Market with Memory: pricing and sensitivity analysis

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Giulia Di Nunno
CMA, University of Oslo

Based on work with:
F. Proske and D. Ruiz-Baños (Univ. Oslo)
M. Moschetta and E. L'Aurora (Univ. Roma Tor Vergata)

Market with memory

Analysis on different market data shows effects that can be explained by taking memory into account in the modeling. Here we can refer to:

- The study of volatility in financial markets as well as commodity markets shows evidence of a dependence on time with no deterministic pattern. See e.g. Akgiray (1989), Bates (1996), Bernard and Thomas (1989), Rubinstein (1994), Scott (1987).
- Studies on equity market microstructure, order books, with focus on liquidity, as Bouchaud et al. (2009) and Gatheral et al. (2010) reveal the presence of permanent effects mostly due to information releases and the impact of large traders. These lead to questions of optimal execution.
- Commodity markets have evidence of specific periodic fluctuations that can be caused by time-delay influences: e.g. the time to transport or time to construct has impact on prices. This delays enter directly the market dynamics as they both effect and are partially motivated by demand and supply rules. See Hale and Verduyn Lunel (1993).

So to address some of these features models with delay, memory (or hereditary structure) were introduced. Here we can refer to:

- M. Arriojas, Y. Hu, S-E A. Mohammed, and G. Pap (2007), M-H. Chang and R. K. Youree (1999, 2007), Y. Kazmerchuk, A. Swishchuk, and J. Wu (2005), G. Stoica (2005)
- Fruth et al. (2011), Huberman and Stanzi (2005), Malo and Pennanen (2010)
- Küchler and Platen (2007)

All these models are represented by *stochastic functional differential equations* (SFDE).

Outlines

1. The model via SFDEs

2. Risk-neutral approach: pricing and sensitivity analysis

3. Benchmark approach: pricing and sensitivity analysis

References

1. The model via SFDEs

Complete probability space $(\Omega, \mathcal{F}, P)$
A m-dim dimensional Brownian motion $W(t)$, $t \in [0, T]$
Its P-augmented natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$. We set $\mathcal{F} = \mathcal{F}_T$.

Define $M_2 := L^2(\Omega, \mathcal{M}_2)$, where

$$\mathcal{M}_2 := \mathcal{M}_2^0 := \mathbb{R}^d \oplus L^2([-r,0], \mathbb{R}^d)$$

with the norm

$$\|(v,\varphi)\|_{\mathcal{M}_2} = \left(|v|^2 + \|\varphi\|^2_{L^2([-r,0],\mathbb{R}^d)}\right)^{1/2}, \varphi \in \mathcal{M}_2.$$

Notice: we say that $\eta \in M_2$ when

$$E\left(|\eta(0)|^2 + \|\eta\|^2_{L^2([-r,0],\mathbb{R}^d)}\right) < \infty.$$

We consider a *risk-less bond*

$$dB(t) = B(t)\kappa(t)dt \quad \text{with } B(0) = 1$$

where $\kappa \in L^1([0,T], \mathbb{R}^+)$, and a *risky asset* modeled by an SFDE:

$$\begin{cases} dS(t) = S(t)\{\mu(t, S_t)dt + \sigma(t, S_t)dW(t)\}, & t \in (0,T] \\ S_0 = \eta & (\mathcal{F}_0\text{-measurable in } M_2) \end{cases}$$

Here $S.$ represents the *segment* of the past history:

$$\forall(t,\omega) \qquad S_t(\omega, u) := S(\omega, t+u),\ u \in [-r, 0] \quad \in \mathcal{M}_2$$

Denote, for each ω,

$$f(\cdot, S.) := S(\cdot)\mu(\cdot, S.) : [0,T] \times \mathcal{M}_2 \to \mathbb{R}^d$$

and

$$g(\cdot, S.) := S(\cdot)\sigma(\cdot, S.) : [0,T] \times \mathcal{M}_2 \to \mathbb{R}^{d\times m}$$

For every t, the matrix $g(t, S_t)$ is assumed to have full rank.

