# Conditional Quantiles, Conditional Weighted Expected Shortfall and Application to Risk Capital Allocation

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# Risk measures

1. Risk measures can be used to derive *capital requirements* in order to quantify risks associated with positions in financial and insurance markets.
2. The *coherent risk measures* form an important class, first introduced in Artzner et al. (1997) [2] (e.g. expected shortfall, or also called average value-at-risk).
3. As it is interesting to deal with partial information, *conditional risk measures* arise.
4. Given a filtration, the theory of conditional risk measures can be used to consider the evaluation of risk over time which leads to *dynamic risk measures*.

# Our setup throughout the presentation

(a) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with sub-$\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$.

(b) $X$ and $Y$ are $\mathcal{F}$-measurable losses/gains where losses are positive.

(c) Given a $\mathcal{G}$-measurable level $\delta$ with $0 \leq \delta \leq 1$ a.s.

(d) We use a general version of conditional expectation based on r.v. which are $\sigma$-integrable w.r.t. $\mathcal{G}$ (He et al. [5, Chapter I.4]).

## Definition (Conditional risk measure)

A map $\rho\colon \mathcal{L}_0(\mathbb{P}) \to \mathcal{L}_{\mathcal{G},0}(\mathbb{P})$ which is normalised, translation (or cash) invariant and monotone is called a *conditional risk measure* where $\mathcal{L}_0(\mathbb{P})$ and $\mathcal{L}_{\mathcal{G},0}(\mathbb{P})$ denote the equivalence classes of all $\mathcal{F}$- and $\mathcal{G}$-measurable real-valued r.v.

# $\mathcal{G}$-measurable upper envelope

**Definition**

For $X$ define $X^{\mathcal{G}}$ as the $\mathcal{G}$-*measurable upper envelope* of $X$, i.e., as the essential infimum of all $\mathcal{G}$-measurable r.v. $Z\colon \Omega \to \mathbb{R} \cup \{\infty\}$ satisfying $\mathbb{P}(X \leq Z) = 1$.

**Properties (among others):**

(a) $X^{\mathcal{G}}$ is $\mathcal{G}$-measurable and satisfies $X^{\mathcal{G}} \geq X$ a.s.

(b) If $X$ is $\mathcal{G}$-measurable, then $X^{\mathcal{G}} = X$ a.s.

(c) If $X \leq Y$ a.s., then $X^{\mathcal{G}} \leq Y^{\mathcal{G}}$ a.s.

(d) $(X + Y)^{\mathcal{G}} \leq X^{\mathcal{G}} + Y^{\mathcal{G}}$ a.s.

(e) If $X, Y \geq 0$ a.s., then $(X\,Y)^{\mathcal{G}} \leq X^{\mathcal{G}} Y^{\mathcal{G}}$ a.s.

(f) If $Z$ is $\mathcal{G}$-measurable, then $(X + Z)^{\mathcal{G}} = X^{\mathcal{G}} + Z$ a.s.

(g) If $Z \geq 0$ is $\mathcal{G}$-measurable, then $(X\,Z)^{\mathcal{G}} = X^{\mathcal{G}} Z$ a.s.

# Properties of the upper envelope

**Lemma**

The upper envelope $X^{\mathcal{G}}$ is a coherent conditional risk measure.

**Lemma**

Given a filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$. Then, the upper envelope is time-consistent, i.e., for any two stopping times $\sigma$ and $\tau$ with $\sigma \leq \tau$ a.s., we have that $X^{\mathcal{F}_\tau} \leq Y^{\mathcal{F}_\tau}$ a.s. implies $X^{\mathcal{F}_\sigma} \leq Y^{\mathcal{F}_\sigma}$ a.s.

# Definition of conditional lower quantiles

**Definition**

Define the lower $\delta$-*quantile* $q_{\mathcal{G},\delta}(X)$ of $X$ given $\mathcal{G}$ as the essential infimum of all $\mathcal{G}$-measurable r.v. $Z\colon \Omega \to \mathbb{R} \cup \{\infty\}$ satisfying $\mathbb{P}(X \le Z \,|\, \mathcal{G}) \ge \delta$ a.s.

**Remarks**

(a) Usually $\delta = 0.9, 0.95, 0.995, \ldots$.

(b) Since the level of risk aversion depends on previous developments in the market, $\delta$ can be chosen $\mathcal{G}$-measurable.

(c) $q_{\mathcal{G},0}(X) = -\infty$ and $q_{\mathcal{G},1}(X) = X^{\mathcal{G}}$, both a.s.

(d) $q_{\mathcal{G},\delta}(X)$ is $\mathcal{G}$-measurable and satisfies $\mathbb{P}(q_{\mathcal{G},\delta}(X) \ge X \,|\, \mathcal{G}) \overset{\text{a.s.}}{\ge} \delta$.

(e) For trivial $\mathcal{G}$, the definition above corresponds to usual lower quantiles.

# Conditional stochastic order

## Definition

(a) *First order conditional stochastic dominance:* If $\mathbb{E}[h(X)|\mathcal{G}] \leq \mathbb{E}[h(Y)|\mathcal{G}]$ a.s., for all increasing functions $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(X)^-$ and $h(Y)^-$ are $\sigma$-integrable w.r.t. $\mathcal{G}$, then we define $X \prec_{\mathrm{SD}(1,\mathcal{G})} Y$.

