Rare Events, Asymmetric Correlation and Under-diversification

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Background and Motivation

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Documented empirical finding of asymmetric correlation

- Asymmetric correlations have been widely observed in financial markets.
- Evidence of asymmetry is also found in covariances, volatilities and betas of returns.
- Asymmetric correlations can be understood simply as a phenomenon that asset returns tend to be more correlated in two circumstances:
  - bear markets
  - volatile markets
- The pattern of asymmetry is complicated and hard to measure.

Why do we care about asymmetry?

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- Ignoring asymmetry will lead to significant losses for investors believing in diversification
- Hedging strategy will be sensitive to asymmetry risk
  - Hedging crucially depends on correlations between assets hedged and hedging instruments
  - It will suffer heavily if the correlation structure is asymmetric and stochastic
- Asymmetry can also provide a new aspect to many economic puzzles
  - Home-Bias puzzle and international under-diversification

A Simple Measure to Identify Asymmetry

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Suppose we have two raw return series x and y, and corresponding standardised returns $\tilde{x}$ and $\tilde{y}$ are used in:

$$\rho(\theta) = \begin{cases} \rho(\theta)^{+} = corr(\tilde{x}, \tilde{y} | \tilde{x} > \theta, \tilde{y} > \theta) & \text{if} \quad \theta \geq 0 \\ \rho(\theta)^{-} = corr(\tilde{x}, \tilde{y} | \tilde{x} < \theta, \tilde{y} < \theta) & \text{if} \quad \theta \leq 0 \end{cases} \tag{1}$$

$\rho(\theta)$ is named as exceedance correlation with exceedance level θ. Note, we might have two different values $\rho(0)^{+}$ and $\rho(0)^{-}$.

A First Glance at Asymmetry

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What's the typical pattern of asymmetry?

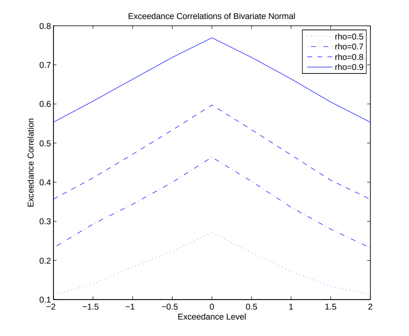

Figure: An exceedance plot of bivariate normal distribution

Pattern of exceedance correlation

1. Symmetric curve

A First Glance at Asymmetry

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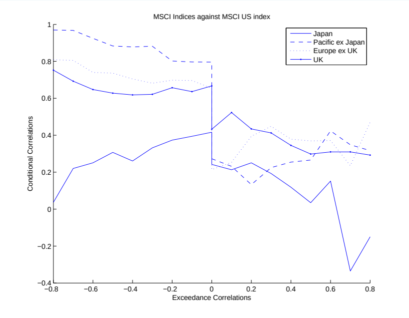

Figure: An exceedance plot of real data (MSCI indices)

Pattern of exceedance correlation

1. Symmetric curve
2. Asymmetric curve

Existing Literature

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- How to test if data is of asymmetry and if a special model can capture asymmetry
- Ang and Bekaert (2002), Longin and Solnik (2001), Ang and Chen (2002), Bae, Karolyi and Stulz (2003)
  - H-test: based on benchmark of normality, model-related, Ang and Chen (2002)
  - J-test: model-free, Hong, Tu and Zhou (2007)
  - Contour plot: more flexible and can exhibit non-linear dependence
- Existing models
  - GARCH model
  - Regime-switching model
  - Jump-diffusion model
  - Copula model

Modelling

Fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. Suppose we have N risky assets, and $\{S_t\}_{0 \le t \le T}$ and $\{V_t\}_{0 \le t \le T}$ represent the spot price and the stochastic variance. The present model can be represented as:

$$dS_{i,t} = S_{i,t-}\Big[(r+\mu_i\sqrt{V_{i,t}})dt + \sqrt{V_{i,t}}\sum_{j=1}^{N}\sigma_{i,j}dz_{j,t} + \sqrt{V_{i,t}}\left(\sigma^q_{i,1}\left(\exp(Y_u)-1\right)dN^u_t + \sigma^q_{i,2}\left(\exp(-Y_d)-1\right)\right)dN^d_t\Big]$$

$$dV_{i,t} = \kappa_i(1-V_{i,t})dt + \sigma_{v,i}\sqrt{V_{i,t}}\left(\rho_i dz_{i,t} + \sqrt{1-\rho_i^2}dw_t\right)$$

where $dz_{j,t}$ and dw_t are independent Brownian motions, Y_u and Y_d are exponential distributed random variables, and N^u_t and N^d_t are independent poisson processes.