SFDEs: existence and uniqueness

The SFDE:

$$(1) \qquad \begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), & t \in (0, T] \\ x_0 = \eta & (\mathcal{F}_0\text{-measurable in } M_2) \end{cases}$$

Hypotheses (E):

(i) (Local Lipschitzianity): $\forall n \geqslant 0$, $\exists L_n > 0$ such that:

$$|f(t, \varphi_1) - f(t, \varphi_2)|_{\mathbb{R}^d} + \|g(t, \varphi_1) - g(t, \varphi_2)\|_{\mathbb{R}^{d\times m}} \leqslant L_n \|\varphi_1 - \varphi_2\|_{\mathcal{M}_2}$$

for all $t \in [0, T]$ and $\varphi_1, \varphi_2 \in \mathcal{M}_2$: $\|\varphi_1\|_{\mathcal{M}_2} \leqslant n$, $\|\varphi_2\|_{\mathcal{M}_2} \leqslant n$.

(ii) (Linear growth): $\exists C > 0$ such that:

$$|f(t, \psi)|_{\mathbb{R}^d} + \|g(t, \psi)\|_{\mathbb{R}^{d\times m}} \leqslant C\,(1 + \|\psi\|_{\mathcal{M}_2})$$

for all $t \in [0, T]$ and $\psi \in \mathcal{M}_2$.

Existence and uniqueness of solution:
Under hypotheses (E), and given η, the SFDE (1) has a solution ${}^{\eta}x = {}^{\eta}x^0$ that is a stochastic process:

$$ {}^{\eta}x : \Omega \times [-r, T] \longrightarrow \mathbb{R}^d $$

with ${}^{\eta}x(t) = \eta(t)$ for all $t \in [-r, 0]$ P-a.s. such that

$$ {}^{\eta}x \in L^2(\Omega, \mathcal{M}_2^T), \qquad \mathcal{M}_2^T := \mathbb{R}^d \oplus L^2([-r, T], \mathbb{R}^d) $$

Moreover it admits an adapted modification:

$$ \forall t, \quad {}^{\eta}x(t, \cdot) \text{ is } \mathcal{F}_t\text{-measurable.} $$

Given η, the solution is unique as element in $L^2(\Omega, \mathcal{M}_2^T)$ admitting adapted representative.

From now on we assume the existence of a solution to the given SFDEs.

We can consider the SFDE starting at s:

$$(2) \qquad \begin{cases} dx(t) = f(t, x_t)dt + g(t, x_t)dW(t), & t \in (s, T] \\ x_s = \eta & (\mathcal{F}_s\text{-measurable in } M_2) \end{cases}$$

Under assumption (E) the solution of (2) exists and it is unique as element in $L^2(\Omega, \mathcal{M}_2^T)$ admitting adapted representative.

The solution is denoted: ${}^{\eta}x^s = {}^{\eta}x^s(t)$, $t \in [s - r, T]$.

Price functionals and derivatives

Consider a bounded function $\Phi : \mathcal{M}_2 \to \mathbb{R}^+$ and a fixed time $T > 0$.
Price functionals. Our targets are functionals of the form:

$$p(\eta) = E\left[\Phi({}^{\eta}x(T), {}^{\eta}x_T)\right], \quad \eta \in \mathcal{M}_2$$

where ${}^{\eta}x$ is the solution of an SFDE with initial condition η.

