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Modelling Default Risk

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Default Risk, Credit risk: The event in which companies or individuals will be unable to make the required payments on their debt obligations. Lenders and investors are exposed to default risk in virtually all forms of credit extensions. The loss may be complete or partial and can arise in a number of circumstances. For example:

- A consumer may fail to make a payment due on a mortgage loan, credit card, line of credit, or other loan
- A business or consumer does not pay a trade invoice when due
- A business does not pay an employee's earned wages when due
- A business or government bond issuer does not make a payment on a coupon or principal payment when due
- An insolvent insurance company does not pay a policy obligation

http://www.investopedia.com and Wikipedia

Measuring the risk: a Modern Approach J. Hilscher, R. A. Jarrow, and D. R. van Deventer (2008)

• In **structural models**, the default is triggered when the firm value reaches a predetermined level. The firm's **probability of default** depends primarily on two factors:
the size of the firm's asset value relative to the face value of debt if the debt/equity ratio is very high, default is likely;
and how volatile the firm's asset value is

• The idea of the **reduced-form approach** is straightforward; corporate default may be triggered by many different factors, and default may happen at any point in time. The reduced-form **probability of default** is calculated using **all available information** at a given point in time. An advantage of the reduced-form approach is that it can be used to estimate the probability of default not only over the next month but over any period of time; the next year or even the next five years.

The common tool is the **Probability of default** (PD), i.e., the likelihood of a default over a particular time horizon. It provides an estimate of the likelihood that a client of a financial institution will be unable to meet its debt obligations.

PD is a key parameter used in the calculation of economic capital or regulatory capital under Basel II for a banking institution.

To date, the most successful use of credit spread data that we are aware of has been in the cross-sectional estimation of credit spread curves. These curves describe the typical market spread for a given level of credit quality. These data can be combined with a structural model to estimate expected recovery in the event of default. In this way, both modeling approaches can be used to produce better credit analysis tools. Cross-sectional in this context means **combining data from many different firms and issues**.

Moody's KMV

This presentation is devoted to **modeling** default risk. No calibration issues will be discussed, even if this is a main issue.

1. Single Default
2. Multidefaults

Defaultable Claims

Let us first describe a generic defaultable claim on a single default $(X, \widetilde{X}, A, Z, \tau)$:

- **Default** of an entity occurs at time τ. Default may be bankruptcy or other financial distress.
- At maturity T the **promised payoff** X is paid only if the default did not occurred.
- The **promised dividends** A are paid up to default time.
- The **recovery claim** $\widetilde{X}$ is received at time T, if default occurs prior to or at the claim's maturity date T.
- The **recovery process** Z: the r.v. Z_τ specifies the recovery payoff at time of default, if default occurs prior to or at the maturity date T.

The discounted (ex-dividend) price of the defaultable claim is

$$\mathbb{E}(B_T^{-1}(X \mathbb{1}_{T<\tau} + \widetilde{X} \mathbb{1}_{t<\tau\leq T}) + (\int_t^{T\wedge\tau} B_s^{-1} dA_s + (B_\tau)^{-1} Z_\tau) \mathbb{1}_{t<\tau\leq T} | \mathcal{G}_t)$$

where B_t represents the value of the savings account $B_t = \exp \int_0^t r_s ds$, the expectation being under some e.m.m. and $\mathcal{G}_t$ the information available at time t.

Single Default

1. Structural Approach
2. Intensity Based Approach
3. Generalisation
4. Density Approach

Structural Approach

Assuming that some process X (e.g., the value of the firm) is observable, the default time is defined as an $\mathbb{F}^X$ stopping time (e.g., an hitting time for a deterministic boundary).

If the process X is a traded asset and if the associated market (which includes a savings account) is complete, the computation of the price requires the conditional law of τ given $\mathbb{F}^X$, i.e.

$$\mathbb{P}(\tau > u|\mathcal{F}_t^X)$$

If X is continuous (more precisely if all $\mathbb{F}$ martingales are continuous), τ is predictable and, for $T > t$, the quantity $\mathbb{P}(\tau > T|\mathcal{F}_t^X)$ goes to 0 when $t \to \tau, t < \tau < T$, fact that is not observed on the data.

If X is observable (and not continuous) the default time may be not predictable, and closed form formulae for hitting times are difficult to obtain.

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Partial Information.

If the value of the firm is not observable, one has to compute $\mathbb{P}(\tau > u|\mathcal{F}_t^Y)$ where Y is the observation process: this leads to (difficult) **filtering problem** (Runggaldier)

One can assume that the value of the firm is observed only at discrete times (Duffie and Lando, J and Valchev) or observed with some noise (Coculescu et al.)

Guo et al. study a structural model with **delayed information**. Let

$$X_t = x + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dW_s$$

the natural augmentation of the filtration generated by X is $\mathbb{F}$, and τ is the $\mathbb{F}$-predictable stopping time defined by

$$\tau = \inf\{t > 0, X_t \leq b\},$$

If $\delta > 0$, they introduce, for $t > \delta$ the σ -algebra $\widetilde{\mathcal{F}}_t = \mathcal{F}_{t-\delta} \subset \mathcal{F}_t$, and for $0 < t < \delta$, $\widetilde{\mathcal{F}}_t$ is equal to the trivial σ-algebra.

Denoting by $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$ the delayed information

$$\widetilde{G}_t := \mathbb{P}(\tau > t | \widetilde{\mathcal{F}}_t) = K_t \Phi(Y_t, \delta, b)$$

where $K_t = 1\!\!1_{\inf_{s \leq t-\delta} X_s > b}$, $Y_t = X_{t-\delta}$ and$\Phi(x, u, y) = \mathbb{P}_x(\inf_{s \leq u} X_s > y)$.

$$d\widetilde{G}_t \quad = \quad K_t \partial_1 \Phi(Y_t, \delta, b) dY_t + \frac{1}{2} K_t \partial_{1,1} \Phi(Y_t, \delta, b) d\left\langle Y \right\rangle_t.$$

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$$d\widetilde{G}_t = K_t \partial_1 \Phi(Y_t, \delta, b) dY_t + \frac{1}{2} K_t \partial_{1,1} \Phi(Y_t, \delta, b) d\langle Y \rangle_t .$$

Intensity Based Approach

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F}$.
An increasing nonnegative $\mathbb{F}$-adapted process Λ is given.
We assume that there exists a random variable Θ, independent of $\mathcal{F}_\infty$, with an exponential law: $\mathbb{P}(\Theta \geq t) = e^{-t}$.
We define the random time τ as the first time when the process Λ is above the random level Θ, i.e.,

$$\tau = \inf\{t \geq 0 \,:\, \Lambda_t \geq \Theta\}.$$

In particular, $\{\tau > s\} = \{\Lambda_s < \Theta\}$ and $\mathbb{P}(\tau > t | \mathcal{F}_t) = e^{-\Lambda_t}$.

