Duality methods for pricing contingent claims

Kristina Rognlien Dahl

Department of Mathematics, University of Oslo

June 10th, AMaMeF 2013

Plan

- Introduction.
- Conjugate duality.
- The pricing problem: No short-selling and inside information.
- Analysis of the problem via conjugate duality.
- Consequences.

Plan

- Introduction.
- Conjugate duality.
- The pricing problem: No short-selling and inside information.
- Analysis of the problem via conjugate duality.
- Consequences.

Plan

- Introduction.
- Conjugate duality.
- The pricing problem: No short-selling and inside information.
- Analysis of the problem via conjugate duality.
- Consequences.

Plan

- Introduction.
- Conjugate duality.
- The pricing problem: No short-selling and inside information.
- Analysis of the problem via conjugate duality.
- Consequences.

Plan

- Introduction.
- Conjugate duality.
- The pricing problem: No short-selling and inside information.
- Analysis of the problem via conjugate duality.
- Consequences.

Plan

- Introduction.
- Conjugate duality.
- The pricing problem: No short-selling and inside information.
- Analysis of the problem via conjugate duality.
- Consequences.

Topic

Duality methods in mathematical finance: Hot topic!

Duality methods used in:

- Pricing problems.
- Arbitrage problems.
- Utility maximization problems.
- Convex risk measures.

Recent work by King, Kramkov and Schachermayer, Pennanen, and Rogers.

Topic

Duality methods in mathematical finance: Hot topic!

Duality methods used in:

- Pricing problems.
- Arbitrage problems.
- Utility maximization problems.
- Convex risk measures.

Recent work by King, Kramkov and Schachermayer, Pennanen, and Rogers.

Topic

Duality methods in mathematical finance: Hot topic!

Duality methods used in:

- Pricing problems.
- Arbitrage problems.
- Utility maximization problems.
- Convex risk measures.

Recent work by King, Kramkov and Schachermayer, Pennanen, and Rogers.

Topic

Duality methods in mathematical finance: Hot topic!

Duality methods used in:

- Pricing problems.
- Arbitrage problems.
- Utility maximization problems.
- Convex risk measures.

Recent work by King, Kramkov and Schachermayer, Pennanen, and Rogers.

A main idea of duality

Beginning with a difficult primal optimization problem, derive a dual problem which gives bounds on the optimal value.

A main idea of duality

Beginning with a difficult primal optimization problem, derive a dual problem which gives bounds on the optimal value.

Conjugate duality

Introduced by Rockafellar (1974).

Let X be a linear space, and $f : X \to \mathbb{R}$ a function.

Primal problem:

$$\min_{x \in X} f(x)$$

Find function $F : X \times U \to \bar{\mathbb{R}}$ (where U is a linear space: the perturbation space) s.t.

$$f(x) = F(x, 0),$$

F is called the perturbation function.

Would like to choose U and F s.t. F is a closed, jointly convex function of x and u.

Conjugate duality

Introduced by Rockafellar (1974).
Let X be a linear space, and $f : X \to \mathbb{R}$ a function.

Primal problem:

$$\min_{x \in X} f(x)$$

Find function $F : X \times U \to \bar{\mathbb{R}}$ (where U is a linear space: the perturbation space) s.t.

$$f(x) = F(x, 0),$$

F is called the perturbation function.

Would like to choose U and F s.t. F is a closed, jointly convex function of x and u.

Conjugate duality

Introduced by Rockafellar (1974).
Let X be a linear space, and $f : X \to \mathbb{R}$ a function.

Primal problem:

$$\min_{x \in X} f(x)$$

Find function $F : X \times U \to \bar{\mathbb{R}}$ (where U is a linear space: the perturbation space) s.t.

$$f(x) = F(x, 0),$$

F is called the perturbation function.

Would like to choose U and F s.t. F is a closed, jointly convex function of x and u.

Conjugate duality

Introduced by Rockafellar (1974).
Let X be a linear space, and $f : X \to \mathbb{R}$ a function.

Primal problem:

$$\min_{x \in X} f(x)$$

Find function $F : X \times U \to \bar{\mathbb{R}}$ (where U is a linear space: the perturbation space) s.t.

$$f(x) = F(x, 0),$$

F is called the perturbation function.

Would like to choose U and F s.t. F is a closed, jointly convex function of x and u.

Conjugate duality

Introduced by Rockafellar (1974).
Let X be a linear space, and $f : X \to \mathbb{R}$ a function.

Primal problem:

$$\min_{x \in X} f(x)$$

Find function $F : X \times U \to \bar{\mathbb{R}}$ (where U is a linear space: the perturbation space) s.t.

$$f(x) = F(x, 0),$$

F is called the perturbation function.