Delta. We will compute quantities of type:

$$\Delta_\eta := Dp(\eta) = E\left[D\Phi({}^{\eta}x(T), {}^{\eta}x_T)\right]. \tag{3}$$

However, $Dp(\eta)$ is a functional. If we want an index, then there is choice.
To explain, one can choose a directional derivative:

$$\Delta_h := \lim_{\varepsilon \to 0} \frac{p(\eta + \varepsilon h) - p(\eta)}{\varepsilon} = \frac{d}{d\varepsilon} p(\eta + \varepsilon h)\Big|_{\varepsilon=0}, \quad h \in \mathcal{M}_2$$

and this represents the rate of change near η along the direction h. Or one could consider a form of "worst case scenario" of type:

$$\Delta := |||Dp(\eta)||| := \sup_{\substack{\psi \in \mathcal{M}_2 \\ \psi \neq 0}} \frac{|Dp(\eta)(\psi)|}{\|\psi\|_{\mathcal{M}_2}}.$$

SFDEs: differentiability

References:

- ω-wise Fréchet differentiability of ${}^{\eta}x(t)$ and ${}^{\eta}x_t$ is discussed in Mohammed and Schetzow (2003).
- Malliavin differentiability of ${}^{\eta}x(t)$ and ${}^{\eta}x_t$ is discussed in Yan and Mohammed (2005), Pronk and Veraar (2004), and Carmona and Tehranchi (2006).

Goal: we want to draw a relationship between the two types of derivative.

To this aim we introduce the following operators:

$$X_t^s(\eta,\omega) := \left({}^{\eta(\omega)}x^s(t)(\omega),\ {}^{\eta(\omega)}x_t^s(\omega) \right) \in \mathcal{M}_2 \tag{4}$$

for all $s \leqslant t$, $\eta \in M_2$, $\omega \in \Omega$.

Semigroup property: $X_t^0 = X_t^s \circ X_s^0$.

Under the following assumptions (D), X represents a Fréchet differentiable stochastic flow associated to the SFDEs above.

Hypotheses (D):

(i) the functionals f is jointly continuous and it has continuous Fréchet partial derivatives with respect to the second argument, i.e. for all $t \in [0, T]$, the following bounded linear operators exist:

$$\begin{aligned} Df(t,\cdot) : \mathcal{M}_2 &\longrightarrow L(\mathcal{M}_2, \mathbb{R}^d) \\ \varphi \quad &\longrightarrow Df(t,\varphi) \end{aligned}$$

$$\begin{aligned} Dg(t,\cdot) : \mathcal{M}_2 &\longrightarrow L(\mathcal{M}_2, \mathbb{R}^{d\times m}) \\ \varphi \quad &\longrightarrow Dg(t,\varphi) \end{aligned}$$

(ii) $Df(t,\cdot)$ is continuous uniformly with respect to t, $Dg(t,\cdot)$ is continuous and globally bounded

Result from [MS2003]: Under Hypotheses (D), for coefficients f such that there exist some constants $C := C(T) > 0$ and $\gamma := \gamma(T) \in [0, 1)$ such that

$$|f(t, \varphi)| \leqslant C \left(1 + \|\varphi\|_{M_2}^{\gamma}\right)$$

we can guarantee the Fréchet differentiability of $X_t^s(\varphi, \omega)$ with respect to $\varphi \in \mathcal{M}_2$:

$$X_t^s(\varphi, \omega) \in \mathcal{M}_2$$
$$DX_t^s(\varphi, \omega) \in L(\mathcal{M}_2, \mathcal{M}_2)$$

this gives the differentiability of the solution and the segment of the SFDE.

Remark: alternative sufficient conditions can be given.

Result see [CT2006], [PV2004], [YM2005]: Under assumption (D), we have also differentiability of the solution ${}^{\eta}x(t)$ and its segments ${}^{\eta}x_t$ in the sense of Malliavin $\mathcal{D}$.