Let $\mathbb{G}$ be the observation filtration, i.e. $\mathcal{G}_t = \cap_{s>t} \mathcal{F}_s \vee \sigma(s \wedge \tau)$, then if $X \in \mathcal{F}_T$

$$\mathbb{E}(X \mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} e^{\Lambda_t} \mathbb{E}(X e^{-\Lambda_T} | \mathcal{F}_t).$$

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$$\mathbb{E}(X 1\!\!1_{\{\tau > T\}} | \mathcal{G}_t) = 1\!\!1_{\{\tau > t\}} \, e^{\Lambda_t} \mathbb{E}(X e^{-\Lambda_T} | \mathcal{F}_t).$$

In that case, immersion property holds, i.e., any $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale.

If Λ is continuous (or simply $\mathbb{F}$-predictable), the process $M_t^\tau := \mathbb{1}_{\tau \le t} - \Lambda_{t\wedge\tau}$ is a $\mathbb{G}$-martingale.

In the case where $\mathbb{F}$ is a Brownian filtration, any $\mathbb{G}$ martingale Y can be written as

$$dY_t = \varphi_t dW_t + \psi_t dM_t^\tau$$

Setting $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = e^{-\Lambda_t}$ and, for a (bounded) predictable process h,

$$Y_t = \mathbb{E}(h_\tau|\mathcal{G}_t) =: h_\tau \mathbb{1}_{\tau \le t} + \mathbb{1}_{t<\tau}\frac{X_t}{G_t}$$

one has

$$dY_t = (h_t - \frac{X_t}{G_t})dM_t^\tau + \varphi_t dW_t$$

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Example where Λ is not continuous.

Let $\mathbb{F}$ be the filtration of a Poisson process N and $\Lambda_t = e^{-N_t}$, and let M be the compensated martingale $M_t = N_t - \lambda t$. The process $M_t^\tau := 1\!\!1_{\tau \leq t} - (1 - \frac{1}{e})\lambda(t \wedge \tau)$ is a $\mathbb{G}$-martingale. Any $\mathbb{G}$ martingale Y can be written as

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$$dY_t = (h_t - \frac{X_t}{G_t}) dM_t^\tau + \varphi_t dM_t$$

Density Approach

Here, we assume that there exists $p_t(u)$ such that

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty p_t(u) f(u) du$$

where f is the density law of τ. Then,

- $G_t := \mathbb{P}(\tau > t | \mathcal{F}_t) = m_t - \int_0^t p_s(s) f(s) ds$ where m is an $\mathbb{F}$-martingale
- $1\!\!1_{\tau \le t} - \int_0^t \frac{p_s(s)}{G_{s-}} f(s) ds =: 1\!\!1_{\tau \le t} - \Lambda_{t \wedge \tau}$ is a $\mathbb{G}$ martingale
- For $X \in \mathcal{F}_T$, the pricing formula is

$$\mathbb{E}(X 1\!\!1_{T<\tau} | \mathcal{G}_t) = 1\!\!1_{t<\tau} \frac{1}{G_t} \mathbb{E}(X G_T | \mathcal{F}_t)$$

- This model corresponds to the case $\tau = \inf\{t : \Lambda_t \ge \Theta\}$ where Θ IS NOT independent from $\mathbb{F}$.

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• This model corresponds to the case $\tau = \inf\{t : \Lambda_t \geq \Theta\}$ where Θ IS NOT independent from $\mathbb{F}$.

- Y is a $\mathbb{G}$ martingale ($Y_t = y_t 1\!\!1_{t<\tau} + y_t(\tau) 1\!\!1_{\tau \le t}$) if and only if
  - (i) for any u, the processes $(y_t(u)p_t(u), t \ge u)$ are $\mathbb{F}$ martingales
  - (ii) $\mathbb{E}(Y_t|\mathcal{F}_t) = y_t G_t + \int_t^\infty y_t(s)p_t(s)f(s)ds$ is an $\mathbb{F}$ martingale
- If X is an $\mathbb{F}$-martingale, then

$$X_t = \widetilde{X}_t + \int_0^{t\wedge\tau} \frac{d\langle X, G\rangle_s}{G_{s-}} + \int_{t\wedge\tau}^t \frac{d\langle X, f_.(\tau)\rangle_s}{f_{s-}(\tau)},$$

where $\widetilde{X}$ is an $\mathbb{G}$-martingale.

- Immersion property is equivalent to $f_t(u) = f_u(u)$ for $t > u$.
- G admits a multiplicative decomposition as $G_t = m_t e^{-\Lambda_t}$ where m is an $\mathbb{F}$-local martingale. If immersion holds, $G_t = e^{-\Lambda_t}$.

Random times with the same intensity

If $G_t = m_t e^{-\Lambda_t}$ where m is an $\mathbb{F}$-local martingale and Λ an $\mathbb{F}$-predictable increasing process, then Λ is the intensity process, i.e., $\mathbb{1}_{\tau \le t} - \Lambda_{t \wedge \tau}$ is a $\mathbb{G}$ martingale, and, if $\Lambda_t = \int_0^t \lambda_s ds$, the intensity rate λ satisfies

$$\lambda_t = \lim \frac{1}{h} \mathbb{P}(t < \tau \le t + h | \mathcal{G}_t)$$

Here, we shall construct three random times having the same intensity (in the same filtration)

Let W be a Brownian motion on the probability space $(\Omega, \mathbb{P})$ and $\mathbb{F}$ be its natural filtration.