Would like to choose U and F s.t. F is a closed, jointly convex function of x and u.

Optimal value function and paired spaces

Correspondingly, define the optimal value function

$$\varphi(u) = \inf_{x \in X} F(x, u), \quad u \in U.$$

If the perturbation function F is jointly convex, $\varphi(\cdot)$ is convex as well.

Pairing of two linear spaces U and Y: A real valued bilinear form $\langle \cdot, \cdot \rangle$ on $U \times Y$.

Two linear spaces are paired if they have a pairing and compatible topologies (i.e. locally convex topologies s.t. $\langle \cdot, v \rangle$ is continuous, and any continuous linear function on U is of this form for some $y \in Y$).

Optimal value function and paired spaces

Correspondingly, define the optimal value function

$$\varphi(u) = \inf_{x \in X} F(x,u), \quad u \in U.$$

If the perturbation function F is jointly convex, $\varphi(\cdot)$ is convex as well.

Pairing of two linear spaces U and Y: A real valued bilinear form $\langle \cdot, \cdot \rangle$ on $U \times Y$.

Two linear spaces are paired if they have a pairing and compatible topologies (i.e. locally convex topologies s.t. $\langle \cdot, v \rangle$ is continuous, and any continuous linear function on U is of this form for some $y \in Y$).

Optimal value function and paired spaces

Correspondingly, define the optimal value function

$$\varphi(u) = \inf_{x \in X} F(x, u), \quad u \in U.$$

If the perturbation function F is jointly convex, $\varphi(\cdot)$ is convex as well.

Pairing of two linear spaces U and Y: A real valued bilinear form $\langle \cdot, \cdot \rangle$ on $U \times Y$.

Two linear spaces are paired if they have a pairing and compatible topologies (i.e. locally convex topologies s.t. $\langle \cdot, v \rangle$ is continuous, and any continuous linear function on U is of this form for some $y \in Y$).

Optimal value function and paired spaces

Correspondingly, define the optimal value function

$$\varphi(u) = \inf_{x \in X} F(x, u), \quad u \in U.$$

If the perturbation function F is jointly convex, $\varphi(\cdot)$ is convex as well.

Pairing of two linear spaces U and Y: A real valued bilinear form $\langle \cdot, \cdot \rangle$ on $U \times Y$.

Two linear spaces are paired if they have a pairing and compatible topologies (i.e. locally convex topologies s.t. $\langle \cdot, v \rangle$ is continuous, and any continuous linear function on U is of this form for some $y \in Y$).

Optimal value function and paired spaces

Correspondingly, define the optimal value function

$$\varphi(u) = \inf_{x \in X} F(x, u), \quad u \in U.$$

If the perturbation function F is jointly convex, $\varphi(\cdot)$ is convex as well.

Pairing of two linear spaces U and Y: A real valued bilinear form $\langle \cdot, \cdot \rangle$ on $U \times Y$.

Two linear spaces are paired if they have a pairing and compatible topologies (i.e. locally convex topologies s.t. $\langle \cdot, v \rangle$ is continuous, and any continuous linear function on U is of this form for some $y \in Y$).

The Lagrange function and the dual problem

Choose linear space Y paired with U and V paired with X. Define the Lagrange function $K : X \times Y \to \bar{\mathbb{R}}$ by

$$K(x, y) = \inf\{F(x, u) + \langle u, y \rangle : u \in U\}.$$

Theorem

If $F(x, u)$ is closed and convex in u, then

$$f(x) = \sup_{y \in Y} K(x, y).$$

Motivated by this, define the dual problem,

$$\max_{y \in Y} g(y)$$

where $g(y) := \inf_{x \in X} K(x, y)$.

The Lagrange function and the dual problem

Choose linear space Y paired with U and V paired with X. Define the Lagrange function $K : X \times Y \to \bar{\mathbb{R}}$ by

$$K(x, y) = \inf\{F(x, u) + \langle u, y \rangle : u \in U\}.$$

Theorem

If $F(x, u)$ is closed and convex in u, then

$$f(x) = \sup_{y \in Y} K(x, y).$$

Motivated by this, define the dual problem,

$$\max_{y \in Y} g(y)$$

where $g(y) := \inf_{x \in X} K(x, y)$.

