Markovian fluid queues in ALM

6th AMaMeF and Banach Center Conference

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1. Motivation

Asset and Liability Management

$X(\cdot)$

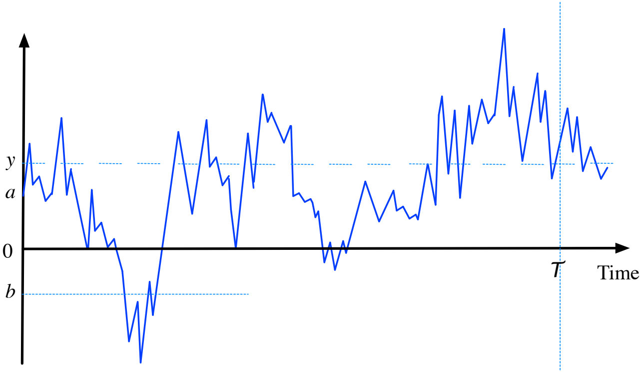

Figure: $X(t) = \ln \frac{A(t)}{L(t)}$ where $\{A(t)\}_{t\geq 0}$ is the asset process and $\{L(t)\}_{t\geq 0}$ is the liability process

Goal:

$$\mathbb{P}\left[\min_{0\leq t\leq \mathcal{T}} X(t) \geq b \text{ and } X(\mathcal{T}) > y | X(0) = a\right]$$

1. Motivation

- Markov modulated Brownian motion

$$X(t) = X(0) + \int_0^t \mu\left(\varphi(s)\right) ds + \int_0^t \sigma^2(\varphi(s)) dB(s)$$

where $B(t)$ is a Brownian motion and where

- $\varphi(s) \in \{1, ..., m\}$
- $\mu\left(\varphi(s)\right) \in \{\mu_1, ..., \mu_m\}$
- $\sigma^2(\varphi(s)) \in \{\sigma_1^2, ..., \sigma_m^2\}$

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- Markov modulated Brownian motion

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- Markovian fluid queue (G. Latouche, G. Nguyen, *The morphing of fluid queues into Markov-modulated Brownian motion*)

$$X(t) = X(0) + \int_0^t \mu(\varphi(s))\, ds + \sqrt{\lambda} \int_0^t (-1)^{\beta(s)} \sigma^2(\varphi(s)) ds$$

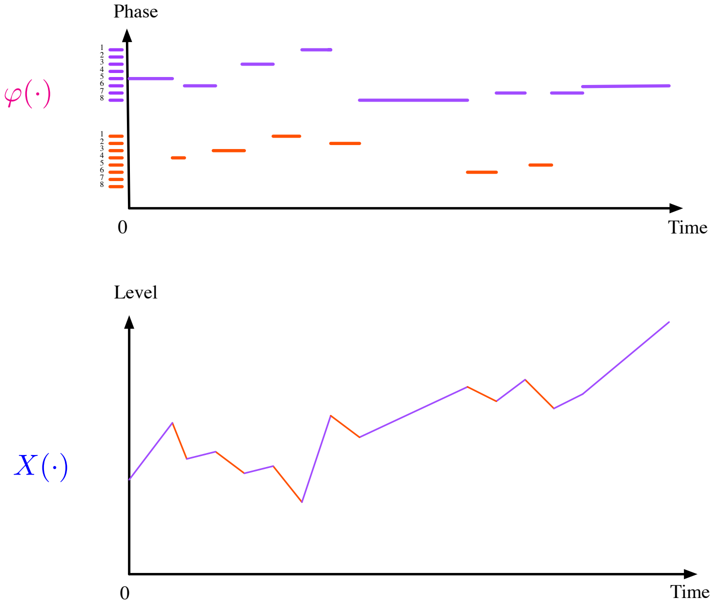
where

- $\lambda > 0$,
- $\beta(s) \in \{1, 2\}$

Outline

1. Motivation
2. Markovian fluid queues
3. Erlangization method
4. Application

2. Markovian fluid queues



2. Markovian fluid queues

Markov process $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$

- $X(t) \in \mathbb{R}^+$ is the continuous level
- $\varphi(t) \in \mathcal{S}$ is the phase : state of a discrete Markov chain with state space $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ and infinitesimal generator T,