1) Let ϑ a random variable independent of $\mathcal{F}_\infty$, with an exponential law of parameter λ. Then, λ is the $\mathbb{F}$-intensity rate of ϑ

2) Let us assume that N is an $\mathbb{F}$ local martingale such that $N_t e^{-\lambda t}$ is valued in $[0,1]$. Then, it is possible to construct a random time τ such that

$$Z_t := \mathbb{P}(\tau > t | \mathcal{F}_t) = N_t e^{-\lambda t}$$

To do that, one needs to construct a candidate to be the conditional law of τ given $\mathcal{F}_t$. This can be done setting

$$\mathbb{P}(\tau \leq \theta | \mathcal{F}_t) = (1 - Z_t) \exp - \int_\theta^t \frac{Z_s}{1 - Z_s} \lambda \, ds$$

It remains to give an example of N which is a martingale such that $N_t e^{-\lambda t}$ is valued in $[0,1]$

We set $N_t = e^{\lambda t} e^{-2Y_t}$ with

$$dY_t = (Y_t + \frac{\lambda}{2}) dt + \sqrt{Y_t} dW_t, \; Y_0 = 0$$

3) Another random time $\widehat{\tau}$ can be constructed so that $Z_t = e^{-\lambda t}$.

Let M^u be the solution of

$$\begin{aligned} dM^u_t &= M^u_t(M^u_t - (1 - e^{-\lambda t}))dW_t, \text{for } t \geq u \\ M^u_u &= Z_u = e^{-\lambda u} \end{aligned}$$

Then, one can construct $\widehat{\tau}$ so that, for $u < t$, $\mathbb{P}(\widehat{\tau} \leq u|\mathcal{F}_t) = M^u_t$

This time $\widehat{\tau}$ is such that any bounded $\mathbb{F}$ martingale stopped at $\widehat{\tau}$ is a $\mathbb{G}$ martingale.

Examples

It is not easy to find examples on a given probability space:
The quantities $P(\tau > u|\mathcal{F}_t) = M_t^u$ are a family of **martingales**, depending of a parameter, **increasing with respect to that parameter**

The quantities $p_t(u)$ are positive martingales such that $\int_0^\infty p_t(u)f(u)du = 1$

A possible construction is to start with a model where τ is independent from $\mathcal{F}_\infty$, with law η and to set $d\mathbb{Q} = \beta_t(\tau)d\mathbb{P}$. Then the $\mathbb{F}$-conditional density of τ under $\mathbb{Q}$ is

$$\mathbb{Q}(\tau \in du|\mathcal{F}_t) = \frac{1}{m_t^\beta}\beta_t(u)f(u)du$$

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Questions, Comments

Pricing of an $\mathcal{F}_T$ measurable payoff X (case $r = 0$)

• Is the price of X equal to $\mathbb{E}(X|\mathcal{F}_t)$ or to $\mathbb{E}(X|\mathcal{G}_t)$? This question is of main importance in case of several defaults.
The price is $\mathbb{E}(X|F_t)$ if there exists an hedging portfolio, hence the second question is does there exist an $\mathbb{F}$-adapted price process? If yes, this price process must be as well a $(\mathbb{P}, \mathbb{F})$ martingale and a $(\mathbb{P}, \mathbb{G})$ martingale under the pricing measure $\mathbb{P}$, and immersion holds true (therefore $\mathbb{E}(X|\mathcal{F}_t) = \mathbb{E}(X|\mathcal{G}_t)$).

• If τ is a totally inaccessible $\mathbb{G}$ stopping time, how to compute the intensity of τ?

• If there is no default-free asset, what is the meaning of $\mathbb{F}$? Pricing of a $\mathcal{G}_T$ measurable payoff of the form $X\mathbb{1}_{T<\tau}$ for $X \in \mathcal{G}_T$ (case $r = 0$) reduces to pricing of $\tilde{X}\mathbb{1}_{T<\tau}$ for $\tilde{X} \in \mathcal{F}_T$ since there exists such a $\tilde{X}$ so that $X\mathbb{1}_{T<\tau} = \tilde{X}\mathbb{1}_{T<\tau}$.

• One can ask the following question. Let $\mathbb{G}$ be a filtration and τ a $\mathbb{G}$ stopping time. Is it possible to find $\mathbb{F}$ such that $\mathbb{G}$ is the enlarged filtration with $\mathbb{F}$ not equal to $\mathbb{G}$. No answer.

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Density approach is useful (only?) for life after default

Assume that

$$\mathbb{P}(\tau \in du|\mathcal{F}_t) = p_t(u)du$$

where $p(u)$ is a family of strictly positive martingales and $G_t = N_t e^{-\Lambda_t}$.

Let $\mathbb{P}^*$ be defined as

$$d\mathbb{P}^*|_{\mathcal{G}_t} = L_t^* d\mathbb{P}|_{\mathcal{G}_t}$$

where L^* is the $(\mathbb{P}, \mathbb{G})$-martingale defined as

$$L_t^* = 1\!\!1_{\{t<\tau\}} + 1\!\!1_{\{t\geq\tau\}} \lambda_\tau e^{-\Lambda_\tau} \frac{N_t}{p_t(\tau)}$$

It can be proved that

$$d\mathbb{P}^*|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t}$$

and that $\mathbb{P}^*$ and $\mathbb{P}$ coincide on $\mathcal{G}_\tau$. Moreover, immersion holds true under $\mathbb{P}^*$, and the intensity of τ is the same under $\mathbb{P}$ and $\mathbb{P}^*$. It follows that

$$\mathbb{E}_{\mathbb{P}}(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = \mathbb{E}^*(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \frac{1}{e^{-\Lambda_t}} \mathbb{E}^*(e^{-\Lambda_T} X|\mathcal{F}_t)$$

Let us now study the pricing of a recovery. Let Z be an $\mathbb{F}$-predictable bounded process.

