Q \in \mathcal{M}^e \;\middle|\; \frac{dQ}{dP} = \mathcal{E}\left(\lambda^Q \cdot W\right) \text{ with } \lambda^Q \in C \right\}.$$

- $\mathcal{Q}^{\mathrm{ngd}}$ is multiplicatively stable, which implies nice dynamic properties of $\pi_\cdot^u(X) := \operatorname*{esssup}_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E_\cdot^Q[X]$ as follows...

Dynamic properties of $\pi^u_\cdot(\cdot)$

Theorem: The mappings $X \mapsto \pi^u_t(X)$ from $L^\infty \to L^\infty(\mathcal{F}_t)$ satisfy

- (Nice paths) There exists a càdlàg version Y of $\pi^u_\cdot(X)$ s.t.

$$Y_\tau = \operatorname*{esssup}_{Q \in \mathcal{Q}^{\mathrm{ngd}}} E^Q_\tau[X] =: \pi^u_\tau(X) \quad \forall \tau \leq T \text{ stopping time.}$$

- (Dynamic coherent risk measure) For any stopping time $\tau \leq T$, $\forall X_1, X_2 \in L^\infty(\mathcal{F})$, $m_\tau, \lambda_\tau \in L^\infty(\mathcal{F}_\tau)$ with $\lambda_\tau \geq 0$,
  - **Monotonicity:** $X_1 \geq X_2 \text{ implies } \pi^u_\tau(X_1) \geq \pi^u_\tau(X_2)$
  - **Subadditivity:** $\pi^u_\tau(X_1 + X_2) \leq \pi^u_\tau(X_1) + \pi^u_\tau(X_2)$
  - **Positive Homogeneity:** $\pi^u_\tau(\lambda_\tau X) = \lambda_\tau \pi^u_\tau(X)$
  - **Translation Invariance:** $\pi^u_\tau(X + m_\tau) = \pi^u_\tau(X) + m_\tau$.

- (Supermartingale property) $\forall Q \in \mathcal{Q}^{\mathrm{ngd}}$,

$$\forall \sigma \leq \tau \leq T \text{ stopping times,} \quad \pi^u_\sigma(X) \geq E^Q_\sigma\left[\pi^u_\tau(X)\right].$$

Financial market model

More specific market model:

- Stock price vector $S = (S^i)_{i=1}^d$ is a non-Markovian Itô process:

$$\begin{cases} dS_t &= \mathrm{diag}(S_t)\sigma_t(\xi_t dt + dW_t) =: \mathrm{diag}(S_t)\sigma_t d\widehat{W}_t \\ S_0 &\in (0,\infty)^d \end{cases}$$

for bounded market price of risk $\xi \in \mathrm{Im}\ \sigma^{\mathrm{tr}}$, $\mathbb{R}^{d\times n}$-valued volatility matrix σ of full rank ($\rightsquigarrow$ incomplete market).

- $Q \in \mathcal{M}^e$ iff $\lambda^Q = -\xi + \eta^Q$ predictable and $\eta_t^Q \in \mathrm{Ker}\ \sigma_t$.
- $\ldots \rightsquigarrow Q \in \mathcal{Q}^{\mathrm{ngd}} \subseteq \mathcal{M}^e$ iff $\lambda^Q \in \Lambda$, where Λ is the correspondence given by $\Lambda(t,\cdot) = C_t \cap (-\xi_t + \mathrm{Ker}\ \sigma_t)$.