$$T = \begin{bmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{bmatrix}$$

where $T_{++} : \mathcal{S}_+ \rightsquigarrow \mathcal{S}_+$, $T_{+-} : \mathcal{S}_+ \rightsquigarrow \mathcal{S}_-$, $T_{-+} : \mathcal{S}_- \rightsquigarrow \mathcal{S}_+$, $T_{--} : \mathcal{S}_- \rightsquigarrow \mathcal{S}_-$

Evolution of the level, varies linearly according to the phase

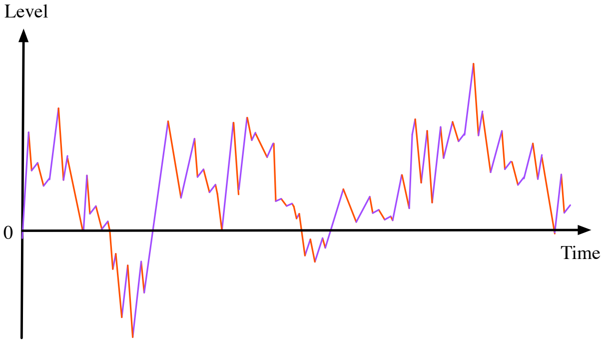
$$X(t) = X(0) + \int_0^t c_{\varphi(s)} ds$$

Matrix of the rates

$$C = \text{diag}(c_i : i \in \mathcal{S})$$

2. Markovian fluid queues

The matrix Ψ of the first return probability

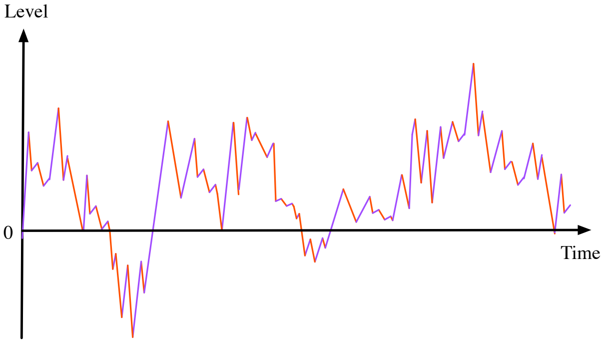


$$\Psi_{ij} = \mathbb{P}[\theta(0) < \infty, \varphi(\theta(0)) = j | X(0) = 0, \varphi(0) = i]$$

where $i \in \mathcal{S}_+, j \in \mathcal{S}_-$ and $\theta(0) = \inf\{t > 0 : X(t) = 0\}$

2. Markovian fluid queues

The matrix Ψ of the first return probability



$$\Psi_{ij} = \mathbb{P}[\theta(0) < \infty, \varphi(\theta(0)) = j | X(0) = 0, \varphi(0) = i]$$

where $i \in \mathcal{S}_+, j \in \mathcal{S}_-$ and $\theta(0) = \inf\{t > 0 : X(t) = 0\}$

$$\hat{\Psi}_{ij} = \mathbb{P}[\theta(0) < \infty, \varphi(\theta(0)) = j | X(0) = 0, \varphi(0) = i]$$

where $i \in \mathcal{S}_-, j \in \mathcal{S}_+$

2. Markovian fluid queues

Theorem

The matrix Ψ *of first return probabilities is the solution of the Riccati equation*

$$C_+^{-1}T_{++}\Psi + C_+^{-1}T_{+-} + \Psi\left|C_-^{-1}\right|T_{--} + \Psi\left|C_-^{-1}\right|T_{-+}\Psi = 0$$

where $C_+ = diag(c_i : i \in \mathcal{S}_+)$ *and* $C_- = diag(c_i : i \in \mathcal{S}_-)$