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Z_\tau 1\!\!1_{\{t<\tau\le T\}}|\mathcal{G}_t) &= 1\!\!1_{\{t<\tau\}}\frac{1}{G_t}\mathbb{E}_{\mathbb{P}}(-\int_t^T Z_u dG_u|\mathcal{F}_t) \\
&= 1\!\!1_{\{t<\tau\}}\frac{1}{G_t}\mathbb{E}_{\mathbb{P}}(\int_t^T Z_u N_u \lambda_u e^{-\Lambda_u} du|\mathcal{F}_t) \\
&= \mathbb{E}^*(Z_\tau 1\!\!1_{\{t<\tau\le T\}}|\mathcal{G}_t) \\
&= 1\!\!1_{\{t<\tau\}}\frac{1}{e^{-\Lambda_t}}\mathbb{E}^*(\int_t^T Z_u \lambda_u e^{-\Lambda_u} du|\mathcal{F}_t)
\end{aligned}$$

The problem is different for pricing a recovery paid at maturity. If both quantities $\mathbb{E}_{\mathbb{P}}(X 1\!\!1_{\tau<T}|\mathcal{G}_t)$ and $\mathbb{E}^*(X 1\!\!1_{\tau<T}|\mathcal{G}_t)$ are the same, this would imply that immersion holds under $\mathbb{P}$.

Hence, non-immersion property is important while evaluating recovery paid at maturity ($\mathbb{P}^*$ and $\mathbb{P}$ do not coincide on $\mathcal{F}_\infty$) or while evaluating equity derivatives. It is also important for multi-default setting, where, in general, immersion does not hold between the various filtrations.

Arbitrages:

under which conditions on τ the $\mathbb{G}$ market is arbitrage free? For answer, attend to C. Fontana talk for honest times in the Brownian case and A. Aksamit for a jump case. Roughly speaking, there are no arbitrages (of the first kind) before τ, there may have arbitrages immediately before τ and immediately after τ. Under specific conditions (in a model with jumping $\mathbb{F}$ martingales), there are no arbitrages (of the first kind) after τ.

Multi Defaults

1. Intensity Based Approach
2. Marked Point Processes
3. Ranked Default Times
4. Ranked Default Times with Reference Filtration
5. A General Construction
6. Exemples

In literature and in practice, default correlation is one key subject in the credit risk analysis.

- bottom-up approach for individual credit names and their dependence
  - intensity correlation models (Duffie and Garleanu) with extension of contagious impact (Jarrow and Yu, Herbertsson)
  - latent variable models with copulas (Li, Frey and McNeil, Laurent et al.)
  - markov dynamic copula models (Bielecki et al.)
- top-down approach for cumulative losses
  - ordered defaults and loss process dynamics (Arnsdorf and Halperin, Cont and Minca, Filipovic and Overbeck, Giesecke et al., Schönbucher...)

Intensity Based Approach

- A first class of models consist of

$$\tau_i = \inf\{t \,:\, \Lambda^i_t \geq \Theta_i\}$$

where Λ^i are $\mathbb{F}$ adapted increasing processes, Θ^i's independent of $\mathbb{F}$ and correlated

- Another class of models (contagion effect)

$$\tau_i = \inf\{t \,:\, \Lambda^i_t \geq \Theta_i\}$$

where Λ^i depends on $(\tau_j, j \neq i)$

Marked Point Processes

We recall some results on **Marked Point Processes** which will be useful for top-down approach for cumulative losses.

A MPP $\mathbb{M}$ is a sequence $(\sigma_k, Y_k)_{k\geq 1}$ where

1. The random variables σ_k satisfy $0 < \sigma_k < \sigma_{k+1}$, $\sigma_0 = 0$
2. The r.vs Y_k (the marks) are valued in $\mathbb{R}^d$

We note $(\mathcal{M}_t, t \geq 0)$ the history of $\mathbb{M}$ (the marked point process filtration generated by $\mathbb{M}$) so that $\mathcal{M}_{\sigma_k} = \sigma\{(\sigma_1, Y_1), \dots, (\sigma_k, Y_k)\}$.

To any MPP, we associate the random measure μ defined as, for $C \in \mathcal{B}(\mathbb{R}^d \setminus 0)$,

$$\mu(]0, t] \times C) = \sum_k \mathbb{1}_{\{(\sigma_k, Y_k) \in]0,t] \times C\}}$$

For any integrable r.v. U, setting $\sigma_0 = 0$, one has

$$\mathbb{E}(U|\mathcal{M}_t) = \sum_{k\geq 0} \mathbb{1}_{\{\sigma_k < t \leq \sigma_{k+1}\}} \frac{\mathbb{E}(\mathbb{1}_{\{t<\sigma_{k+1}\}} U|\mathcal{M}_{\sigma_k})}{\mathbb{P}(t < \sigma_{k+1}|\mathcal{M}_{\sigma_k})}$$

An important tool is $\eta^{k+1|k}(dt, dy)$, **the regular version of the conditional distribution of** (σ_{k+1}, Y_{k+1}) w.r.t. $\mathcal{M}_{\sigma_k}$.

The compensator of the point process $\mathbb{M}$ is the (unique) random measure $\nu(dt, dy)$ such that for any (bounded) predictable function K, the process $K \star (\mu - \nu)$ is a local martingale, where

$$(K \star (\mu - \nu))_t = \int_{]0,t] \times \mathbb{R}^d} K(\cdot; s, y)(\mu(\cdot; ds, dy) - \nu(\cdot; ds, dy))$$

given by

$$\begin{aligned} \nu(dt, dy) &= \sum_{k \geq 0} \mathbb{1}_{\{\sigma_k \leq t < \sigma_{k+1}\}} \frac{\mathbb{P}((\sigma_{k+1}, Y_{k+1}) \in (dt, dx) | \mathcal{M}_{\sigma_k})}{\mathbb{P}(\sigma_{k+1} \geq t | \mathcal{M}_{\sigma_k})} \\ &= \sum_{k \geq 0} \mathbb{1}_{\{\sigma_k \leq t < \sigma_{k+1}\}} \frac{\eta^{k+1|k}(dt, dy)}{\eta^{k+1|k}([t, \infty[\times \mathbb{R}^d)} \end{aligned}$$

Ranked Default Times

We restrict our attention to a finite number of ranked default times $(\sigma_k, k \leq n)$. We set $\sigma_0 = 0, \sigma_{n+1} = \infty$ and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$. This is a MPP (without marks!)