Modelling

We have

$$\Sigma_b = \begin{pmatrix} \sqrt{V_{1,t}}\sigma_{1,1} & \dots & \sqrt{V_{1,t}}\sigma_{1,N} \\ \dots & \dots & \dots \\ \sqrt{V_{N,t}}\sigma_{N,1} & \dots & \sqrt{V_{N,t}}\sigma_{N,N} \end{pmatrix}, \quad \Sigma_b^0 = \begin{pmatrix} \sigma_{1,1} & \dots & \sigma_{1,N} \\ \dots & \dots & \dots \\ \sigma_{N,1} & \dots & \sigma_{N,N} \end{pmatrix}$$

and

$$\Sigma_q = \begin{pmatrix} \sqrt{V_{1,t}}\sigma^q_{1,1} & \sqrt{V_{1,t}}\sigma^q_{1,N} \\ \dots & \dots \\ \sqrt{V_{N,t}}\sigma^q_{N,1} & \sqrt{V_{N,t}}\sigma^q_{N,N} \end{pmatrix}, \quad \Sigma_q^0 = \begin{pmatrix} \sigma^q_{1,1} & \sigma^q_{1,N} \\ \dots & \dots \\ \sigma^q_{N,1} & \sigma^q_{N,N} \end{pmatrix}$$

Modelling

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What we have done:

- We employ a multi-variate jump-diffusion model equipped with stochastic volatility, in order to capture asymmetric correlation.
- Leverage effect is incorporated as $S_{i,t}$ and $V_{i,t}$ has dependence.
- We have systematic jumps that are split into two: positive and negative.
- Jump sizes are controlled by asymmetric distributions (exponential dist.).
- Lévy jumps are also supported.

Modelling

What we got now:

- The special pattern of asymmetry can be captured: flat left tail and dropping right tail.
- It provides better performance than existing benchmark models including regime-switching models.
- No need to test artificial ‘regimes’.
- A large number of state variables can be adopted based on Xing and Zhang (2012) so as to enhance the model performance: macro factor, interest rate and volatility
- With parsimoniousity, we have stochastic correlation and volatility automatically.

Solving portfolio allocation with CRRA utility

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Consider an investor with CRRA utility endowed with initial wealth $W_0 = \omega_0$. She can allocate her wealth between one risk-less asset and N risky assets. Investment constraints may apply. The traditional Merton's problem is to maximize:

$$u(W_0) = \max_{\pi_i} J(W_0, \pi) = E[U(W_T)] \tag{2}$$

where

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} \\ -\infty \end{cases} \tag{3}$$

Following Merton (1971), optimal portfolio problem can be solved by solving HJB equation if no jump exists.

How to solve a jump-diffusion framework?

Following Liu (2007) and Xing and Zhang (2012), we can

- decompose the optimal portfolio choice problem in a jump-diffusion market into those in a pure-diffusion market and a set of pure-jump markets.
- use Σ to denote the matrix $\begin{pmatrix} \Sigma_b & \Sigma_q \\ \mathbf{0}_{2\times N} & \mathbf{I}_{2\times 2} \end{pmatrix}$, and assume Σ is invertible.
- for a trading strategy π, use $\pi_b = (\pi_{b1}, \ldots, \pi_{bN})$ and $\pi_q = (\pi_{q1}, \pi_{q2})$ to denote $\pi\Sigma_b$ and $\pi\Sigma_q$.
- $\pi = \pi\Sigma\Sigma^{-1} = (\pi_b, \pi_q)\Sigma^{-1}$
- solve the optimal exposure π_b and π_q, then obtain the optimal strategy π via using a rotation Σ^{-1}.