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Admissible trading strategies

- Trading strategies $\varphi = (\varphi^i)_{i=1}^d$ are the amounts φ^i of wealth invested into stocks of prices S^i, $i = 1 \ldots, d$.
- Corresponding wealth process V^φ for φ:

$$dV_t^\varphi := \varphi_t^{\mathrm{tr}} \frac{dS_t}{S_t} = \varphi_t^{\mathrm{tr}} \sigma_t d\widehat{W}_t.$$

- Re-parameterize trading strategy φ as $\phi := \sigma^{\mathrm{tr}} \varphi \in \mathrm{Im}\ \sigma^{\mathrm{tr}}$, such that

$$V_t^\phi = V_0^\phi + \int_0^t \phi_t^{\mathrm{tr}} d\widehat{W}_t.$$

- Set of admissible trading strategies:

$$\Phi = \left\{ \phi \;\middle|\; \phi \text{ predictable, } \phi \in \mathrm{Im}\ \sigma^{\mathrm{tr}} \text{ and } E \int_0^T |\phi_t|^2 dt < \infty \right\}.$$

Good-deal valuation and hedging tools

Main tool: Use backward stochastic differential equations (BSDEs) to describe good-deal valuation bounds and their corresponding hedging strategies.

Good-deal valuation with C uniformly bounded

- For $X \in L^2(\mathcal{F})$ and $Q \in \mathcal{Q}^{\text{ngd}}$, the process $Y = E^Q_{\cdot}[X]$ solves linear lipschitz BSDE

$$- dY_t = Z_t^{\text{tr}} \lambda_t^Q dt - Z_t^{\text{tr}} dW_t, \quad Y_T = X.$$

- Let (Y, Z) be the solution to the Lipschitz BSDE

$$-dY_t = Z_t^{\text{tr}} \bar{\lambda}_t(Z) dt - Z_t^{\text{tr}} dW_t, \quad Y_T = X.$$

where $\bar{\lambda} = \bar{\lambda}(Z)$ with $Z_t^{\text{tr}} \bar{\lambda}_t = \operatorname*{esssup}_{Q \in \mathcal{Q}^{\text{ngd}}} Z_t^{\text{tr}} \lambda_t^Q$.

- By the comparison theorem for Lipschitz BSDEs, we have $\pi^u_{\cdot}(X) := \operatorname*{esssup}_{Q \in \mathcal{Q}^{\text{ngd}}} E_t^Q[X] = Y$
- . . . and there is a worst case measure $\bar{Q} \in \mathcal{Q}^{\text{ngd}}$ with $\lambda^{\bar{Q}} := \bar{\lambda}(Z)$ s.t. $\pi_t^u(X) = E_t^{\bar{Q}}[X]\ \forall t$.

Good-deal hedging problem

Consider the set $\mathcal{P}^{\mathrm{ngd}} \supseteq \mathcal{Q}^{\mathrm{ngd}}$ defined by

$$\mathcal{P}^{\mathrm{ngd}} := \left\{ Q \sim P \;\middle|\; \frac{dQ}{dP} = \mathcal{E}\left(\lambda^Q \cdot W\right),\ \text{with } \lambda^Q \in C \right\}.$$

- Associated upper bound is given by $\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}^{\mathrm{ngd}}} E_t^Q[X]$.
- $\mathcal{P}^{\mathrm{ngd}}$ is m-stable and convex $\Longrightarrow (\rho_t(\cdot))_{t \le T}$ is a dynamic coherent time-consistent risk measure.
- Hedging problem: Find an admissible strategy $\bar{\phi} \in \Phi$ such that

$$\pi_t^u(X) = \rho_t\left(X - \int_t^T \bar{\phi}_s^{\mathrm{tr}} d\widehat{W}_s\right) = \operatorname*{ess\,inf}_{\phi \in \Phi} \rho_t\left(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s\right).$$

Ellipsoidal setting with bounded correspondence C

Beyond radial restrictions consider ellipsoid correspondences for explicit results:

- Bounded predictable process $h > 0$
- Predictable $\mathbb{R}^{n\times n}$-valued process A, uniformly elliptic i.e.

$$\exists c > 0 \text{ s.t. } x^{\mathrm{tr}} A_t(\omega) x \geq c\,|x|^2 \quad P \otimes dt\text{-a.a.}$$

- Compact-convex-valued, predictable and uniformly bounded correspondence C given by ellipsoids

$$C(t,\omega) = \left\{ x \in \mathbb{R}^n \mid x^{\mathrm{tr}} A_t(\omega) x \leq h_t^2(\omega) \right\}.$$