(b) *Second order conditional stochastic dominance:* If $\mathbb{E}[h(X)|\mathcal{G}] \leq \mathbb{E}[h(Y)|\mathcal{G}]$ a.s., for all increasing and convex functions $h\colon \mathbb{R} \to \mathbb{R}$ such that $h(X)^-$ and $h(Y)^-$ are $\sigma$-integrable w.r.t. $\mathcal{G}$, then we define $X \prec_{\mathrm{SD}(2,\mathcal{G})} Y$.

# Conditional comonotonicity

## Definition

$X$ and $Y$ are *conditionally comonotonic w.r.t.* $\mathcal{G}$ if $\mathbb{P}(X \le x, Y \le y \,|\, \mathcal{G}) = \min\{\mathbb{P}(X \le x \,|\, \mathcal{G}), \mathbb{P}(Y \le y \,|\, \mathcal{G})\}$ a.s., for all $x, y \in \mathbb{R}$.

**Remarks**

(a) For trivial $\mathcal{G}$ the definition corresponds to the definition of comonotonicity.

(b) We avoid a definition via *conditional distributions* and corresponding transition kernels since their existence heavily depends on the structure of $(\Omega, \mathcal{F})$.

# Properties of conditional lower quantiles I

**Lemma**

Let $\{\delta_t\}_{t\in[0,\infty)}$ be a $\mathcal{G}$-measurable $[0,1]$-valued process with left-continuous and increasing paths. Then, there exists a version of $\{q_{\mathcal{G},\delta_t}(X)\}_{t\in[0,\infty)}$ with left-continuous and increasing paths.

**Remark:** From now on we always use this 'nice' version of conditional lower quantiles.

# Properties of conditional lower quantiles II

**Lemma**

Let $Z$ be a $\mathcal{G}$-measurable real-valued r.v. Then, conditional lower quantiles satisfy the following conditional properties:

(a) *Positive homogeneity:* If $Z \geq 0$ a.s., then $q_{\mathcal{G},\delta}(ZX) = Z\, q_{\mathcal{G},\delta}(X)$ a.s.

(b) *Translation (or cash) invariance:* $q_{\mathcal{G},\delta}(X+Z) \overset{\text{a.s.}}{=} q_{\mathcal{G},\delta}(X) + Z$.

(c) *Comonotonic additivity:* If $X$ and $Y$ are conditionally comonotonic w.r.t. $\mathcal{G}$, then $q_{\mathcal{G},\delta}(X+Y) \overset{\text{a.s.}}{=} q_{\mathcal{G},\delta}(X) + q_{\mathcal{G},\delta}(Y)$.

(d) *Monotonicity:* If $X \prec_{\mathrm{SD}(1,\mathcal{G})} Y$, then $q_{\mathcal{G},\delta}(X) \leq q_{\mathcal{G},\delta}(Y)$ a.s.

(e) *Law-invariance:* If $\mathbb{E}[f(X)|\mathcal{G}] = \mathbb{E}[f(Y)|\mathcal{G}]$ a.s., for all bounded and continuous $f\colon \mathbb{R} \to \mathbb{R}$, then $q_{\mathcal{G},\delta}(X) \overset{\text{a.s.}}{=} q_{\mathcal{G},\delta}(Y)$.

# Definition of conditional expected shortfall

## Definition

Define $f_{\mathcal{G},\delta,X}\colon \Omega \to [0,1]$ by

$$f_{\mathcal{G},\delta,X} := 1_{\{X>q_{\mathcal{G},\delta}(X)\}} + \beta_{\mathcal{G},\delta,X} 1_{\{X=q_{\mathcal{G},\delta}(X)\}}$$

where $\beta_{\mathcal{G},\delta,X}\colon \Omega \to [0,1]$ is a $\mathcal{G}$-measurable r.v. satisfying

$$\beta_{\mathcal{G},\delta,X} \overset{\text{a.s.}}{=} \begin{cases} \frac{\mathbb{P}(X\le q_{\mathcal{G},\delta}(X)\,|\,\mathcal{G})-\delta}{\mathbb{P}\big(X=q_{\mathcal{G},\delta}(X)\,|\,\mathcal{G}\big)} & \text{on } \{\mathbb{P}(X=q_{\mathcal{G},\delta}(X)\,|\,\mathcal{G})>0\}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

**Remark:** $\mathbb{E}[f_{\mathcal{G},\delta,X}\,|\,\mathcal{G}] = 1-\delta$ a.s.

# Definition of conditional expected shortfall (CES)

## Definition

Then, *CES* of $X$ at level $\delta$ given $\mathcal{G}$ is defined by

$$\mathrm{ES}_\delta[X\,|\,\mathcal{G}] = \begin{cases} X^{\mathcal{G}} & \text{on } \{\delta = 1\}\,, \\ \frac{1}{1-\delta}\,\mathbb{E}[f_{\mathcal{G},\delta,X}X\,|\,\mathcal{G}] & \text{on } \{\delta \in (0,1)\}\,, \\ \operatorname{ess\,inf}_{\delta' \in (0,1)} \frac{1}{1-\delta'}\mathbb{E}[f_{\mathcal{G},\delta',X}X\,|\,\mathcal{G}] & \text{otherwise}\,. \end{cases}$$

**Remark:** Conditional expected shortfall can be defined using acceptance sets. Under some continuity condition, conditional convex risk measures have a robust representation in terms of a penalty function (cf. Acciaio and Penner [1, Chapters 1.2 and 1.3]).