We assume that the law of the vector $\boldsymbol{\sigma}$ has a density $\eta(\boldsymbol{u})$, i.e.,

$$\mathbb{E}[f(\boldsymbol{\sigma})] = \int_{\mathbb{R}_+^n} f(\boldsymbol{u})\eta(\boldsymbol{u})d\boldsymbol{u},$$

Here, we make use of the following notation

- $\boldsymbol{u} = (u_1, \cdots, u_n), \quad \boldsymbol{u}_{(k:p)} = (u_k, \cdots, u_p), \quad \boldsymbol{u}_{(p)} = \boldsymbol{u}_{(1:p)}$
- $d\boldsymbol{u} = du_1 \cdots du_n, \quad d\boldsymbol{u}_{(k:p)} = du_k \ldots du_p$
- $\boldsymbol{u} > \boldsymbol{\theta}$ stands for $u_i > \theta_i$ for all $i \in \{1, \cdots, n\}$
- $\int_{]\boldsymbol{t},+\infty[} f(\boldsymbol{u}_{(k:n)})d\boldsymbol{u}_{(k:n)} := \int_{]t,+\infty[} du_k \cdots \int_{]t,+\infty[} du_n\, f(u_k, \ldots, u_n).$

The (marginal) density of $\boldsymbol{\sigma}_{(k)}$ is

$$\eta^{(k)}(\boldsymbol{u}_{(k)}) = \int_{\mathbb{R}_+^{n-k}} \eta(\boldsymbol{u})d\boldsymbol{u}_{(k+1:n)}$$

Furthermore, on $\sigma_k \leq t < \sigma_{k+1}$

$$\mathbb{P}(\sigma_{k+1} > \theta | \mathcal{M}_t) = \int_\theta^\infty \eta^{k+1|k}(s)ds$$

where

$$\eta^{k+1|k}(s) = \frac{1}{\eta^{(k)}(\boldsymbol{\sigma}_{(k)})} \int_{\mathbb{R}_+^{n-(k+2)}} \eta\big(\boldsymbol{\sigma}_{(k)}, s, \boldsymbol{u}_{(k+2:n)}\big)\, d\boldsymbol{u}_{(k+2:n)}$$

It follows that

$$\mathbb{E}(f(\boldsymbol{\sigma})|\mathcal{M}_t) = \int_{\mathbb{R}_+^n} f(\boldsymbol{u})\eta_t^{\mathcal{M}}(d\boldsymbol{u})$$

where, on the set $\sigma_k \leq t < \sigma_{k+1}$

$$\eta_t^{\mathcal{M}}(d\boldsymbol{u}) = \frac{\mathbb{1}_{\{\boldsymbol{t}<\boldsymbol{u}_{(k+1:n)}\}}}{\int_t^\infty \eta^{k+1|k}(s)ds} \delta_{\boldsymbol{\sigma}_{(k)}}(d\boldsymbol{u}_{(k)})\; \eta(\boldsymbol{u}_{(k)}, \boldsymbol{u}_{(k+1:n)})d\boldsymbol{u}_{(k+1:n)}$$

Let $N_t = \sum_{k=1}^{n} 1\!\!1_{\{\sigma_k \le t\}}$. The compensator of N is

$$\Lambda_t = \int_0^{t\wedge\sigma_n} \lambda_s ds = \sum_{k=0}^{n-1} \int_{\sigma_k}^{t\wedge\sigma_{k+1}} \frac{1}{\int_s^\infty \eta^{k+1|k}(y)dy} \eta^{k+1|k}(s)ds$$

where

$$\eta^{k+1|k}(s) = \frac{1}{\eta^{(k)}(\boldsymbol{\sigma}_{(k)})} \int_{\mathbb{R}_+^{n-(k+2)}} \eta(\boldsymbol{\sigma}_{(k)}, s, \boldsymbol{u}_{(k+2:n)})\, d\boldsymbol{u}_{(k+2:n)}$$

Ranked Default Times with Reference Filtration

We assume now that a reference filtration $\mathbb{F}$ is given and that there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable functions $(\omega, \boldsymbol{u}) \to \eta_t(\omega, \boldsymbol{u})$ such that

$$\mathbb{E}[f(\boldsymbol{\sigma})|\mathcal{F}_t] = \int_{\mathbb{R}_+^n} f(\boldsymbol{u})\eta_t(\boldsymbol{u})d\boldsymbol{u},$$

We denote by $\mathbb{G}$ the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{M}_t$.

It can be useful to keep in mind that, if one defines

$$d\mathbb{Q}|_{\mathcal{F}_t \vee \sigma(\boldsymbol{\sigma})} = \frac{1}{\eta_t(\boldsymbol{\sigma})} d\mathbb{P}|_{\mathcal{F}_t \vee \sigma(\boldsymbol{\sigma})}$$

then, $\mathbb{F}$ and $\boldsymbol{\sigma}$ are independent under $\mathbb{Q}$, and $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$.

Ranked Default Times with Reference Filtration

We assume now that a reference filtration $\mathbb{F}$ is given and that there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^n_+)$-measurable functions $(\omega, \boldsymbol{u}) \to \eta_t(\omega, \boldsymbol{u})$ such that

$$\mathbb{E}[f(\boldsymbol{\sigma})|\mathcal{F}_t] = \int_{\mathbb{R}^n_+} f(\boldsymbol{u})\eta_t(\boldsymbol{u})d\boldsymbol{u},$$

We denote by $\mathbb{G}$ the filtration $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{M}_t$.

It can be useful to keep in mind that, if one defines

$$d\mathbb{Q}|_{\mathcal{F}_t \vee \sigma(\boldsymbol{\sigma})} = \frac{1}{\eta_t(\boldsymbol{\sigma})} d\mathbb{P}|_{\mathcal{F}_t \vee \sigma(\boldsymbol{\sigma})}$$

then, $\mathbb{F}$ and $\boldsymbol{\sigma}$ are independent under $\mathbb{Q}$, and $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$.