Define the relative risk premium as

$$\theta = \begin{pmatrix} \theta^b \\ \theta^q \end{pmatrix} = \Sigma^{-1}\left(b - r + \widetilde{\Sigma}_q(\lambda \cdot \alpha)\right) \tag{4}$$

where $\widetilde{\Sigma}_q = \begin{pmatrix} \Sigma_q \\ \mathbf{I}_{2\times 2} \end{pmatrix}$,
$b = \left(r + \mu_1\sqrt{V_{1,t}}, ..., r + \mu_N\sqrt{V_{N,t}}, r + \nu_1, r + \nu_2\right)'$, and λ is the vector of jump intensities, and $\alpha_k = \int z\Phi_k(dz)$ represents the expected size of k-th jump.

Optimal portfolio solution

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The optimal portfolio is $\pi = \left(\frac{\pi_1}{\sqrt{V_{1,t}}}, ..., \frac{\pi_N}{\sqrt{V_{N,t}}}\right) = \frac{\pi_b' \Sigma^{-1}}{\sqrt{V_t}}$, where

$$\pi_b = \frac{\Sigma^{-1}(\mu' - \Sigma_q^0 \nu')}{\gamma}$$
$$\pi_q = \pi_b' (\Sigma_b^0)^{-1} \Sigma_q^0$$

with the constraints

$$\max_{\pi_{q,1}} \pi_{q,1}\nu_1 + \frac{\lambda_1}{1-\gamma} \int [(1 + \pi_{q,1} z)^{1-\gamma} - 1] \Phi_1(dz)$$
$$\max_{\pi_{q,2}} \pi_{q,2}\nu_2 + \frac{\lambda_2}{1-\gamma} \int [(1 + \pi_{q,2} z)^{1-\gamma} - 1] \Phi_2(dz)$$

Capturing Asymmetry with Jump-diffusion Model

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Many empirical literature has demonstrated the common existence of asymmetry in many markets. For example:

- US market: S&P 500, NASDAQ, Fama-French data
- International markets: MSCI indices

Only Regime-switching models can provide acceptable performance on capturing asymmetry; however, transitions between regimes are not easy to deal with. The claim of existence of regimes is also suspicious.

Fama-French data sets are selected from July 1963 to December 1998. Weekly excess returns are used to reduce non-synchronous effect based on US 1 month T-bill data.

	Jump model	RS GARCH (Ang and Chen (2002))	Our model
Industry	11/13	2/13	1/13
Size	3/5	1/5	0/5
Book-to-Market	3/5	0/5	0/5
Momentum	4/5	1/5	0/5
Overall	21/28	4/28	1/28

Table: Summary of rejections based on Fama-French data set (H-test)

A comparison of contour plots between real data and simulation data

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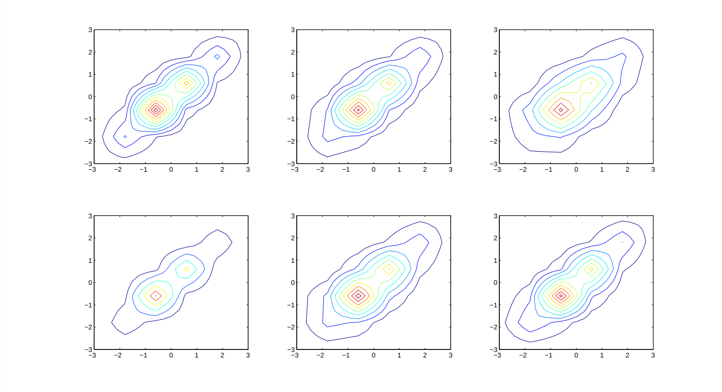

Figure: Contour plot of real data

A comparison of contour plots between real data and simulation data

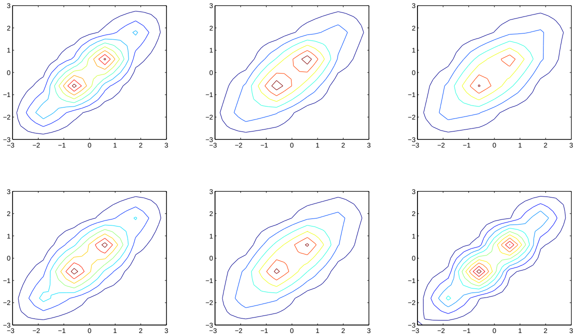

Figure: Contour plot of simulation data

Following Hong, Tu and Zhou (2007), we present a similar example to compare frequency tables between real data and simulation data.