Theorem

Given the assumptions before. For any $0 \leqslant s \leqslant t \leqslant T]$ and $\eta \in \mathcal{M}_2$, we have that $X_t^0(\eta) = \left({}^{\eta}x(t), {}^{\eta}x_t\right)$ is Malliavin differentiable and $X_t^s(\eta, \omega) = \left({}^{\eta}x^s(t, \omega), {}^{\eta}x_t^s(\omega)\right)$ is Fréchet differentiable at η for all ω. Moreover, we have the following relationship:

$$DX_t^s\left(X_s^0(\eta,\omega),\omega\right) = \left(\mathcal{D}_s X_t^0(\eta,\cdot)\right)(\omega)\, g_R^{-1}\left(s, X_s^0(\eta,\omega)\right)\rho_0 \qquad \omega - a.e.$$

where $\rho_0 : \mathcal{M}_2 \longrightarrow \mathbb{R}^d$ is the "evaluation at 0", i.e. $\rho_0(\varphi(0), \varphi) = \varphi(0)$. Here g_R^{-1} denotes the right-inverse of the $(d \times m)$-matrix g. The term $DX_t^s\left(X_s^0(\eta,\omega),\omega\right)$ stands for the Fréchet derivative of X_t^s, given ω, evaluated at $X_s^0(\eta,\omega)$.

Proof. Based on chain rules for Malliavin derivative and Fréchet derivatives, the semigroup property of $\{X_t^s\}_{s \leqslant t}$ plus unicity of the solution.

Corollary. In the conditions of the Theorem we also have:

$$DX_t^0(\eta,\omega) = \left(\mathcal{D}_s X_t^0(\eta,\cdot)\right)(\omega)\, g_R^{-1}\left(s, X_s^0(\eta,\omega)\right)\rho_0 \circ DX_s^0(\eta,\omega)$$

Sensitivity analysis: Delta

Our targets are functionals of the form:

$$p(\eta) = E\left[\Phi({}^{\eta}x(T), {}^{\eta}x_T)\right] = E\left[\Phi(X_T^0(\eta))\right], \quad \eta \in \mathcal{M}_2$$

The Delta operator is of the form:

$$\Delta_\eta := Dp(\eta) = E\left[D\Phi(X_T^0(\eta))\right].$$

Proposition.
Let $a : [0, T] \longrightarrow \mathbb{R}$ integrating to 1, and assume Φ Fréchet differentiable C^1 with bounded derivatives, then in the setting above we have:

$$\Delta_\eta \quad = \quad E\Big[\Phi(X_T^0(\eta)) w^\Delta(\eta)\Big]$$

where

$$w^\Delta(\eta)(\psi) = \int_0^T \Big[a(s) g_R^{-1}(s,\ X_s^0(\eta))\rho_0 \circ DX_s^0(\eta)\Big](\psi) dW(s)$$

Notice: Assumptions on Φ are softened to $\Phi(X_T^0(\eta)) \in L_2(\Omega)$, by approximation argument.

2. Risk-neutral approach: pricing and sensitivity analysis

Recall the model...

A *risk-less bond*

$$dB(t) = B(t)\kappa(t)dt \quad \text{with } B(0) = 1$$

A *risky asset*:

$$\begin{cases} dS(t) = S(t)\{\mu(t, S_t)dt + \sigma(t, S_t)dW(t)\}, & t \in [0, T] \\ S_0 = \eta & (\mathcal{F}_0\text{-measurable in } M_2) \end{cases}$$

Assume that

$$Z(t) := \exp\left\{-\int_0^t \theta_d(u, S_u)dW(u) - \frac{1}{2}\int_0^t \theta_d^2(u, S_u)du\right\}$$

with $\theta_d(t, S_t) := \frac{\mu(t,S_t)-\kappa(t)}{\sigma(t,S_t)}$, is a martingale.

Then $dQ = Z(T)dP$ is a risk-neutral measure.

We consider a path dependent option of type $\Phi({}^{\eta}S(T), {}^{\eta}S_T)$. The price for the option is given by:

$$p_Q(\eta) = E_Q\left[\frac{\Phi({}^{\eta}S(T), {}^{\eta}S_T)}{B(T)}\right]$$

And the sensitivity to the initial condition is given by:

$$\Delta_\eta^Q(\psi) := Dp_Q(\eta)(\psi) = \frac{1}{B(T)} E_Q\Big[\Phi({}^{\eta}S(T), {}^{\eta}S_T) w^{\Delta}(\psi)\Big],$$

where the weight is:

$$w^{\Delta}(\psi) = \int_0^T a(s)\sigma_R^{-1}(s, ({}^{\eta}S(s), {}^{\eta}S_s)) DX_s^0(\eta)(\psi(0)) dW(s),$$

Notation: $DX_s^0(\eta)(\psi(0)) := \rho_0 \circ DX_s^0(\eta)(\psi)$.