The marginal density of $\boldsymbol{\sigma}_{(k)}$ with respect to $\mathcal{F}_t$ is given by

$$\eta_t{}^{(k)}\left(\boldsymbol{u}_{(k)}\right) = \int_{\mathbb{R}_+^{n-k}} \eta_t(\boldsymbol{u}) d\boldsymbol{u}_{(k+1:n)}$$

and, on $\sigma_k \leq t < \sigma_{k+1}$,

$$\mathbb{P}(\sigma_{k+1} > \theta | \mathcal{G}_t) = \int_\theta^\infty \eta_t^{k+1|k}(s) ds$$

where

$$\eta_t^{k+1|k}(s) = \frac{1}{\eta_t^{(k)}\left(\boldsymbol{\sigma}_{(k)}\right)} \int_{\mathbb{R}^{n-(k+2)}} \eta_t\left(\boldsymbol{\sigma}_{(k)}, s, \boldsymbol{u}_{(k+2,n)}\right) d\boldsymbol{u}_{(k+2,n)}$$

It follows that

$$\mathbb{E}(f(\boldsymbol{\sigma})|\mathcal{G}_t) = \int_{\mathbb{R}_+^n} f(\boldsymbol{u}) \mu_t^{\mathcal{G}}(d\boldsymbol{u})$$

where, on the set $\sigma_k \leq t < \sigma_{k+1}$

$$\mu_t^{\mathcal{G}}(d\boldsymbol{u}) = \frac{\mathbb{1}_{\{t<\boldsymbol{u}_{(k+1,n)}\}}}{\int_t^\infty \eta_t{}^{k+1|k}(s)ds} \delta_{\boldsymbol{\sigma}_{(k)}}\left(d\boldsymbol{u}_{(k)}\right) \eta_t(\boldsymbol{u}) \, d\boldsymbol{u}_{(k+1,n)}$$

Furthermore, for $Y_T(\boldsymbol{u})$ a family of positive $\mathcal{F}_T$ adapted random variables,

$$\mathbb{E}(Y_T(\boldsymbol{\sigma})|\mathcal{G}_t) = \int_{\mathbb{R}_+^n} \frac{1}{\eta_t(\boldsymbol{u})} \mathbb{E}(Y_T(\boldsymbol{u})\eta_T(\boldsymbol{u})|\mathcal{F}_t)\mu_t(d\boldsymbol{u})$$

$$= \sum_{k=0}^{n-1} \frac{\mathbb{1}_{\{\sigma_k \leq t < \sigma_{k+1}\}}}{\int_t^\infty \eta_t^{k+1|k}(s)ds} \int_{\boldsymbol{t}}^\infty \mathbb{E}(Y_T(\boldsymbol{u})\eta_T(\boldsymbol{u})|\mathcal{F}_t)|_{\boldsymbol{u}_{(k)}=\boldsymbol{\sigma}_{(k)}} d\boldsymbol{u}_{(k+1,n)}$$

Let $N_t = \sum_{k=1}^{n} \mathbb{1}_{\{\sigma_k \le t\}}$. The compensator of N in the filtration $\mathbb{G}$ is

$$\Lambda_t = \int_0^{t\wedge\sigma_n} \lambda_s ds = \sum_{k=0}^{n-1} \int_{\sigma_k}^{t\wedge\sigma_{k+1}} \frac{1}{\int_s^\infty \eta_s^{k+1|k}(y)dy} \eta_s^{k+1|k}(s)ds = \sum_{k=0}^{n-1} \int_{\sigma_k}^{t\wedge\sigma_{k+1}} \lambda_s^k ds$$

Note that λ_s^k depends on $\boldsymbol{\sigma}_{(k)}$.

A General Construction

The random variable Ξ is a random variable of law η taking values in a complete metric space E with countable base and equipped with Borel σ-algebra $\mathcal{B}(E)$. The main example is $\Xi = (\tau_k, Y_k)_{1\leq k\leq n}$ where $\boldsymbol{\tau}$ is a sequence (not necessarily ranked) of random times and Y_k some marks.

Without loss of generality, we assume that Ξ is the canonical map from E in E, defined as $\Xi(\chi) = \chi$ so that $\mathbb{E}(f(\Xi)) = \int_E f(\chi)\eta(d\chi)$ where η is the law of Ξ.

- The "default-free" information is represented by a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$.
- We denote be $\boldsymbol{\sigma}$ the ranked sequence of times, the filtration $\mathcal{M}_t$ is the one of the associated MMP $\mathbb{M} = (\sigma_k, Y_{\sigma_k})_k$.
- The filtration $\mathbb{F}$ is considered as well on Ω or on the product space.
- The filtration $\mathbb{J}$ is defined as $\mathcal{J}_t = \mathcal{F}_t \otimes \sigma(\Xi)$.
- The filtration $\mathbb{G}$ is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{M}_t$.

All the filtrations are defined in such a way that they satisfy usual conditions.

We start with the fundamental case where the two sources of risks are independent (i.e., the random variable Ξ is independent from $\mathcal{F}_\infty$), the probability measure is the product measure $\overline{\mathbb{P}}^0(d\omega, d\chi) = \mathbb{P}(d\omega) \otimes \eta(d\chi)$.

The conditional law of Ξ given $\mathcal{M}_t$ is denoted by $\eta_t^{\mathcal{M}}$.

Given a non-negative measurable function Y on $\Omega \times E$,

$$\overline{\mathbb{E}}^0[Y(.,\Xi)|\mathcal{F}_\infty \vee \mathcal{M}_t] = \int_E Y(.,\chi)\eta_t^{\mathcal{M}}(d\chi) =: \eta_t^{\mathcal{M}}(Y)$$

which is $\mathcal{F}_\infty \vee \mathcal{M}_t$-measurable.

One should take care about the notation: $\eta^{\mathcal{M}}$ refers to the filtration $\mathcal{F}_\infty \vee \mathcal{M}_t$ and not to $\mathcal{M}_t$.

Note that, from the independence assumption, $\overline{\mathbb{E}}^0(f(\Xi)|\mathcal{M}_t) = \overline{\mathbb{E}}^0(f(\Xi)|\mathcal{M}_t \vee \mathcal{F}_\infty)$.

Given a non-negative measurable function Y on $\Omega \times E$ (that is $(\omega, \chi) \to Y(\omega, \chi)$), there exists a family of $\mathbb{F}$-adapted processes, parameterized by χ, say $Y^{\mathcal{F}}(\chi)$, such that $\mathbb{P}$-a.s, for any $\chi \in E$ and for any $t \geq 0$, $Y_t^{\mathcal{F}}(\chi) = \mathbb{E}[Y(\cdot, \chi)|\mathcal{F}_t]$.