Real data					
2	6	13	48	83	165
1	14	48	63	107	83
5	34	68	93	70	47
12	73	89	82	45	16
71	123	84	22	12	4
228	68	14	6	0	1
Simulation data					
0	0	20	44	78	165
3	24	44	74	101	80
9	48	65	82	80	33
12	77	80	71	45	12
67	114	63	46	20	6
235	53	35	10	2	1

Table: Frequency Table of Real Data

In order to investigate if asymmetry has significant impacts on portfolio allocation, a benchmark model that does not admit asymmetry is used to derive a sub-optimal portfolio allocation. An optimal portfolio allocation is derived based on our model. Strong evidences of under-diversification is found.

our model	π_1	π_2	π_3	π_4	π_5
Size	1.5621	3.8828	-7.1703	2.9269	-1.2033
Book-2-Market	2.3432	4.6914	-7.8410	1.3774	-0.9721
Momentum	3.1243	5.8372	-8.2335	1.3592	-1.2731
benchmark					
Size	3.4021	9.5851	-12.2868	-0.0862	3.6752
Book-2-Market	4.0220	10.8273	-13.3259	-0.5102	3.8252
Momentum	4.8252	12.0352	-14.3222	-0.9320	3.9272

Table: A comparison of portfolio weights ($\gamma = 5$)

Economic loss of ignoring asymmetry

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- Impacts of ignoring asymmetry can be quantified by calculating economic losses if a sub-optimal strategy is chosen by investors
  - Computing Certain Equivalent Cost for a sub-optimal portfolio
- The cost is defined as

$$\omega = \exp\left(\frac{(\kappa(\pi^*) - \kappa(\pi))T}{1-\gamma} - 1\right)$$

- Intuitively, the economic cost/loss measures how much an investor needs to be paid if he is asked to ignore asymmetry risk.

Economic loss of ignoring asymmetry

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Economic cost/loss is presented in annual rates in the figure below.

γ	5	6	7	8	20
R_w	0.45	0.41	0.40	0.39	0.21

Table: Certain Equivalent Costs of ignoring asymmetry based on size portfolio (annual rate)

- The cost of ignoring asymmetry is huge and decreases as the level of risk-aversion increases.
- Investors will lower down the leverage as becoming more risk-averse, which will reduce the difference between optimal and sub-optimal portfolio strategies.

Sensitivity with respect to jump-risk

- The reason and driven factors of asymmetry are still open questions
- It is believed that negative jump risk plays an important role
- Observing differences of portfolio weights given different jump intensity can provide an image of how much investors care about negative jump risk

λ_d	π_1	π_2	π_3	π_4	π_5
13.1243	0.8526	2.3887	-5.7572	1.3449	-1.1345
13.9053	1.0352	3.0293	-6.5640	2.0394	-1.1608
14.6864	1.5621	3.8828	-7.1703	2.9269	-1.2033
15.4675	2.3262	4.8869	-8.8655	3.4998	-1.4760
16.2485	3.2034	6.2323	-10.4894	4.2426	-1.5649

Table: Impact of negative jump activity on portfolio allocation: unconstrained problem

A new explanation of home-bias puzzle

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- Existing literature has tried to explain this puzzle from :
  - barriers of international investments
  - transaction costs
  - information asymmetry and estimation uncertainty
  - political or corporate governance risks
- Asymmetry risk provides a new aspect to explain this puzzle.

A new explanation of home-bias puzzle

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- MSCI indices data are used, including Japan, Europe ex UK, US, Asia ex Japan and UK.
- Data is selected from Jan. 1975 to Dec. 2005.
- Short-selling is prohibited, in order to match with most of results in existing literature.