# Conditional optimality of $f_{\mathcal{G},\delta,X}$

## Definition

Let $Y \geq 0$ a.s. and assume that $Y$ is $\sigma$-integrable with respect to $\mathcal{G}$. Define

$$\mathcal{F}^{Y}_{\mathcal{G},\delta,X} := \Big\{ f\colon \Omega \to [0,1] \,\Big|\, f \text{ is } \mathcal{F}\text{-measurable and } \mathbb{E}[f\,Y\,|\,\mathcal{G}] = \mathbb{E}[f_{\mathcal{G},\delta,X}\,Y\,|\,\mathcal{G}] \text{ a.s.} \Big\},$$

where $f_{\mathcal{G},\delta,X}$ is defined as before.

## Lemma

If $1_{\{\delta=0\}} X^-$ and $Y \geq 0$ are $\sigma$-integrable w.r.t. $\mathcal{G}$, then $\mathbb{E}[f_{\mathcal{G},\delta,X} X\,Y\,|\,\mathcal{G}]$ is well-defined and

$$\operatorname*{ess\,sup}_{f \in \mathcal{F}^{Y}_{\mathcal{G},\delta,X}} \mathbb{E}[f\,X\,Y\,|\,\mathcal{G}] = \mathbb{E}[f_{\mathcal{G},\delta,X} X\,Y\,|\,\mathcal{G}] \quad \text{a.s.}$$

# Additional definitions

## Definition

Let $0 < \delta < 1$ a.s. Then, define

$$\mathcal{F}_{\mathcal{G},\delta} := \left\{ f\colon \Omega \to [0,\infty) \,\middle|\, \mathbb{E}[f \,|\, \mathcal{G}] = 1 \text{ a.s., } f \le \frac{1}{1-\delta} \text{ a.s.} \right\}$$

and

$$\mathcal{F}_{\mathcal{G},\delta,X} := \left\{ f \in \mathcal{F}_{\mathcal{G},\delta} \,\middle|\, \mathbb{E}[X^+ f \,|\, \mathcal{G}] < \infty \text{ a.s. or } \mathbb{E}[X^- f \,|\, \mathcal{G}] < \infty \text{ a.s.} \right\}.$$

**Remark:** The definitions above are similar as in Schmock [7]. Note that, for $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$, we have $\mathcal{F}_{\mathcal{G},\delta} \subset \mathcal{F}_{\mathcal{H},\delta}$. Further, $\mathcal{F}_{\mathcal{G},\delta'} \subset \mathcal{F}_{\mathcal{G},\delta}$, for $\delta' \le \delta$ a.s.

# Quantile representation of conditional expected shortfall

**Lemma**

Let $0 < \delta < 1$ a.s. Then, CES at level $\delta$ satisfies

$$\mathrm{ES}_\delta[X \,|\, \mathcal{G}] = \frac{1}{1-\delta} \int_{[\delta,1)} q_{\mathcal{G},t}(X)\, dt \quad \text{a.s.}$$

**Idea of proof:** Let $U$ be uniformly distributed on $[0,1]$ and independent of $\mathcal{G}$ and show, for $\delta' := \mathbb{P}(X \le q_{\mathcal{G},\delta}(X) \,|\, \mathcal{G})$,

$$\begin{aligned}\int_{[\delta,1)} q_{\mathcal{G},t}(X)\, dt &= \mathbb{E}\big[q_{\mathcal{G},U}(X) 1_{\{U>\delta'\}} \,\big|\, \mathcal{G}\big] + q_{\mathcal{G},\delta}(X)(\delta' - \delta) \\ &= \mathbb{E}\big[X 1_{\{X > q_{\mathcal{G},\delta}(X)\}} \,\big|\, \mathcal{G}\big] \\ &\quad + \mathbb{E}\big[q_{\mathcal{G},\delta}(X)\big(1_{\{X \le q_{\mathcal{G},\delta}(X)\}} - \delta\big) \,\big|\, \mathcal{G}\big]\end{aligned}$$

# Alternative representation of conditional expectation

**Corollary**

Assume that $X^-$ is $\sigma$-integrable w.r.t. $\mathcal{G}$. Then,

$$\mathbb{E}[X \,|\, \mathcal{G}] = \int_{(0,1)} q_{\mathcal{G},t}(X)\, dt \quad \text{a.s.}$$

# Properties of conditional expected shortfall I

## Lemma

Let $Z$ be a $\mathcal{G}$-measurable real-valued r.v. Then, CES at level $\delta$ satisfies the following conditional properties:

(a) *Positive homogeneity:* If $Z \geq 0$ a.s. is $\mathcal{G}$-measurable, then $\mathrm{ES}_\delta[ZX \,|\, \mathcal{G}] = Z\,\mathrm{ES}_\delta[X \,|\, \mathcal{G}]$ a.s.

(b) *Translation (or cash) invariance:* If $Z$ is $\mathcal{G}$-measurable, then $\mathrm{ES}_\delta[X + Z \,|\, \mathcal{G}] = \mathrm{ES}_\delta[X \,|\, \mathcal{G}] + Z$ a.s.

(c) *Subadditivity:* $\mathrm{ES}_\delta[X + Y \,|\, \mathcal{G}] \leq \mathrm{ES}_\delta[X \,|\, \mathcal{G}] + \mathrm{ES}_\delta[Y \,|\, \mathcal{G}]$ a.s., where $\infty - \infty := \infty$.

(d) *Comonotonic additivity:* If $X$ and $Y$ are conditionally comonotonic w.r.t. $\mathcal{G}$, then $\mathrm{ES}_\delta[X + Y \,|\, \mathcal{G}] = \mathrm{ES}_\delta[X \,|\, \mathcal{G}] + \mathrm{ES}_\delta[Y \,|\, \mathcal{G}]$ a.s., where $\infty - \infty := \infty$.