The Lagrange function and the dual problem

Choose linear space Y paired with U and V paired with X. Define the Lagrange function $K : X \times Y \to \bar{\mathbb{R}}$ by

$$K(x, y) = \inf\{F(x, u) + \langle u, y\rangle : u \in U\}.$$

Theorem

If $F(x, u)$ is closed and convex in u, then

$$f(x) = \sup_{y \in Y} K(x, y).$$

Motivated by this, define the dual problem,

$$\max_{y \in Y} g(y)$$

where $g(y) := \inf_{x \in X} K(x, y)$.

The Lagrange function and the dual problem

Choose linear space Y paired with U and V paired with X. Define the Lagrange function $K : X \times Y \to \bar{\mathbb{R}}$ by

$$K(x,y) = \inf\{F(x,u) + \langle u, y\rangle : u \in U\}.$$

Theorem

If $F(x,u)$ is closed and convex in u, then

$$f(x) = \sup_{y \in Y} K(x,y).$$

Motivated by this, define the dual problem,

$$\max_{y \in Y} g(y)$$

where $g(y) := \inf_{x \in X} K(x,y)$.

Conjugate duality: the dual problem

Hence we have a primal problem

$$\min_{x \in X} f(x),$$

and a corresponding dual problem

$$\max_{y \in Y} g(y).$$

The optimal value of this dual problem gives a lower bound on the optimal value of the primal problem.

Can sometimes show no duality gap.

Conjugate duality: the dual problem

Hence we have a primal problem

$$\min_{x \in X} f(x),$$

and a corresponding dual problem

$$\max_{y \in Y} g(y).$$

The optimal value of this dual problem gives a lower bound on the optimal value of the primal problem.

Can sometimes show no duality gap.

Conjugate duality: the dual problem

Hence we have a primal problem

$$\min_{x \in X} f(x),$$

and a corresponding dual problem

$$\max_{y \in Y} g(y).$$

The optimal value of this dual problem gives a lower bound on the optimal value of the primal problem.

Can sometimes show no duality gap.

Conjugate duality: the dual problem

Hence we have a primal problem

$$\min_{x \in X} f(x),$$

and a corresponding dual problem

$$\max_{y \in Y} g(y).$$

The optimal value of this dual problem gives a lower bound on the optimal value of the primal problem.

Can sometimes show no duality gap.

Duality result

Important duality result by Rockafellar:

Theorem

The function g in (D) is closed and concave. Also

$$\sup_{y \in Y} g(y) = \mathrm{cl}(\mathrm{co}(\varphi))(0)$$

and

$$\inf_{x \in X} f(x) = \varphi(0).$$

The theorem implies: If φ is convex, the lower semi-continuity of φ is sufficient for the absence of duality gap.

The financial market model

- Probability space $(\Omega, \mathcal{F}, P)$.
- $N+1$ assets: N risky assets, one bond.
- Price processes (stochastic): $S_0(t,\omega)$ (bond), $S_1(t,\omega), \dots, S_N(t,\omega)$.
- Assume that the price processes are discounted.
- Time: $t = 0, 1, \dots, T$.

The financial market model

- Probability space $(\Omega, \mathcal{F}, P)$.
- $N+1$ assets: N risky assets, one bond.
- Price processes (stochastic): $S_0(t,\omega)$ (bond), $S_1(t,\omega), \dots, S_N(t,\omega)$.
- Assume that the price processes are discounted.
- Time: $t = 0, 1, \dots, T$.

The financial market model

- Probability space $(\Omega, \mathcal{F}, P)$.
- $N + 1$ assets: N risky assets, one bond.
- Price processes (stochastic): $S_0(t, \omega)$ (bond), $S_1(t, \omega), \ldots, S_N(t, \omega)$.
- Assume that the price processes are discounted.
- Time: $t = 0, 1, \ldots, T$.

The financial market model

- Probability space $(\Omega, \mathcal{F}, P)$.
- $N + 1$ assets: N risky assets, one bond.
- Price processes (stochastic): $S_0(t, \omega)$ (bond), $S_1(t, \omega), \dots, S_N(t, \omega)$.
- Assume that the price processes are discounted.
- Time: $t = 0, 1, \dots, T$.

The financial market model

- Probability space $(\Omega, \mathcal{F}, P)$.
- $N + 1$ assets: N risky assets, one bond.
- Price processes (stochastic): $S_0(t, \omega)$ (bond), $S_1(t, \omega), \dots, S_N(t, \omega)$.
- Assume that the price processes are discounted.
- Time: $t = 0, 1, \dots, T$.

The seller

Consider a seller in this market, selling a contingent claim: B.

Associated to this seller there is a filtration: $(\mathcal{G}_t)_t$.

The prices S are assumed to be adapted to this filtration: Seller knows prices, and maybe something more.