Example: Küchler and Platen model. Commodity prices presenting cyclical fluctuations:

$$S(t) = \alpha_1 \exp\{\alpha_2 Y(t) + \alpha_3 t\}, \quad t \in [0, T]$$

where α_1, α_2 and α_3 are adequate parameters and

$$dY(t) = \begin{cases} -\mu Y(t-r)dt + \sigma dW(t), & t \in (0, T] \\ \eta(t), & t \in [-r, 0]. \end{cases}$$

Explicitly, the price process is:

$$S(t) = \alpha_1 \exp\left\{\alpha_2 \left(\eta(0) - \mu \int_0^t Y(u-r)du + \sigma W(t)\right) + \alpha_3 t\right\}$$

Consider $\Phi({}^{\eta}S(T), {}^{\eta}S_T)$ with risk-neutral price

$$p_Q(\eta) = \frac{1}{B(T)} E_Q\left[\Phi({}^{\eta}S(T), {}^{\eta}S_T)\right].$$

The delta operator is

$$\Delta_\eta^Q = \frac{1}{B(T)} E_Q\Big[\Phi({}^\eta S(T), {}^\eta S_T)\, w^\Delta(\eta)\Big],$$

with weight

$$w^\Delta(\eta) = \frac{1}{\alpha_1 \alpha_2 \sigma} \int_0^T a(s)\rho_0\big(DX_s^0(\eta)\big)\, dW(s)$$

Then the Fréchet derivative above is given by

$$D\rho_0\left(X_s^0(\eta)\right) = {}^\eta S(s)\alpha_2 \left[\rho_0 - \mu \int_0^s D^\eta Y(u-r)du\right].$$

For a close expression, we assume that $t \in [0, r]$, then $Y(t-r) = \eta(t-r)$ and so $D^\eta Y(t-r) = \rho_{t-r}$. Then

$$w^\Delta(\eta) = \frac{1}{\alpha_1 \sigma} \int_0^T a(s)^\eta S(s)\big[\rho_0 - \mu \int_0^s \rho_{u-r} du\big]\, dW(s).$$

3. Benchmark approach: pricing and sensitivity analysis

Benchmark pricing in short.
This pricing method exploit the relationship that exists between *numéraire* and pricing measure. We refer to Platen and Heath (2006) for details (plus related works).

The idea is to find the "appropriate" process G such that the discounted prices $\frac{S(t)}{G(t)}$, $t \in [0, T]$, is a martingale under P.

The process G is called *benchmark*.

A self-financing portfolio π is *fair* if the benchmarked value $\frac{V^{\pi}(t)}{G(t)}$, $t \in [0, T]$, is a martingale under P.

How to find G?

In this set-up one can find G solving the optimization problem:

$$V^{\pi^*}(T) = \sup_{\pi \in \mathcal{A}} E[\log V^{\pi}(T)],$$

where $\mathcal{A}$ denotes the set of self-financing portfolios with V strictly positive. Then $G := V^{\pi^*}$.
In our framework, the SFDE for the value process is

$$\begin{cases} \frac{dV(t)}{V(t)} = [\kappa(t) + (\mu(t, S_t) - \kappa(t))\pi(t, S_t)]dt + \sigma(t, S_t)\pi(t, S_t)dW(t), \\ \frac{dS(t)}{S(t)} = \mu(t, S_t)dt + \sigma(t, S_t)dW(t), \;\; t \in [0, T] \\ V(0) = x, \quad S_0 = \eta \end{cases}$$

and one can show that the G corresponds to:

$$\begin{cases} \frac{dG(t)}{G(t)} = [\kappa(t) + \theta_d^2(t, S_t)]dt + \theta_d(t, S_t)dW(t), \;\; t \in [0, T] \\ \frac{dS(t)}{S(t)} = \mu(t, S_t)dt + \sigma(t, S_t)dW(t), \quad t \in [0, T] \\ G(0) = 1, \;\; S_0 = \eta \end{cases}$$

Pricing formula.