# Properties of conditional expected shortfall II

## Lemma (continued)

(e) *Monotonicity:* If $X \prec_{\mathrm{SD}(2,\mathcal{G})} Y$ and if $X 1_{\{\delta=1\}} \prec_{\mathrm{SD}(1,\mathcal{G})} Y 1_{\{\delta=1\}}$, then $\mathrm{ES}_\delta[X\,|\,\mathcal{G}] \le \mathrm{ES}_\delta[Y\,|\,\mathcal{G}]$ a.s.

(f) *Convexity:* If $Z$ is $\mathcal{G}$-measurable with $0 \le Z \le 1$ a.s., then
$$\mathrm{ES}_\delta[ZX + (1-Z)Y\,|\,\mathcal{G}] \overset{\text{a.s.}}{\le} Z\,\mathrm{ES}_\delta[X\,|\,\mathcal{G}] + (1-Z)\,\mathrm{ES}_\delta[Y\,|\,\mathcal{G}].$$

(g) *Law-invariance:* If $\mathbb{E}[f(X)|\mathcal{G}] \overset{\text{a.s.}}{=} \mathbb{E}[f(Y)|\mathcal{G}]$, for all bounded and continuous $f\colon \mathbb{R} \to \mathbb{R}$, then $\mathrm{ES}_\delta[X\,|\,\mathcal{G}] = \mathrm{ES}_\delta[Y\,|\,\mathcal{G}]$ a.s.

(h) *Regularity:* If $A \in \mathcal{G}$, then $X 1_A = Y 1_A$ a.s. implies $\mathrm{ES}_\delta[X\,|\,\mathcal{G}]\,1_A = \mathrm{ES}_\delta[Y\,|\,\mathcal{G}]\,1_A$ a.s.

(i) *Bounds:* Define $\mathbb{E}[X^+\,|\,\mathcal{G}]\,/0 = \infty$, then $q_{\mathcal{G},\delta}(X) \le \mathrm{ES}_\delta[X\,|\,\mathcal{G}] \le \min\left\{X^{\mathcal{G}}, \frac{\mathbb{E}[X^+\,|\,\mathcal{G}]}{1-\delta}\right\}$ a.s.

# Properties of conditional expected shortfall III

**Lemma (continued)**

If in addition $0 < \delta < 1$ a.s., then:

(j) *Scenario representations:*

(1) $\mathrm{ES}_\delta[X|\mathcal{G}] = \frac{1}{1-\delta}\operatorname{ess\,sup}_{f\in\mathcal{F}^1_{\mathcal{G},\delta,X}} \mathbb{E}[fX|\mathcal{G}]$ a.s.

(2) $\mathrm{ES}_\delta[X|\mathcal{G}] = \operatorname{ess\,sup}_{f\in\mathcal{F}_{\mathcal{G},\delta,X}} \mathbb{E}[fX|\mathcal{G}]$ a.s.

(3) If either $\mathbb{E}[X^+|\mathcal{G}] < \infty$ or $\mathbb{E}[X^-|\mathcal{G}] < \infty$, then we have $\mathrm{ES}_\delta[X|\mathcal{G}] = \operatorname{ess\,sup}_{f\in\mathcal{F}_{\mathcal{G},\delta}} \mathbb{E}[fX|\mathcal{G}]$ a.s.

(4) $\mathrm{ES}_\delta[X|\mathcal{G}] = \operatorname{ess\,inf}_{Z\in\mathcal{L}_{\mathcal{G},0}(\mathbb{P})}(Z + \frac{1}{1-\delta}\,\mathbb{E}[(X-Z)^+|\mathcal{G}])$ a.s., where $\mathcal{L}_{\mathcal{G},0}(\mathbb{P})$ denotes the set of all $\mathcal{G}$-measurable real-valued r.v., $Z = q_{\mathcal{G},\delta}(X)$ takes this essential infimum.

(k) *Fatou:* Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of r.v. bounded from below by some $\mathcal{G}$-measurable r.v. $C$. Then, $X := \liminf_{n\to\infty} X_n$ satisfies $\mathrm{ES}_\delta[X|\mathcal{G}] \le \liminf_{n\to\infty} \mathrm{ES}_\delta[X_n|\mathcal{G}]$ a.s.

# Properties of conditional expected shortfall IV

## Corollary

Let $\{\delta_t\}_{t\in[0,\infty)}$ be a $\mathcal{G}$-measurable $[0,1]$-valued process with increasing and continuous paths. Then, there exists a version of $\{\mathrm{ES}_{\delta_t}[X\,|\,\mathcal{G}]\}_{t\in[0,\infty)}$ with increasing and continuous paths on $\overline{\mathbb{R}}$.

## Corollary

Given a filtration $\mathbb{F}=\{\mathcal{F}_t\}_{t\in[0,\infty)}$ and assume that $X$ is $\sigma$-integrable w.r.t. $\mathcal{F}_0$. Let $\{\delta_t\}_{t\in[0,\infty)}$ be a $[0,1]$-valued $\mathbb{F}$-adapted process with decreasing paths. Then, $\{\mathrm{ES}_{\delta_t}[X\,|\,\mathcal{F}_t]\}_{t\in[0,\infty)}$ is a supermartingale.

# Remarks on conditional expected shortfall

(a) The properties above imply that CES is a coherent risk measure (see also [1, Example 1.10]).