An useful example is $Y = Xh(\Xi)$ where $X \in \mathcal{F}_\infty$.

We shall call $Y^{\mathcal{F}}$ the universal version of conditional expectation.

One has $\overline{\mathbb{E}}^0(Y|\mathcal{J}_t) = Y_t^{\mathcal{F}}(\Xi)$ and, for any $\mathcal{J}_t$-measurable r.v. Y_t

$$\overline{\mathbb{E}}^0(Y_t|\mathcal{G}_t) = \int_E Y_t(\chi)\eta_t^{\mathcal{M}}(d\chi) =: \eta_t^{\mathcal{M}}(Y_t).$$

Consider now a non-negative measurable random variable Y on $\Omega \times E$. The calculation of its conditional expectation w.r.t. $\mathcal{G}_t$ can be done in two different ways as shown below:

On the one hand, using the notation of the universal martingale

$$\overline{\mathbb{E}}^0[Y|\mathcal{G}_t] = \overline{\mathbb{E}}^0[\overline{\mathbb{E}}^0[Y|\mathcal{H}_t]\,|\,\mathcal{G}_t] = \overline{\mathbb{E}}^0[Y_t^{\mathcal{F}}\,|\,\mathcal{G}_t] = \eta_t^{\mathcal{M}}(Y_t^{\mathcal{F}})$$

On the other hand, using the intermediary σ-algebra $\mathcal{F}_\infty \vee \mathcal{M}_t$

$$\overline{\mathbb{E}}^0[Y|\mathcal{G}_t] = \overline{\mathbb{E}}^0[\overline{\mathbb{E}}^0[Y|\mathcal{F}_\infty \vee \mathcal{M}_t]\,|\,\mathcal{G}_t] = \overline{\mathbb{E}}^0[\eta_t^{\mathcal{M}}(Y)|\mathcal{G}_t] = (\eta_t^{\mathcal{M}}(Y))_t^{\mathcal{F}}$$

In the general case, we characterize the dependence between Ξ and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ by a change of probability w.r.t. the probability measure $\overline{\mathbb{P}}^0$.

We suppose that there exist a positive $\mathbb{F} \otimes \sigma(\Xi)$-martingale $\beta_t(\omega, \Xi)$ with expectation under $\overline{\mathbb{P}}^0$ equal to 1 and we define the probability measure $\overline{\mathbb{P}}$ on $\mathcal{J}_t$ by

$$\overline{\mathbb{P}}(d\omega, d\chi) = \beta_t(\omega, \chi)\overline{\mathbb{P}}^0(d\omega, d\chi)$$

In the following, we suppose the process $\beta^{\mathcal{F}} > 0$ where $\beta_t^{\mathcal{F}}(\chi) = \overline{\mathbb{E}}^0(\beta_T(\cdot, \chi)|\mathcal{J}_t)$.

We can generate different types of density processes depending on the structure information:

$$
\begin{aligned}
\beta_t^{\mathcal{J}} &= \overline{\mathbb{E}}^0[\beta_T|\mathcal{J}_t] = \beta_t^{\mathcal{F}}(\Xi) \\
\beta_t^{\mathcal{M}} &= \overline{\mathbb{E}}^0[\beta_T|\mathcal{F}_\infty \vee \mathcal{M}_t] = \eta_t^{\mathcal{M}}(\beta_T) \\
\beta_t^{\mathcal{G}} &= \overline{\mathbb{E}}^0[\beta_T|\mathcal{G}_t] = (\beta_t^{\mathcal{M}})_t^{\mathcal{F}} = \eta_t^{\mathcal{M}}(\beta_t^{\mathcal{F}})
\end{aligned}
$$

Then

$$
\begin{aligned}
\overline{\mathbb{E}}(f(\Xi)|\mathcal{M}_t \vee \mathcal{F}_\infty) &= \int_E f(\chi)\overline{\eta}_t^{\mathcal{M}}(d\chi) = \int_E f(\chi)\frac{\beta_T(\chi)\eta_t^{\mathcal{M}}(d\chi)}{\beta_t^{\mathcal{M}}} \\
\overline{\mathbb{E}}(f(\Xi)|\mathcal{G}_t) &= \int_E f(\chi)\overline{\eta}_t^{\mathcal{G}}(d\chi) = \int_E f(\chi)\frac{\beta_t^{\mathcal{F}}(\chi)\eta_t^{\mathcal{M}}(d\chi)}{\beta_t^{\mathcal{G}}}
\end{aligned}
$$

and, for any integrable $\mathcal{G}_T$ measurable random variable Y_T

$$
\overline{\mathbb{E}}[Y_T \mid \mathcal{G}_t] = \frac{\overline{\mathbb{E}}^0[Y_T\beta_T|\mathcal{G}_t]}{\overline{\mathbb{E}}^0[\beta_T|\mathcal{G}_t]} = \frac{\eta_t^{\mathcal{M}}((Y_T\beta_T)_t^{\mathcal{F}})}{\eta_t^{\mathcal{M}}(\beta_t^{\mathcal{F}})}
$$

$\mathcal{J}_t = \mathcal{F}_t \vee \sigma(\Xi)$

Examples

Giesecke Example

Let $\tau_k, k = 1, \ldots, N$ be a family of independent random times, with a unit exponential law, independent of $\mathbb{F}$ and, for any i,

$$\mathbb{G}^{(-i)} = \mathbb{F} \vee \mathbb{H}^k, k \neq i$$

where $\mathcal{H}_t^k = \sigma(\tau_k \wedge t)$. The processes $M_t^k = H_t^k - (t \wedge \tau_k)$ are $\mathbb{G}$ martingales. We assume that λ^i is a $\mathbb{G}^{-i}$ adapted non negative process

$$d\lambda_t^i = -\alpha^i(\lambda_t^i - \bar{\lambda}^i)dt + \sigma^i dW_t + dL_t^{-i}$$

where $L_t^{(-i)} = \sum_{k \neq i} \mathbb{1}_{\tau_k \leq t}$.

We define ζ as $d\zeta_t = \zeta_{t-} \sum(\lambda_t^i - 1)dM_t^i$, and $d\mathbb{Q} = \zeta_t d\mathbb{P}$.