For the European option $\Phi({}^{\eta}S(T), {}^{\eta}S_T) \in L^2(\Omega, \mathbb{R}^+)$ we have that the benchmark price is

$$p(\eta) := E\left[\frac{\Phi({}^{\eta}S(T), {}^{\eta}S_T)}{{}^{\eta}G(T)}\right], \quad \eta \in M_2.$$

Note that the numéraire depends on the initial path.

Sensitivity analysis: Delta.

In this case the formula of the delta operators is given by:

$$Dp(\eta) = E\left[\Phi({}^{\eta}S(T), {}^{\eta}S_T)\, w^{\Delta}\right],$$

where,

$$w^{\Delta} = \delta\left(\frac{a(\cdot)g_R^{-1}(\cdot, {}^{\eta}S(\cdot), {}^{\eta}S_{\cdot})\rho_0 \circ DX_{\cdot}^{0}(\eta)}{{}^{\eta}G(T)}\right) - \frac{D\log {}^{\eta}G(T)}{{}^{\eta}G(T)}$$

and

$$D\log {}^{\eta}G(T) = \int_0^T \frac{\theta_d(t, {}^{\eta}S_t)}{\sigma(t, {}^{\eta}S_t)}\left[D\mu(t, {}^{\eta}S_t) - \theta_d(t, {}^{\eta}S_t)D\sigma(t, {}^{\eta}S_t)\right] \circ DX_t^0(\eta)dt$$
$$+ \int_0^T \left[\frac{D\mu(t, {}^{\eta}S_t) - \theta_d(t, {}^{\eta}S_t)D\sigma(t, {}^{\eta}S_t)}{\sigma(t, {}^{\eta}S_t)}\right] \circ DX_t^0(\eta)dW(t)$$

Example: Classical Black and Scholes market.

$$\begin{cases} dS(t) = \mu S(t)dt + \sigma S(t)dW(t), & t \in (0, T] \\ S(t) = \eta(t), & t \in [-r, 0] \end{cases}$$

Observe that the initial condition is just relevant at the point $t = 0$.

In this case: $G(t) = e^{(\kappa + \frac{\theta^2}{2})t + \theta W(t)}$ and:

$$Dp(\eta)(\psi) E\left[\Phi({}^{\eta}S(T), {}^{\eta}S_T)\, w^{\Delta}(\psi)\right],$$

where $(a \equiv 1/T)$

$$\begin{aligned} w^{\Delta}(\eta)(\psi) &= \delta\left(\frac{\psi(0)}{T\eta(0)\sigma G(T)}\right) \\ &= \frac{1}{G(T)}\delta\left(\frac{\psi(0)}{T\eta(0)\sigma}\right) - \frac{\psi(0)}{T\eta(0)\sigma}\int_0^T \mathcal{D}_s \frac{1}{G(T)} ds. \end{aligned}$$

Then $w^{\Delta}(\eta)(\psi) = \frac{1}{G(T)}\frac{\psi(0)}{T\eta(0)\sigma}\left(W(T) + \theta T\right)$.
Also we consider the worst case evaluation with:

$$\Delta = \sup_{\substack{\psi \in M_2 \\ \|\psi\|_{M_2}=1}} |Dp(\eta)(\psi)| = \left| E\left[\frac{1}{G(T)} \frac{\Phi({}^{\eta}S(T), {}^{\eta}S_T)}{T\eta(0)\sigma}\left(W(T) + \theta T\right)\right]\right|$$