- $\Pi_t(\cdot)$ and $\Pi_t^{\perp}(\cdot)$ denote resp. projections onto $\mathrm{Im}\ \sigma_t^{\mathrm{tr}}$ and $\mathrm{Ker}\ \sigma_t = (\mathrm{Im}\ \sigma_t^{\mathrm{tr}})^{\perp}$

Good-deal valuation by BSDEs

- Optimal Girsanov kernel $\bar{\lambda}_t(Z) := \underset{\lambda_t \in \Lambda_t}{\text{argmax}} \; \lambda_t^{\text{tr}} Z_t$ is given by

$$\bar{\lambda}_t = -\xi_t + \frac{\sqrt{h_t^2 - \xi_t^{\text{tr}} A_t \xi_t}}{\sqrt{\Pi_t^{\perp}(Z_t)^{\text{tr}} A_t^{-1} \Pi_t^{\perp}(Z_t)}} A_t^{-1} \Pi_t^{\perp}(Z_t)$$

- ... hence $\pi_\cdot^u(X) = Y$ for (Y, Z) solving the Lipschitz BSDE

$$-dY_t = f^{\bar{\lambda}}(t, Z_t)dt - Z_t^{\text{tr}} dW_t, \quad Y_T = X,$$

with

$$f^{\bar{\lambda}}(t, Z_t) := -\xi_t^{\text{tr}} \Pi_t(Z_t) + \sqrt{h_t^2 - \xi_t^{\text{tr}} A_t \xi_t} \sqrt{\Pi_t^{\perp}(Z_t)^{\text{tr}} A_t^{-1} \Pi_t^{\perp}(Z_t)}.$$

Good-deal hedging strategy via BSDEs

- Kuhn-Tucker arguments yields formula for the hedging strategy:

$$\bar{\phi}_t = \sqrt{\frac{\Pi_t^{\perp}(Z_t)^{\mathrm{tr}} A_t^{-1} \Pi_t^{\perp}(Z_t)}{h_t^2 - \xi_t^{\mathrm{tr}} A_t \xi_t}}\; A_t \xi_t + \Pi_t(Z_t),$$

for (Y, Z) solution to the π^u-BSDE.

Robustness of the good-deal hedging strategy

- For $\phi \in \Phi$, define the associated hedging (or tracking) error

$$L_t^{\phi} := \underbrace{\pi_t^u(X) - \pi_0^u(X)}_{\text{capital requirement}} - \underbrace{\int_0^t \phi_s^{\mathrm{tr}} d\widehat{W}_s}_{\text{Gain/loss from trading}}.$$

- **Super-mean-self-financing:** hedging error $L^{\bar{\phi}}$ of the good-deal hedging strategy $\bar{\phi}$ is a Q-supermartingale $\forall Q \in \mathcal{P}^{\mathrm{ngd}}$

- ... $\rightsquigarrow$ "robustness" of hedging strategy w.r.t. generalized scenarios corresponding to probability measures in $\mathcal{P}^{\mathrm{ngd}}$.

Example: option on a non-tradable asset

- Black-Scholes with one stock and a non-tradable asset:

$$\begin{cases} \frac{dS_t}{S_t} = & \sigma dW_t^S, \quad S_0 > 0 \\ \frac{dH_t}{H_t} = & \gamma dt + \beta \left(\rho dW_t^S + \sqrt{1-\rho^2} dW_t^H \right), \quad H_0 > 0, \end{cases}$$

  with correlation coefficient $\rho \in (-1, 1)$ and volatility $\sigma > 0$.
- Consider Call option $X = (H_T - K)^+$ on non-tradable asset H
- ...and ellipsoidal restriction $C_t = \{x : \ x^{\mathrm{tr}} A x \le h^2\}$, with $h = \mathrm{const} > 0$ and $A = \mathrm{diag}(a, b)$, with $a, b > 0$.
- ... $\rightsquigarrow$ then explicit form of the good-deal bound:

$$\pi_t^u(X) = \alpha * \textbf{Black-Scholes-Call-Price} \left(t, \ \mathrm{Strike:} \ \frac{K}{\alpha}, \mathrm{vol:} \ \beta \right),$$

where $\alpha = \exp \left(T \left(\gamma + \beta \sqrt{1-\rho^2} \frac{h}{\sqrt{b}} \right) \right) > 0$