(b) In general, CES is not time-consistent. Note that at level $\delta = 1$, CES is time-consistent, even in a continuous-time setting.

(c) The second scenario representation is equivalent to the widely used dual definition of CES.

# Definition of contributions to conditional expected shortfall

Let $\mathcal{L}_0(\mathbb{P})$ denote the vector space of all $\mathcal{F}$-measurable real-valued r.v. and let $\mathcal{L}_{\mathcal{G},1}(\mathbb{P})$ and $\mathcal{L}^{-}_{\mathcal{G},1}(\mathbb{P})$ denote the cone of those $X \in \mathcal{L}_0(\mathbb{P})$ such that $X$ and $X^-$, resp., are $\sigma$-integrable w.r.t. $\mathcal{G}$.

## Definition

For a portfolio loss $L \in \mathcal{L}_0(\mathbb{P})$ consider a subportfolio loss $X \in \mathcal{L}_0(\mathbb{P})$ with $X 1_{\{L \ge q_{\mathcal{G},\delta}(L)\}} 1_{\{\delta > 0\}} \in \mathcal{L}^{-}_{\mathcal{G},1}(\mathbb{P})$. Then, the *CES contribution* of the subportfolio loss $X$ to $L$ at level $\delta$ is defined by

$$\mathrm{ES}_\delta[X, L|\mathcal{G}] = \begin{cases} X^{\mathcal{G}} & \text{on } \{\delta = 1\}, \\ \frac{1}{1-\delta}\,\mathbb{E}[f_{\mathcal{G},\delta,L} X \,|\, \mathcal{G}] & \text{on } \{\delta \in (0,1)\}, \\ \operatorname{ess\,inf}_{\delta' \in (0,1)} \frac{1}{1-\delta'}\,\mathbb{E}\big[f_{\mathcal{G},\delta',L} X \,|\, \mathcal{G}\big] & \text{otherwise}. \end{cases}$$

# Properties of contributions to conditional expected shortfall I

## Lemma

Let $L, X, Y \in \mathcal{L}_0(\mathbb{P})$ with $X 1_{\{L \geq q_{\mathcal{G},\delta}(L)\}} 1_{\{\delta > 0\}} \in \mathcal{L}^-_{\mathcal{G},1}(\mathbb{P})$ and $Y 1_{\{L \geq q_{\mathcal{G},\delta}(L)\}} 1_{\{\delta > 0\}} \in \mathcal{L}^-_{\mathcal{G},1}(\mathbb{P})$. Then, we get the following conditional properties:

(a) *Consistency with CES:* $\mathrm{ES}_\delta[L, L|\mathcal{G}] = \mathrm{ES}_\delta[L|\mathcal{G}]$.

(b) *Diversification:* $\mathrm{ES}_\delta[X, L|\mathcal{G}] \leq \mathrm{ES}_\delta[X|\mathcal{G}]$ a.s.

(c) *Linearity:* If $Z_1, Z_2 \geq 0$ a.s. are $\mathcal{G}$-measurable, then $\mathrm{ES}_\delta[Z_1 X + Z_2 Y, L|\mathcal{G}] = Z_1\,\mathrm{ES}_\delta[X, L|\mathcal{G}] + Z_2\,\mathrm{ES}_\delta[Y, L|\mathcal{G}]$ a.s. on $\{\delta < 1\}$. On $\{\delta = 1\}$ we have '$\leq$' instead.

(d) *Translation (or cash) invariance:* If $Z$ is $\mathcal{G}$-measurable, then $\mathrm{ES}_\delta[X + Z, L|\mathcal{G}] = \mathrm{ES}_\delta[X, L|\mathcal{G}] + Z$ a.s.

# Properties of contributions to conditional expected shortfall II

## Lemma (continued)

(e) *Monotonicity:* If $X \overset{\text{a.s.}}{\leq} Y$, then $\mathrm{ES}_\delta[X, L|\mathcal{G}] \overset{\text{a.s.}}{\leq} \mathrm{ES}_\delta[Y, L|\mathcal{G}]$.

(f) *Independence:* If $\delta < 1$ a.s. and if $X$ and $f_{\mathcal{G},\delta,L}$ are conditionally uncorrelated given $\mathcal{G}$, $\mathrm{ES}_\delta[X, L|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}]$ a.s.

(g) *Invariance of portfolio scale:* If $Z > 0$ a.s. is $\mathcal{G}$-measurable, then $\mathrm{ES}_\delta[X, ZL|\mathcal{G}] = \mathrm{ES}_\delta[X, L|\mathcal{G}]$ a.s.

(h) *Subportfolio continuity:* If $\delta < 1$ a.s. and if $Y \in \mathcal{L}_{\mathcal{G},1}(\mathbb{P})$, then

$$\begin{aligned}\bigl|\mathrm{ES}_\delta[X, L|\mathcal{G}] - \mathrm{ES}_\delta[Y, L|\mathcal{G}]\bigr| &\leq \mathrm{ES}_\delta[|X - Y|, L|\mathcal{G}] \\ &\leq \frac{\mathbb{E}[|X - Y||\mathcal{G}]}{1-\delta} \quad \text{a.s.}\end{aligned}$$

# Properties of contributions to conditional expected shortfall III

## Lemma (continued)

If in addition $X \in \mathcal{L}_{\mathcal{G},1}(\mathbb{P})$ and $\mathbb{P}(L \leq q_{\mathcal{G},\delta}(L)\,|\,\mathcal{G}) = \delta$ a.s. or if $X$ is a.s. constant on $\{L = q_{\mathcal{G},\delta}(L)\}$, then the following holds:

(i) *Portfolio continuity:* For every sequence $\{L_n\}_{n\in\mathbb{N}} \subset \mathcal{L}_0(\mathbb{P})$ converging to $L$ in probability, $\mathrm{ES}_\delta[X, L\,|\,\mathcal{G}] = \lim_{n\to\infty} \mathrm{ES}_\delta[X, L_n\,|\,\mathcal{G}]$, in $L^1$.