(L.C.G. Rogers, *Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains)*

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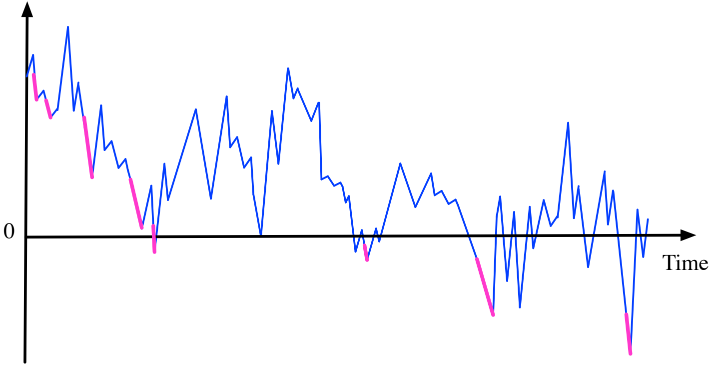
Similar equation for the matrix $\hat{\Psi}$

2. Markovian fluid queues

The matrix $U = \left|C_-^{-1}\right| T_{--} + \left|C_-^{-1}\right| T_{-+} \Psi$



U is the infinitisimal generator of the process observed only during those intervals of time in which

$$X(t) = \min_{0 \leq u \leq t} X(u)$$

3. Erlangization method

Erlang distribution : $Y \sim \mathbf{Erl}(\nu, L)$

- $\mathbb{E}(Y) = \frac{L}{\nu} = \mathcal{T}$
- $\mathbb{V}\mathrm{ar}(Y) = \frac{L}{\nu^2} = \frac{\mathcal{T}^2}{L}$
- Probability density function : $f_Y(x) = \frac{\nu^L x^{L-1}}{(L-1)!} e^{-\nu x}$

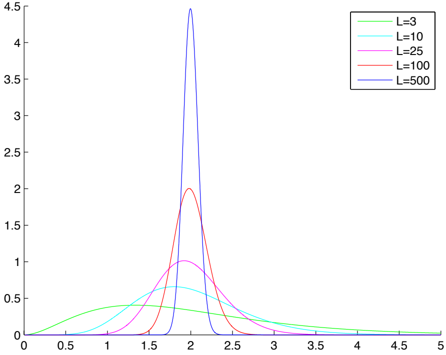
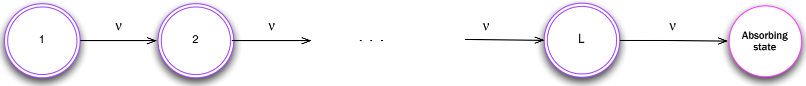


Figure: $\mathrm{Erl}\left(\frac{L}{2}, L\right)$

3. Erlangization method

Erlang distribution $\rightrightarrows$ Continuous time Markov chain
$\{\phi(t) : t \in \mathbb{R}^+\}$



infinitesimal generator $L \times L$

$$N = \begin{bmatrix} -\nu & \nu & & & \\ & -\nu & \ddots & & \\ & & \ddots & & \\ & & & -\nu & \nu \\ & & & & -\nu \end{bmatrix}$$

3. Erlangization method

Erlangization method in Markovian fluid queues

Markovian fluid queue $\{(X(t), \varphi(t)) : t \in \mathbb{R}^+\}$

$+$

Fixed time $\sim$ Erlang Markov chain $\{\phi(t) : t \in \mathbb{R}^+\}$

$\Downarrow$

Erlangized Markovian fluid queue $\{(X(t), \Phi(t)) : t \in \mathbb{R}^+\}$

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Erlangized Markovian fluid queue $\{(X(t), \Phi(t)) : t \in \mathbb{R}^+\}$