Example: option in the Heston model

Heston model with stochastic volatility $\sigma_t = \sqrt{v_t}$:

$$\begin{cases} dS_t &= S_t\sqrt{v_t}dW_t^S, \quad S_0 > 0 \\ dv_t &= (a - bv_t)dt + \beta\sqrt{v_t}\left(\rho dW_t^S + \sqrt{1-\rho^2}dW_t^v\right), \; v_0 > 0, \end{cases}$$

with MRL a, MRS b, volvol β and correlation $\rho \in (-1, 1)$.

- Good-deal radial constraint $C_t = \{x : \; |x| \leq h_t\}$ with $h_t := \frac{\varepsilon}{\sqrt{v_t}}$, and Put option $X = (K - S_T)^+$.
- Obtain pseudo-explicit solution for the good-deal bound:

$$\pi_t^u(X) = \textbf{Heston-Put-Price}(t, \; \mathrm{MRL}: \bar{a}, \; \mathrm{MRS}: b, \; \mathrm{volvol}: \beta),$$

with increased MRL: $\bar{a} := a + \beta\varepsilon\sqrt{1-\rho^2} > a$.

Good-deal theory and model uncertainty

Model ambiguity:

- Unknown real world measure P and market prices of risk
- ... $\rightsquigarrow$ model uncertainty.

Goal: robust valuation and hedging w.r.t uncertainty.

Approach:

- Rather than single reference measure $P = P^0$, consider "confidence region" $\mathcal{R}$ of reference measures:

$$\mathcal{R} := \{P^\nu \mid dP^\nu = \mathcal{E}(\nu \cdot W^0)dP^0, \ \nu \in V\}$$

for some correspondence V.

- Market price of risk under P^ν: $\xi^\nu = \xi^0 + \Pi(\nu) \in \mathrm{Im}\ \sigma^{\mathrm{tr}}$.
- Fix correspondences $\{C^\nu, \ \nu \in V\} \ldots \rightsquigarrow \mathcal{Q}^{\mathrm{ngd}}(P^\nu)$,
- Definition:

$$\mathcal{Q}^{\mathrm{ngd}} := \text{m-stable-convex-hull}\left(\cup_{\nu \in V} \mathcal{Q}^{\mathrm{ngd}}(P^\nu)\right)$$

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Worst case valuation and hedging under uncertainty

Ellipsoid setting with $V_t := \left\{x : \ x^{\mathrm{tr}} A_t x \le \delta_t^2\right\}$, $\delta > 0$, predictable bounded process and $C_t^{\nu} = \left\{x : x^{\mathrm{tr}} A_t x \le h_t^2\right\} - \Pi_t^{\perp}(\nu_t)$:

- $\exists$ "worst case" reference measure $P^{\bar{\nu}}$, such that

$$\mathcal{Q}^{\mathrm{ngd}} = \bigcup_{\nu \in V} \mathcal{Q}^{\mathrm{ngd}}(P^{\nu}) = \mathcal{Q}^{\mathrm{ngd}}(P^{\bar{\nu}})$$

- The measure $P^{\bar{\nu}}$ yields the largest good-deal bound, i.e.

$$\pi_t^u(X) = \operatorname*{esssup}_{\nu \in V} \pi_t^{u,\nu}[X] = \pi_t^{u,\bar{\nu}}(X),$$

- $\ldots \rightsquigarrow$ as in case without uncertainty, one derives a BSDE (under $P^{\bar{\nu}}$), the solution of which describes $\pi_\cdot^u(X) = \pi_\cdot^{u,\bar{\nu}}(X)$ and the "worst case" hedging strategy $\bar{\phi}^{\bar{\nu}}$.