Under $\mathbb{Q}$, the intensity of τ_k is λ^k, and $\tau_k = \inf\{t : \int_0^t \lambda_s^k ds > \Theta_k\}$ where $\Theta_k = \int_0^{\tau_k} \lambda_s^k ds$ are independent r.vs with exponential law.

W is a $(\mathbb{Q}, \mathbb{G})$ Brownian motion.

Gaussian model

Let $f_i, i = 1, \dots, n$ be a family of functions with L^2 norm equal to 1 and $X_i = \int_0^\infty f_i(s)dB_s^i$ where B^i are $\mathbb{F}$-BMs with correlation $\rho^{i,j}$.

Then

$$\mathbb{P}(X_i > \theta_i, \forall i = 1, \dots, n | \mathcal{F}_t)$$

$$= \Phi_n^*(\frac{\theta_1}{\sqrt{1-\rho_1^2}} - m_t^1, \dots, \frac{\theta_n}{\sqrt{1-\rho_n^2}} - m_t^n; \gamma(t))$$

where

- $m_t^i = \int_0^t f_i(s)dB_s^i$
- $\Phi_n^*(x_1, \dots, x_n; \gamma(t)) = \mathbb{P}(G_i^{(t)} > x_i, \forall i = 1, \dots, n)$

where $G^{(t)} = (G_i^{(t)}, i = 1, \dots, n)$ is a Gaussian vector, centered, with covariance matrix $\gamma(t)$ with

$$\gamma_{i,j}(t) = \int_t^\infty f_i(s) f_j(s) \rho^{i,j} ds \,.$$

Let K_i be an increasing function from $\mathbb{R}$ to $\mathbb{R}^+$ with inverse k_i and $\tau_i = K_i(X_i)$.

Then

$$\mathbb{P}(\tau_i > t_i, \forall i = 1, \ldots, n | \mathcal{F}_t)$$

$$= \Phi_n^*\Big(\frac{k_1(t_1)}{\sqrt{1-\rho_1^2}} - m_t^1, \ldots, \frac{k_n(t_n)}{\sqrt{1-\rho_n^2}} - m_t^n; \gamma(t), t\Big)$$

In particular,

$$\mathbb{P}(\tau_i > t_i) = \Phi^*\Big(\frac{k_i(t_i)}{\sqrt{1-\rho_i^2}}\Big)$$

where $\Phi^*(x)$ is the survival function of a standard Gaussian law.

Uniform law (From Kchia and Larson)

One starts with r.v. U_i, with exponential law, independent from $\mathbb{F}$ and R a r.v. with given conditional density $p_t(r)$. Set $\tau_i = RU_i$. Then

$$\mathbb{P}(\tau_i > t_i, i = 1, \ldots, n|\mathcal{F}_t) = \int p_t(r) \prod_{i=1}^{n} (1 - \frac{t_i}{r})^+$$

Hawkes processes

An inhomogeneous Poisson process is a counting process such that $N_t - \int_0^t \lambda(s)ds$ where λ is a deterministic function. A self-exciting process N is a counting process with intensity

$$\lambda_t = \lambda^0(t) + \sum_{s \le t} g(t-s)dN_s = \lambda^0(t) + \sum_{k, T_k \le t} g(t - T_k)$$

In this specification, the intensity of N is updated with default information along the path. The construction of N can be done using change of time procedure.

Joint defaults

The density approach does not allow joint default, which are of main importance in a counterparty approach.

We consider a Markovian model of credit risk in which simultaneous defaults are possible.

We model the pair $H = (H^1, H^2)$ as an inhomogeneous Markov chain with state space $E = \{(0,0),(1,0),(0,1),(1,1)\}$, and generator matrix at time t given by the following matrix $A(t)$, where the first to fourth rows (or columns) correspond to the four possible states $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ of H_t :

$$A(t) = \begin{bmatrix} -(\ell_1(t)+\ell_2(t)+\ell_3(t)) & \ell_1(t) & \ell_2(t) & \ell_3(t) \\ 0 & -(\ell_2(t)+\ell_3(t)) & 0 & \ell_2(t)+\ell_3(t) \\ 0 & 0 & -(\ell_1(t)+\ell_3(t)) & \ell_1(t)+\ell_3(t) \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

where ℓ's are deterministic functions of time.

Let us further introduce the processes $H^{\{1\}}$, $H^{\{2\}}$ and $H^{\{1,2\}}$ standing for the indicator processes of a default of the firm alone, of the counterpart alone, and of a simultaneous default of the firm and the counterpart, respectively.

$$H_t^{\{1\}} = \mathbb{1}_{\tau_1 \leq t, \tau_1 \neq \tau_2} \ , \ H_t^{\{2\}} = \mathbb{1}_{\tau_2 \leq t, \tau_1 \neq \tau_2} \ , \ H_t^{\{1,2\}} = \mathbb{1}_{\tau_1 = \tau_2 \leq t} \ .$$

The $\mathbb{H}$-intensity of H^ι is of the form $q_\iota(t, H_t) = q_\iota(t, H_t^1, H_t^2)$ for a suitable function $q_\iota(t, h)$:

$$\begin{aligned} q_{\{1\}}(t,h) &= 1\!\!1_{h_1=0}\left(1\!\!1_{h_2=0}\ell_1(t) + 1\!\!1_{h_2=1}(\ell_1+\ell_3)(t)\right) \\ q_{\{2\}}(t,h) &= 1\!\!1_{h_2=0}\left(1\!\!1_{h_1=0}\ell_2(t) + 1\!\!1_{e_1=1}(\ell_2+\ell_3)(t)\right) \\ q_{\{1,2\}}(t,h) &= 1\!\!1_{h=(0,0)}\ell_3(t)\ . \end{aligned}$$

The processes M^i defined by, for $i = 1, 2$,

$$M_t^i = H_t^i - \int_0^t (1 - H_s^i)(\ell_i + \ell_3)(s)ds\ ,$$

are $\mathbb{H}$-martingales.

The processes H^1 and H^2 are $\mathbb{H}$-Markov processes

One has,

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = \exp\left(-\int_0^s \ell_1(u)du - \int_0^t \ell_2(u)du - \int_0^{s\vee t} \ell_3(u)du\right)$$

Thank you for your attention