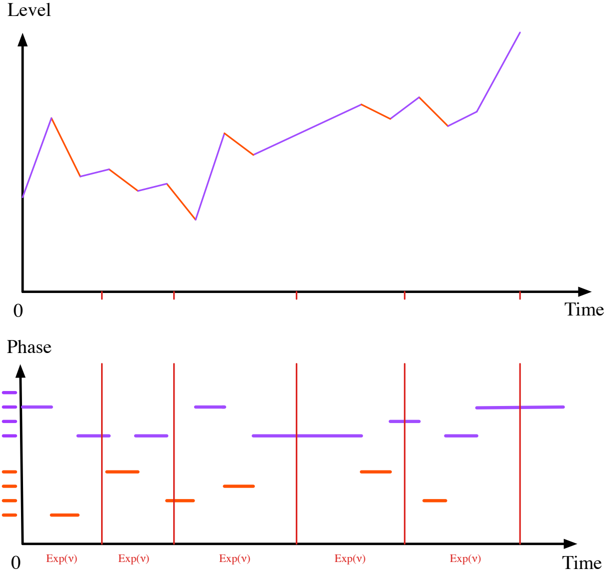
We construct the fluid queue $\{(X(t), \Phi(t)) : t \in \mathbb{R}^+\}$ where

$$\begin{aligned} \Phi(t) &= (\varphi(t), \phi(t)) \\ &= (i, k) \end{aligned}$$

i.e. at time t, the phase is $i \in \mathcal{S}$ and t belongs to the k-th stage of Erlang process $k \in \{1, 2, ..., L\}$.

(S. Asmussen, F. Avram, M. Usábel, *Erlangian approximations for finite time ruin probabilities.)*

3. Erlangization method



3. Erlangization method

T is the infinitesimal generator of $\varphi(t)$

$$T = \begin{bmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{bmatrix}$$

and N is the infinitesimal generator of $\phi(t)$

$$N = \begin{bmatrix} -\nu & \nu & & \\ & -\nu & \ddots & \\ & & \ddots & \\ & & & -\nu \end{bmatrix}$$

The generator of $\Phi(\cdot)$ is Q and has block-matrices

$$Q_{+-} = I \otimes T_{+-} = \begin{bmatrix} T_{+-} & & & \\ & T_{+-} & & \\ & & \ddots & \\ & & & T_{+-} \end{bmatrix}$$

3. Erlangization method

$$Q_{++} = N \otimes I + I \otimes T_{++} = \left[\begin{array}{c|c|c|c} T_{++} - \nu I & \nu I & & \\ \hline & T_{++} - \nu I & & \\ \hline & & \ddots & \\ \hline & & & \nu I \\ \hline & & & T_{++} - \nu I \end{array}\right]$$

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The infinitesimal generator of $\Phi(\cdot)$ is

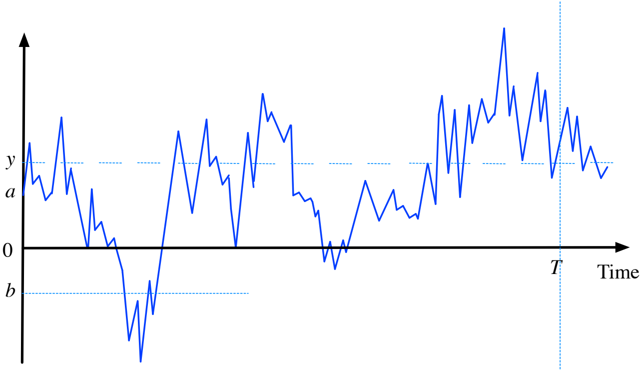
$$Q = \left[\begin{array}{cc|c} N \otimes I + I \otimes T_{++} & I \otimes T_{+-} & -N\mathbf{1}_L \otimes \mathbf{1}_+ \\ I \otimes T_{-+} & N \otimes I + I \otimes T_{--} & -N\mathbf{1}_L \otimes \mathbf{1}_- \\ \hline 0 & 0 & 0 \end{array}\right]$$