(j) *Representation by directional derivative:* Let $\delta < 1$ a.s. Then $\mathrm{ES}_\delta[X, L\,|\,\mathcal{G}] = \lim_{n\to\infty} \frac{\mathrm{ES}_\delta[L+\varepsilon_n X|\mathcal{G}] - \mathrm{ES}_\delta[L|\mathcal{G}]}{\varepsilon_n}$, in $L^1$, where $\varepsilon_n \to 0$, as $n \to \infty$.

# Definition of conditional distortion risk measures I

**Definition**

A function $g\colon [0,1] \to [0,1]$ which is increasing and left-continuous with $g(0) = 0$ and $g(1) = 1$, is called *distortion function*. Define $\bar{g}(\delta) = 1 - g(1-\delta)$, for every $\delta \in [0,1]$.

**Definition**

Let $\mathcal{L}_{g,\mathcal{G}}(\mathbb{P})$ denote the set of all $\mathcal{F}$-measurable real-valued r.v. $X$ with

$$\int_{[0,1]} q^-_{\mathcal{G},\delta}(X)\,\bar{g}(d\delta) < \infty \quad \text{a.s.}$$

# Definition of conditional distortion risk measures II

## Definition

Consider a distortion function $g$ and $X \in \mathcal{L}_{g,\mathcal{G}}(\mathbb{P})$. Then, we define the *conditional g-distortion risk measure* by

$$\rho_g[X\,|\,\mathcal{G}] = \int_{[0,1]} q_{\mathcal{G},\delta}(X)\,\bar{g}(d\delta)\,.$$

**Special cases (with deterministic $\delta \in (0,1)$):**

(a) Conditional lower quantile: $g(t) := \begin{cases} 0 & \text{for } 0 \leq t \leq 1-\delta\,, \\ 1 & \text{for } 1-\delta < t \leq 1\,. \end{cases}$

(b) CES: $g(t) := \min\left\{\frac{t}{1-\delta}, 1\right\}$, for $t \in [0,1]$.

**Remark:** C.f. Dhaene et al. [4] for the unconditional case.

# Properties of conditional distortion risk measures I

## Lemma

Given a distortion function $g$ and let $X, Y \in \mathcal{L}_{g,\mathcal{G}}(\mathbb{P})$. Then, we get the following conditional properties:

(a) *Positive homogeneity:* If $Z \geq 0$ a.s. is $\mathcal{G}$-measurable, then $\rho_g[ZX \,|\, \mathcal{G}] = Z\rho_g[X \,|\, \mathcal{G}]$ a.s.

(b) *Translation (or cash) invariance:* If $Z$ is $\mathcal{G}$-measurable, then $\rho_g[X + Z \,|\, \mathcal{G}] = \rho_g[X \,|\, \mathcal{G}] + Z$ a.s.

(c) *Comonotonic add.:* If $X$ and $Y$ are conditionally comonotonic w.r.t. $\mathcal{G}$, then $\rho_g[X + Y \,|\, \mathcal{G}] = \rho_g[X \,|\, \mathcal{G}] + \rho_g[Y \,|\, \mathcal{G}]$ a.s.

(d) *Monotonicity:* If $X \prec_{\mathrm{SD}(1,\mathcal{G})} Y$, then $\rho_g[X \,|\, \mathcal{G}] \leq \rho_g[Y \,|\, \mathcal{G}]$ a.s.

(e) *Monotonicity under concavity:* If $X \prec_{\mathrm{SD}(2,\mathcal{G})} Y$ and if $g$ is concave, then $\rho_g[X \,|\, \mathcal{G}] \leq \rho_g[Y \,|\, \mathcal{G}]$ a.s.

# Properties of conditional distortion risk measures

## Lemma (continued)

(f) *Subadditivity:* If $X + Y \in \mathcal{L}_{g,\mathcal{G}}(\mathbb{P})$ and if $g$ is concave, then $\rho_g[X + Y \,|\, \mathcal{G}] \leq \rho_g[X \,|\, \mathcal{G}] + \rho_g[Y \,|\, \mathcal{G}]$ a.s.

(g) *Convexity:* If $Z$ is $\mathcal{G}$-measurable with $0 \leq Z \leq 1$ a.s. such that $ZX + (1-Z)Y \in \mathcal{L}_{g,\mathcal{G}}(\mathbb{P})$ and if $g$ is concave, then $\rho_g[ZX + (1-Z)Y \,|\, \mathcal{G}] \leq Z\rho_g[X \,|\, \mathcal{G}] + (1-Z)\rho_g[Y \,|\, \mathcal{G}]$ a.s.

(h) *Law-invariance:* If $\mathbb{E}[f(X)|\mathcal{G}] \overset{\text{a.s.}}{=} \mathbb{E}[f(Y)|\mathcal{G}]$, for all bounded and continuous $f\colon \mathbb{R} \to \mathbb{R}$, then $\rho_g[X \,|\, \mathcal{G}] = \rho_g[Y \,|\, \mathcal{G}]$ a.s.