	Normalised weights				
λ_d	Japan	Europe ex UK	US	Asia ex Japan	UK
19.5805	0.0000	0.1111	0.8140	0.0000	0.0750
20.6916	0.0000	0.1393	0.8049	0.0000	0.0559
21.8028	0.0000	0.1690	0.7953	0.0000	0.0357
22.9139	0.0000	0.2005	0.7851	0.0000	0.0143
23.4694	0.0000	0.2170	0.7798	0.0000	0.0032
24.0250	0.0000	0.2289	0.7711	0.0000	0.0000
25.1361	0.0000	0.2496	0.7504	0.0000	0.0000
26.2472	0.0000	0.2713	0.7287	0.0000	0.0000

Table: Optimal portfolio weights with respect to negative jump risk

	Actual weights				
γ	Japan	Europe ex UK	US	Asia ex Japan	UK
4	0.0000	0.2426	0.8731	0.0000	0.0038
6	0.0000	0.1939	0.6987	0.0000	0.0031
8	0.0000	0.1615	0.5823	0.0000	0.0027
10	0.0000	0.1384	0.4992	0.0000	0.0023
12	0.0000	0.1211	0.4368	0.0000	0.0021
14	0.0000	0.1076	0.3883	0.0000	0.0019
16	0.0000	0.0968	0.3495	0.0000	0.0017
18	0.0000	0.0880	0.3177	0.0000	0.0015
20	0.0000	0.0807	0.2913	0.0000	0.0014
24	0.0000	0.0745	0.2689	0.0000	0.0013

Table: Optimal portfolio weights with respect to risk aversion (actual weights)

	Normalised weights				
γ	Japan	Europe ex UK	US	Asia ex Japan	UK
4	0.0000	0.2167	0.7799	0.0000	0.0034
6	0.0000	0.2165	0.7800	0.0000	0.0035
8	0.0000	0.2164	0.7800	0.0000	0.0036
10	0.0000	0.2163	0.7801	0.0000	0.0036
12	0.0000	0.2162	0.7801	0.0000	0.0037
14	0.0000	0.2162	0.7801	0.0000	0.0037
16	0.0000	0.2161	0.7801	0.0000	0.0037
18	0.0000	0.2161	0.7801	0.0000	0.0038
20	0.0000	0.2161	0.7801	0.0000	0.0038
24	0.0000	0.2161	0.7802	0.0000	0.0038

Table: Optimal portfolio weights with respect to risk aversion (normalised weights)

A new explanation of home-bias puzzle

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- Asymmetry risk causes domestic investor to limit holding positions on foreign indices
- The benefit of international diversification reduces significantly due to asymmetry risk
- An interesting finding is:
  - different risk-averse investors have different risk-aversion levels
  - they should have different holding positions and leverage levels
  - but they tend to have the same holding percentages

Conclusion and Future research

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- A jump-diffusion framework is proposed to capture asymmetric correlation
- Numerical examples show the goodness of asymmetry fitting
- Discussions about the economic loss of sub-optimal portfolios and impacts of investment constraints
- A new explanation of the home bias puzzle from the point of view of asymmetry
- It is still a very preliminary research result on asymmetry
    - the performance of our model can be enhanced by adding up hedging demand
    - our model can be expected to explain the contagion problem
    - effects of investment constraints on portfolio allocation can be quantified and investigated

An enhanced model that we are working on

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$$
\begin{aligned}
dS_{i,t} &= S_{i,t-}\Big[rdt + \sqrt{V_{i,t}}\sum_{j=1}^{N}\sigma_{i,j}(\eta_j^z(X_t)dt + dz_{j,t}) \\
&\quad + \sqrt{V_{i,t}}(\sigma_{i,1}^q(\eta^u\lambda_u(X_t)dt + (\exp(Y_u) - 1)\,dN_t^u) \\
&\quad + \sigma_{i,2}^q(\eta^d\lambda_d(X_t)dt + (\exp(-Y_d) - 1)\,dN_t^d)\Big] \\
dV_{i,t} &= \kappa_i(1 - V_{i,t})dt + \sigma_{v,i}\sqrt{V_{i,t}}\left(\rho_i dz_{i,t} + \sqrt{1-\rho_i^2}dw_t\right) \\
dX_t &= \mu^x(X_t)dt + \sum_{j=1}^{N}\sigma_j dz_{j,t} + \sigma_{N+1}dw_t
\end{aligned}
$$

An enhanced model that we are working on

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- Many choices can be made for the state variable X, and we have chosen the interest rate as suggested.
- Semi-closed form solutions are still obtainable with the new model.
- More features are incorporated:
  - an optimal portfolio can be decomposed into myopic demand and hedging demand
  - investment horizon matters for portfolio allocation

Thanks

Thank you!