4. Application

Joint distribution during the period $[0, T]$

$$\mathbb{P}\left[\min_{0\leq t\leq T} X(t) \geq b \text{ and } X(T) > y | X(0) = a, \varphi(0) = i\right]$$

$X(\cdot)$



Today : $y > a > b$ and $i \in \mathcal{S}_+$

4. Application

Case 1 : $T \sim$ Exponential with parameter ν

The joint probability of the minimum reached during the exponential horizon and the level reached at the end of the exponential horizon given the initial level and phase

$$\mathbb{P}_i^a\left[\min_{0\leq t\leq T} X(t) \geq b \ \& \ X(T) > y\right]$$

is given by the i-th component of the vector

$$\left(I - \Psi e^{U(a-b)}\hat{\Psi}e^{\hat{U}(a-b)}\right)e^{\hat{U}(y-a)}(I - \Psi\hat{\Psi})^{-1}(\mathbf{1} - \Psi\mathbf{1})$$

Notation

$$\mathbb{P}_i^a\left[\min_{0\leq t\leq T} X(t) \geq b \text{ and } X(T) > y\right]$$

denotes $\mathbb{P}\left[\min_{0\leq t\leq T} X(t) \geq b \text{ and } X(T) > y | X(0) = a, \varphi(0) = i\right]$

4. Application

Remark 1 The matrices $\Psi, \hat{\Psi}, U, \hat{U}$ are those of the Erlangized fluid queue.

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Remark 2

$$\mathbb{P}_i^a \left[\min_{0 \le t \le T} X(t) \ge b \text{ and } X(T) > y \right]$$

is equal to

$$\mathbb{P}_i^a \left[X(T) > y \right] - \mathbb{P}_i^a \left[\min_{0 \le t \le T} X(t) < b \text{ and } X(T) > y \right]$$

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Remark 3

$$\mathbb{P} \left[\min_{0 \le t \le T} X(t) < b \right] = \mathbb{P}[\tau(b) < T]$$

where $\tau(b) = \inf\{t > 0 : X(t) < b\}$

4. Application

Computation (keeping track of the phases...)

$$\mathbb{P}_i^a\left[\tau(b) < T \text{ and } X(T) > y\right] = \sum_{j \in \mathcal{S}_-} \mathbb{P}_{ij}^a[\tau(b) < T]\mathbb{P}_j^b[X(T) > y]$$

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where

$$\begin{aligned}
\mathbb{P}_{ij}^a[\tau(b) < T] &= \mathbb{P}[\tau(b) < T, \varphi(\tau(b)) = j | X(0) = a, \varphi(0) = i] \\
&= \sum_{k \in \mathcal{S}_-} \Psi_{ik} \left(e^{U(a-b)}\right)_{kj}
\end{aligned}$$

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and $\mathbb{P}_j^b[X(T) > y]$

$$\begin{aligned} &= \mathbb{P}_j^b[X(T) > y | X(0) = b, \varphi(0) = j] \\ &= \sum_{u \in \mathcal{S}_+} \sum_{v \in \mathcal{S}_+} \hat{\Psi}_{ju} \left\{e^{\hat{U}(y-b)}\right\}_{uv} \left\{(I - \Psi\hat{\Psi})^{-1}(\mathbf{1} - \Psi\mathbf{1})\right\}_v \end{aligned}$$