(i) *Alternative representation:* $\rho_g[X \,|\, \mathcal{G}] \overset{\text{a.s.}}{=} \int_{[0,1]} q_{\mathcal{G},1-\delta}(X)\, g(d\delta)$.

(j) *Derivative representation:* If $g$ is differentiable, then $\rho_g[X \,|\, \mathcal{G}] = \int_{[0,1]} q_{\mathcal{G},\delta}(X)\, \bar{g}'(\delta)\, d\delta$ a.s.

# Definition of conditional weighted expected shortfall

## Definition

Let $G\colon [0,1] \to [0,1]$ be an increasing and right-continuous function with $G(0) = 0$ and $G(1-) = 1$. Let $\mathcal{L}'_{G,\mathcal{G}}(\mathbb{P})$ denote the set of all $\mathcal{F}$-measurable real-valued r.v. $X$ with

$$\int_{[0,1]} \mathrm{ES}^{-}_{\delta}[X\,|\,\mathcal{G}]\; G(d\delta) < \infty \quad \text{a.s.}$$

## Definition

Let $X \in \mathcal{L}'_{G,\mathcal{G}}(\mathbb{P})$ and let $\{\mathrm{ES}_{\delta}[X\,|\,\mathcal{G}]\}_{\delta\in[0,1]}$ denote the version of CES with continuous paths on $\overline{\mathbb{R}}$. Then, *conditional $G$-weighted expected shortfall* is defined by

$$\mathrm{ES}_G[X\,|\,\mathcal{G}] = \int_{[0,1]} \mathrm{ES}_{\delta}[X\,|\,\mathcal{G}]\; G(d\delta)\,.$$

# Conditional weighted expected shortfall as a conditional distortion risk measure

**Lemma**

Let $X \in \mathcal{L}'_{G,\mathcal{G}}(\mathbb{P})$. Then, $\mathrm{ES}_G[X\,|\,\mathcal{G}]$ is a CDRM with concave distortion function

$$g(u) = 1 - \int_{[0,1-u]} \int_{[0,t]} \frac{G(d\delta)}{1-\delta}\, dt\,, \quad u \in [0,1]\,.$$

# Properties of conditional weighted expected shortfall I

## Lemma

If $X \in \mathcal{L}'_{G,\mathcal{G}}(\mathbb{P})$, we get the following conditional properties:

(a) *Bounds:* We have, a.s.,
$\int_{[0,1]} q_{\delta,\mathcal{G}}(X)\, G(d\delta) \le \mathrm{ES}_G[X\,|\,\mathcal{G}] \le \mathbb{E}\big[X^+\,\big|\,\mathcal{G}\big] \int_{[0,1]} \frac{1}{1-\delta}\, G(d\delta)$

(b) *Quantile representations:*

(1) $\mathrm{ES}_G[X\,|\,\mathcal{G}] = \int_{[0,1]} \frac{1}{1-\delta} \int_{[\delta,1]} q_{t,\mathcal{G}}(X)\, dt\, G(d\delta)$ a.s.

(2) $\mathrm{ES}_G[X\,|\,\mathcal{G}] = \int_{[0,1]} q_{t,\mathcal{G}}(X) \int_{[0,t]} \frac{G(d\delta)}{1-\delta}\, dt$ a.s.

(3) $\mathrm{ES}_G[X\,|\,\mathcal{G}] = \int_{[0,1]} q_{t,\mathcal{G}}\big(X \int_{[0,t]} \frac{G(d\delta)}{1-\delta}\big)\, dt$ a.s.

(c) *Distortion representation:* If $\mathcal{G}$ is trivial, then
$\mathrm{ES}_G[X\,|\,\mathcal{G}] = \mathbb{E}[Y]$ a.s. where $Y$ is a real-valued r.v. with distribution function $\bar{g} \circ F$, where $\bar{g}(u) := \int_{[0,u]} \int_{[0,t]} \frac{G(d\delta)}{1-\delta}\, dt$, for $u \in [0,1]$, and $F$ is the distribution function of $X$.

# Properties of conditional weighted expected shortfall II

## Lemma (continued)

(d) *Fatou:* Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of r.v. bounded from below by a $\mathcal{G}$-measurable r.v. $C$. Then, $X := \liminf_{n\to\infty} X_n$ satisfies $\mathrm{ES}_G[X\,|\,\mathcal{G}] \le \liminf_{n\to\infty} \mathrm{ES}_G[X_n\,|\,\mathcal{G}]$ a.s.

(e) Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of r.v. bounded from below by some constant $C$ and converging in probability to a r.v. $X$. Then, (d) holds too.

# Special case: Conditional beta-weighted expected shortfall

## Definition

Let $\alpha, \beta > -1$ with $\alpha > \beta$ and let $G$ denote a beta distribution with parameters $\alpha - \beta$ and $\beta + 1$, i.e., the density of $G$ is given by

$$f_{\mathrm{G}}(x) = \frac{1}{\mathrm{B}(\alpha-\beta, \beta+1)} x^{\alpha-\beta-1}(1-x)^{\beta} 1_{[0,1]}(x), \quad x \in \mathbb{R}\,.$$

Then, *beta-weighted CES* is given by

$$\mathrm{ES}_{\alpha,\beta}[X\,|\,\mathcal{G}] := \mathrm{ES}_G[X\,|\,\mathcal{G}]\ .$$

**Remark:** If $\mathcal{G}$ is trivial, then this definition corresponds to *beta-value-at-risk* (see Cherny and Madan [3]). In this case, fixing $\beta = 1$ results in the so called alpha-value-at-risk.