Note: $(\mathcal{G}_t)_t$ need not be generated by S: Seller has a general level of inside information.

Also: Seller not allowed to short sell in asset 1.

(Asset 1 is chosen to simplify notation: Also holds for arbitrary set of stocks with no short selling.)

The seller

Consider a seller in this market, selling a contingent claim: B.

Associated to this seller there is a filtration: $(\mathcal{G}_t)_t$.

The prices S are assumed to be adapted to this filtration: Seller knows prices, and maybe something more.

Note: $(\mathcal{G}_t)_t$ need not be generated by S: Seller has a general level of inside information.

Also: Seller not allowed to short sell in asset 1.

(Asset 1 is chosen to simplify notation: Also holds for arbitrary set of stocks with no short selling.)

The seller

Consider a seller in this market, selling a contingent claim: B.

Associated to this seller there is a filtration: $(\mathcal{G}_t)_t$.

The prices S are assumed to be adapted to this filtration: Seller knows prices, and maybe something more.

Note: $(\mathcal{G}_t)_t$ need not be generated by S: Seller has a general level of inside information.

Also: Seller not allowed to short sell in asset 1.

(Asset 1 is chosen to simplify notation: Also holds for arbitrary set of stocks with no short selling.)

The seller

Consider a seller in this market, selling a contingent claim: B.

Associated to this seller there is a filtration: $(\mathcal{G}_t)_t$.

The prices S are assumed to be adapted to this filtration: Seller knows prices, and maybe something more.

Note: $(\mathcal{G}_t)_t$ need not be generated by S: Seller has a general level of inside information.

Also: Seller not allowed to short sell in asset 1.

(Asset 1 is chosen to simplify notation: Also holds for arbitrary set of stocks with no short selling.)

The seller

Consider a seller in this market, selling a contingent claim: B.

Associated to this seller there is a filtration: $(\mathcal{G}_t)_t$.

The prices S are assumed to be adapted to this filtration: Seller knows prices, and maybe something more.

Note: $(\mathcal{G}_t)_t$ need not be generated by S: Seller has a general level of inside information.

Also: Seller not allowed to short sell in asset 1.

(Asset 1 is chosen to simplify notation: Also holds for arbitrary set of stocks with no short selling.)

The seller's pricing problem

Which price must the seller demand?

$$\inf_{\{v,H\}} \quad v$$

subject to

$$\begin{array}{rcll} S(0)\cdot H(0) & \leq & v, & \\ S(T)\cdot H(T-1) & \geq & B & \text{for } \omega\in\Omega, \\ S(t)\cdot \Delta H(t) & = & 0 & \text{for } 1\leq t\leq T-1, \text{ and for } \omega\in\Omega, \\ H_1(t) & \geq & 0 & \text{for } 0\leq t\leq T-1, \text{ and for } \omega\in\Omega \end{array}$$

where $v \in \mathbb{R}$ and H is $(\mathcal{G}_t)_t$-adapted.

Hence: Minimize the price of the claim s.t. the seller can pay B at time T from investments in an affordable, self-financing, predictable portfolio, which does not sell short in asset 1.

The seller's pricing problem

Which price must the seller demand?

$$\begin{array}{llll} \inf_{\{v,H\}} & v & & \\ \text{subject to} & & & \\ S(0)\cdot H(0) & \leq & v, & \\ S(T)\cdot H(T-1) & \geq & B & \text{for } \omega \in \Omega, \\ S(t)\cdot \Delta H(t) & = & 0 & \text{for } 1 \leq t \leq T-1, \text{ and for } \omega \in \Omega, \\ H_1(t) & \geq & 0 & \text{for } 0 \leq t \leq T-1, \text{ and for } \omega \in \Omega \end{array}$$

where $v \in \mathbb{R}$ and H is $(\mathcal{G}_t)_t$-adapted.

Hence: Minimize the price of the claim s.t. the seller can pay B at time T from investments in an affordable, self-financing, predictable portfolio, which does not sell short in asset 1.

The seller's pricing problem

Which price must the seller demand?

$$
\begin{array}{lrcll}
\inf_{\{v,H\}} & & v & & \\
\text{subject to} & & & & \\
& S(0)\cdot H(0) & \leq & v, & \\
& S(T)\cdot H(T-1) & \geq & B & \text{for } \omega\in\Omega, \\
& S(t)\cdot \Delta H(t) & = & 0 & \text{for } 1\leq t\leq T-1, \text{ and for } \omega\in\Omega, \\
& H_1(t) & \geq & 0 & \text{for } 0\leq t\leq T-1, \text{ and for } \omega\in\Omega
\end{array}
$$

where $v\in\mathbb{R}$ and H is $(\mathcal{G}_t)_t$-adapted.