References

This presentation is based on:

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By GdN, E.M. L'Aurora, M. Moschetta, F. Proske, and D. Ruiz-Baños.
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Sketch of the proof. Fix ω. For any $\tilde{\eta} \in \mathcal{M}_2$:

$$
\begin{aligned}
&{}^{\tilde{\eta}}x^s(t,\omega) = \rho_0(X_t^s(\tilde{\eta},\omega)) \\
&= \rho_0(\tilde{\eta}) + \int_s^t f(u,\cdot) \circ X_u^s(\cdot,\omega)(\tilde{\eta})du + \int_s^t g(u,\cdot) \circ X_u^s(\cdot,\omega)(\tilde{\eta})dW(u).
\end{aligned}
$$

Compute the Fréchet derivative at $\tilde{\eta}$:

$$
\begin{aligned}
D\,{}^{\tilde{\eta}}x^s(t,\omega) = D\rho_0(\tilde{\eta}) &+ \int_s^t D[f(u,\cdot) \circ X_u^s(\cdot,\omega)](\tilde{\eta})du \\
&+ \int_s^t D[g(u,\cdot) \circ X_u^s(\cdot,\omega)](\tilde{\eta})dW(u).
\end{aligned}
$$

Since ρ_0 is a linear operator, then $D\rho_0(\tilde{\eta}) = \rho_0 \in L(\mathcal{M}_2, \mathbb{R}^d)$.
By use of chain rules, the evaluation at $\tilde{\eta} = X_s^0(\eta,\omega)$, and the semigroup property of the flow $X_u^s \circ X_s^0 = X_u^0$, we obtain:

$$
\begin{aligned}
D\,{}^{\tilde{\eta}}x^s(t,\omega) = \rho_0 &+ \int_s^t D[f(u, X_u^0(\eta,\omega))] \circ DX_u^0(\eta,\omega)du \\
&+ \int_s^t D[g(u, X_u^0(\eta,\omega)))] \circ DX_u^0(\eta,\omega)dW(u)
\end{aligned} \tag{5}
$$

We compute the Malliavin derivative of the solution at the point $0 \leqslant s \leqslant t$ and use of chain rule:

$$
\begin{aligned}
\mathcal{D}_s\, {}^{\eta}x(t) = & \int_s^t D[f(u, X_u^0(\eta)] \circ \mathcal{D}_s(X_u^0(\eta))du + g(s, X_s^0(\eta)) \\
& + \int_s^t D[g(u, X_u^0(\eta))] \circ \mathcal{D}_s(X_u^0(\eta))dW(u).
\end{aligned} \tag{6}
$$

In order to compare the two derivatives (5) and (6), we consider a representative in the Malliavin derivative representation and construct an operator of transfer such that:

$$g(s, X_s^0(\eta,\omega)) \circ \tau_s(\cdot) = \rho_0(\cdot).$$

Indeed for $\varphi \in \mathcal{M}_2$ we have

$$\big(\mathcal{D}_s\,{}^{\eta}x(t)\big) \circ \tau_s(\varphi) = \int_s^t D[f(u, X_u^0(\eta,\omega))] \circ \big(\mathcal{D}_s(X_u^0(\eta))\big)(\omega) \circ \tau_s(\varphi) du$$

$$+\varphi(0) + \int_s^t D[g(u, X_u^0(\eta,\omega))] \circ \big(\mathcal{D}_s(X_u^0(\eta))\big)(\omega) \circ \tau_s(\varphi) dW(u).$$

To complete we have to prove that, for $\tilde{\eta} = X_s^0(\eta)$:

$$\| \big(D\,{}^{\tilde{\eta}}x^s(t))(\varphi)\big)_t - \big(\big(\mathcal{D}_s\,{}^{\eta}x(t)\big) \circ \tau_s(\varphi)\big)_t \|^2_{L^2(\Omega,\mathcal{M}_2)} = 0$$