4. Application

Case 2 : $T \sim$ Erlang with parameters ν and L

The joint probability of the minimum reached during the Erlang horizon and the level reached at the end of the exponential horizon given the initial level and phase

$$\mathbb{P}_i^a\left[\min_{0\le t\le T} X(t) \ge b \;\&\; X(T) > y\right]$$

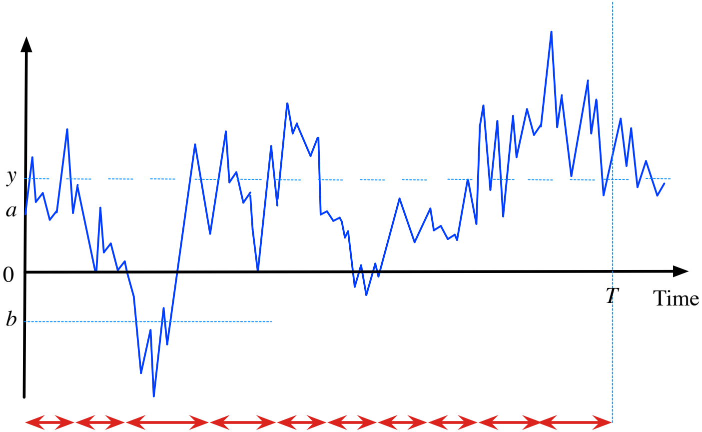
is given by the i-th component of the vector

$$\sum_{k=1}^{L}\left\{e^{U^{(k)}(y-a)}\boldsymbol{h}^{(k^*)}\right\}$$

$$-\sum_{k=1}^{L}\left\{\left[\sum_{n=1}^{k}\Psi^{(n)}e^{U^{(k-n)}(a-b)}\right]\left[\sum_{n=1}^{L-k}\hat{\Psi}^{(n)}\sum_{m=1}^{L-k-n}e^{\hat{U}^{(m)}(y-b)}\boldsymbol{h}^{(m^*)}\right]\right\}$$

with $k^* = L-k$ and $m^* = L-k-n-m$

4. Application



4. Application

Computation (keeping track of the phases and the stages...)

$\mathbb{P}_i^a\left[\tau(b) < T \text{ and } X(T) > y\right] =$

$$\sum_{k=1}^{L} \sum_{j \in \mathcal{S}_-} \mathbb{P}_{ij}^a[\phi(\tau(b)) = k]\mathbb{P}_j^b[X(T) > y | \phi(\tau(b)) = k]$$

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where

$$\mathbb{P}_{ij}^a[\phi(\tau(b)) = k] = \sum_{n=1}^{k} \sum_{l \in \mathcal{S}_-} \Psi_{il}^{(n)} \left(e^{U^{(k-n+1)}(a-b)}\right)_{lj}$$

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and $\mathbb{P}_j^b[X(T) > y|\phi(\tau(b)) = k]$ equals

$$\sum_{n=1}^{L-k+1} \sum_{m=1}^{L-k-n+1} \sum_{u \in \mathcal{S}_+} \sum_{v \in \mathcal{S}_+} \hat{\Psi}_{ju}^{(n)} \left\{e^{\hat{U}^{(m)}(y-b)}\right\}_{uv} \boldsymbol{h}_v^{(L-k-n-m+1)}$$

4. Application

Recurrence equation

$$\boldsymbol{h^{(1)}} = (I - \Psi^{(1)}\hat{\Psi}^{(1)})^{-1}(\mathbf{1} - \Psi^{(1)}\mathbf{1})$$

and for $n > 1$,

$$\boldsymbol{h^{(n)}} = \mathbf{1} - \sum_{k=1}^{n} \Psi^{(k)}\mathbf{1} + \sum_{i+j+l=n+2} \Psi^{(i)}\hat{\Psi}^{(j)}\boldsymbol{h^{(l)}}$$

The i-th component of the vector $\boldsymbol{h^{(n)}}$ is the probability that the process is above the level y after an Erlang horizon time period with n stages, given that the process starts in i an increasing phase, in the level y, i.e.

$$\boldsymbol{h_i^{(n)}} = \mathbb{P}[X(T) > y | X(0) = y, \varphi(0) = i]$$

for $i \in S_+$ and where $T \sim \mathrm{Erl}\big(\frac{n}{T}, n\big)$.

Thanks !

References

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