Worst case hedging is not robust

- **Inconvenience:** hedging strategy $\bar{\phi}^{\bar{\nu}}$ NOT robust w.r.t. uncertainty, i.e. simultaneously w.r.t. all models under $P^{\nu}, \nu \in V$.

Robust hedging with respect to uncertainty

- Definition: $\mathcal{P}^{\mathrm{ngd}} := \cup_{\nu \in V} \mathcal{P}^{\mathrm{ngd}}(P^\nu)$ $(\ldots \rightsquigarrow$ m-stability$)$
- obtain dynamic coherent risk measure $\rho_t(X) = \operatorname*{esssup}_{\nu \in V} \rho_t^\nu(X)$,

  with $\rho_t^\nu(X) = \operatorname*{esssup}_{Q \in \mathcal{P}^{\mathrm{ngd}}(P^\nu)} E_t^Q[X]$
- From hedging without uncertainty we have

$$\pi_t^u(X) = \operatorname*{esssup}_{\nu \in V} \pi_t^{u,\nu}[X] = \operatorname*{esssup}_{\nu \in V} \operatorname*{essinf}_{\phi \in \Phi} \rho_t^\nu \left(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s \right)$$

- For robust hedging, consider the dual bound

$$\bar{\pi}_t^u(X) := \operatorname*{essinf}_{\phi \in \Phi} \operatorname*{esssup}_{\nu \in V} \rho_t^\nu \left(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s \right)$$

Robust hedging with respect to uncertainty. . . (Continued)

- Dual good-deal bound is then obtained by $\bar{\pi}^u_t(X) := Y_t$ for (Y, Z) solution to BSDE under P^0 with parameters (f, X), where

$$f(t, Z_t) = -(\xi^0_t)^{\mathrm{tr}} \Pi_t(Z_t) + h_t \sqrt{\Pi^{\perp}_t(Z_t)^{\mathrm{tr}} A^{-1}_t \Pi^{\perp}_t(Z_t)}$$

- . . . for robust hedging with respect to uncertainty uniformly for generalized scenarios in $\mathcal{P}^{\mathrm{ngd}}$, the strategy is given by $\phi^*_t = \Pi_t(Z_t)$

Comparison of $\pi^u_\cdot(X)$ and $\bar{\pi}^u_\cdot(X)$

- In general $\pi^u(X) \leq \bar{\pi}^u(X)$, i.e. weak duality!
- ... but if $\mathcal{M}^e(P^0) \cap \mathcal{R} \neq \emptyset$, then $\pi^u(X) = \bar{\pi}^u(X)$ and $\bar{\phi}^{\bar{\nu}} = \phi^* = \Pi(Z)$, i.e. strong duality!
- Example: $\xi^{0^{\mathrm{tr}}} A \xi^0 \leq \delta^2 \quad \Longrightarrow \quad P^0 \in \mathcal{M}^e(P^0) \cap \mathcal{R}$

Link to Föllmer-Schweizer risk minimization (Föllmer & Schweizer (1991))

- For a financial risk X, the strategy $\hat{\phi}$ is the Föllmer-Schweizer (F-S) risk minimizing strategy for X w.r.t. $Q \in \mathcal{M}^e(S)$ if

$$\hat{\phi}_t = \operatorname*{argmin}_{\phi} E_t^Q\left[(L_T^\phi - L_t^\phi)^2\right], \quad t \leq T.$$

- If $\mathcal{M}^e(P^0) \cap \mathcal{R} \neq \emptyset$, then robust good-deal hedging $\Leftrightarrow$ F-S risk minimization under some special measure $\bar{Q} \in \mathcal{Q}^{\mathrm{ngd}}$.

Outlook

Possible extensions:

- incorporation of predictable event-risk (jumps, default, ...etc)
- robust good-deal hedging w.r.t. volatility uncertainty?

Thanks for your Attention!