# Special case: Conditional beta-weighted expected shortfall

## Lemma

Let $\mathcal{G}$ be trivial and let $\alpha, \beta \in \mathbb{N}$ with $\beta < \alpha$. Let $(X_1, \ldots, X_\alpha)$ be a vector of $\alpha$ $\mathcal{F}$-measurable independent, identically distributed copies of $X$. Then,

$$\mathrm{ES}_{\alpha,\beta}[X \,|\, \mathcal{G}] = \mathbb{E}\left[\frac{1}{\beta} \sum_{i=\alpha-\beta+1}^{\alpha} X_{(i)}\right]$$

where $(X_{(1)}, \ldots, X_{(\alpha)})$ is the order statistic of $(X_1, \ldots, X_\alpha)$ satisfying $X_{(1)} \leq \cdots \leq X_{(\alpha)}$ a.s.

**Remark:** For alpha-value-at-risk, i.e. $\beta = 1$, we have $\mathrm{ES}_{\alpha,1}[X \,|\, \mathcal{G}] = \mathbb{E}[\max\{X_1, \ldots, X_\alpha\}]$

# A time series example I

Let $\{X_t\}_{t\in\mathbb{Z}}$ be a strictly stationary time series of negative log returns of a portfolio with dynamics

$$X_t = \mu_t + \sigma_t Z_t\,, \quad t \in \mathbb{Z}\,,$$

where $\{Z_t\}_{t\in\mathbb{Z}}$ are i.i.d. with zero mean, unit variance and marginal distribution $F$.

For $t \in \mathbb{Z}$, define $\mathcal{G}_t := \sigma(X_s, s \le t)$, and assume that $\mu_t$ and $\sigma_t > 0$ a.s. are measurable w.r.t. $\mathcal{G}_{t-1}$.

# A time series example II

## Calculation of risk measures

Then, for fixed $t \in \mathbb{Z}$ and $\mathcal{G}_{t-1}$-measurable level $\delta$, we get:

(a) $q_{\delta,\mathcal{G}_{t-1}}(X_t) = \mu_t + \sigma_t F^{\leftarrow}(\delta)$ a.s., for $t \in \mathbb{Z}$.

(b) $\mathrm{ES}_\delta[X_t \,|\, \mathcal{G}_{t-1}] = \mu_t + \sigma_t \, \mathrm{ES}_\delta[Z_t \,|\, \mathcal{G}_{t-1}]$ a.s. and, on $\{0 < \delta < 1\}$, $\mathrm{ES}_\delta[X_t \,|\, \mathcal{G}_{t-1}] = \mu_t + \frac{\sigma_t}{1-\delta} \int_{[\delta,1)} F^{\leftarrow}(u)\, du$ a.s.

(c) The CDRM with distortion function $g$ is given by $\rho_g[X_t \,|\, \mathcal{G}_{t-1}] = \mu_t + \sigma_t \int_{[0,1]} F^{\leftarrow}(u)\, \bar{g}(du)$ a.s.

**Remark:** This example is taken from McNeil and Frey [6]. For GARCH-type models they provide an estimation procedure for conditional lower quantiles and conditional expected shortfall.

# A time series example III

Now, consider a similar setup where

$$X_t = X_{t,1} + X_{t,2}\,, \quad t \in \mathbb{Z}\,,$$

with $X_{t,i} := \mu_{t,i} + \sigma_{t,i} Z_t$ such that $\sigma_{t,1} > \sigma_{t,2} > 0$ a.s. and $\mu_{t,i}$, $\sigma_{t,i}$ are $\mathcal{G}_{t-1}$-measurable, for $i = 1, 2$ and $t \in \mathbb{Z}$.

## Calculation of risk measures

Then,

$$\mathrm{ES}_\delta[X_t \,|\, \mathcal{G}_{t-1}] = (\mu_{t,1} + \mu_{t,2}) + (\sigma_{t,1} + \sigma_{t,2})\,\mathrm{ES}_\delta[Z_t \,|\, \mathcal{G}_{t-1}] \quad \text{a.s.},$$

The conditional expected shortfall contributions of $X_{t,i}$ to $X_t$ is given by

$$\mathrm{ES}_\delta[X_{t,i}, X_t \,|\, \mathcal{G}_{t-1}] = \mu_{t,i} + \sigma_{t,i}\,\mathrm{ES}_\delta[Z_t \,|\, \mathcal{G}_{t-1}] \quad \text{a.s., for } i = 1, 2\,.$$

# A time series example IV

Alternatively, assume $X_{t,2} := \mu_{t,2} + \sigma_{t,2}(-Z_t)$.

## Calculation of risk measures

Then,

$$\mathrm{ES}_\delta[X_t \,|\, \mathcal{G}_{t-1}] = (\mu_{t,1} + \mu_{t,2}) + (\sigma_{t,1} - \sigma_{t,2})\,\mathrm{ES}_\delta[Z_t \,|\, \mathcal{G}_{t-1}] \quad \text{a.s.}$$

and of course

$$\mathrm{ES}_\delta[X_{t,1}, X_t \,|\, \mathcal{G}_{t-1}] = \mu_{t,1} + \sigma_{t,1}\,\mathrm{ES}_\delta[Z_t \,|\, \mathcal{G}_{t-1}] \quad \text{a.s.}$$

**Note:** High sensitivity of conditional expected shortfall contributions to the underlying dependence structure.

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Thank you for your attention!