Hence: Minimize the price of the claim s.t. the seller can pay B at time T from investments in an affordable, self-financing, predictable portfolio, which does not sell short in asset 1.

Deriving the dual problem via conjugate duality

Choose the perturbation function F to be:

$$
\begin{aligned}
F(H,u) &:= S(0)\cdot H(0); \quad && B - S(T)\cdot H(T-1) \leq u_1, \\
& && S(t)\cdot \Delta H(t) = u_2^t \;\forall\; 1 \leq t \leq T-1, \\
& && -H_1(t) \leq u_3^t \;\forall\; 0 \leq t \leq T-1, \\
& && S(0)\cdot H(0) \geq u_4 \text{ and} \\
F(H,u) &:= \infty && \text{otherwise.}
\end{aligned}
$$

where the inequalities hold a.e. and
$u = (u_1, (u_2^t)_{t=1}^{T-1}, (u_3^t)_{t=0}^{T-1}, u_4)$, $u \in \mathcal{L}^p(\Omega, \mathcal{F}, P : \mathbb{R}^{2T+1})$

Get corresponding Lagrange function $K(H,y)$.

Deriving the dual problem via conjugate duality

Choose the perturbation function F to be:

$$
\begin{aligned}
F(H,u) &:= S(0)\cdot H(0); \quad && B - S(T)\cdot H(T-1) \leq u_1, \\
& && S(t)\cdot \Delta H(t) = u_2^t \;\forall\; 1 \leq t \leq T-1, \\
& && -H_1(t) \leq u_3^t \;\forall\; 0 \leq t \leq T-1, \\
& && S(0)\cdot H(0) \geq u_4 \text{ and} \\
F(H,u) &:= \infty && \text{otherwise.}
\end{aligned}
$$

where the inequalities hold a.e. and
$u = (u_1, (u_2^t)_{t=1}^{T-1}, (u_3^t)_{t=0}^{T-1}, u_4)$, $u \in \mathcal{L}^p(\Omega, \mathcal{F}, P : \mathbb{R}^{2T+1})$

Get corresponding Lagrange function $K(H,y)$.

The dual problem

Also get corresponding dual problem:

$$
\begin{array}{llcl}
\sup_{\{y \in Y : y_1 \geq 0\}} & \mathbb{E}[y_1 B] & & \\
\text{s.t.} & & & \\
(i) & \int_A S_i(0)dP & = & \int_A y_2^1 S_i(1)dP, \\
(i)^* & \int_A S_1(0)dP & \geq & \int_A y_2^1 S_1(1)dP, \\
(ii) & \int_A y_2^t S_i(t)dP & = & \int_A y_2^{t+1} S_i(t+1)dP, \\
(ii)^* & \int_A S_1(t) y_2^t dP & \geq & \int_A y_2^{t+1} S_1(t+1)dP, \\
(iii) & \int_A y_2^{T-1} S_i(T-1)dP & = & \int_A y_1 S_i(T)dP, \\
(iii)^* & \int_A y_2^{T-1} S_1(T-1)dP & \geq & \int_A y_1 S_1(T)dP.
\end{array}
$$

where $(i), (ii)$ and (iii) hold for $i \neq 1$, and $(i), (i)^*$ are for $A \in \mathcal{G}_0$, (ii) and $(ii)^*$ for $A \in \mathcal{G}_t, t = 1, \ldots, T-2$, and (iii), $(iii)^*$ for $A \in \mathcal{G}_{T-1}$.

Rewrite the dual problem

Want to make dual problem more interpretable.

Denote by $\bar{\mathcal{M}}_1^a(S, \mathcal{G})$ the set of absolutely continuous probability measures Q s.t. the prices $S_0, S_2, \ldots, S_N$ are Q-martingales and S_1 is a Q-super-martingale (w.r.t. $(\mathcal{G}_t)_t$).

Theorem

The dual problem is equivalent to the following optimization problem.

$$\sup_{Q \in \bar{\mathcal{M}}_1^a(S,\mathcal{G})} \mathbb{E}_Q[B].$$

Idea of proof

Show maximizing sets are equivalent.

Use:

- Change of measure under conditional expectation.
- Radon-Nikodym theorem.
- Induction.
- Double expectation (tower property).
- Martingale/super-martingale definition.

Idea of proof

Show maximizing sets are equivalent.

Use:

- Change of measure under conditional expectation.
- Radon-Nikodym theorem.
- Induction.
- Double expectation (tower property).
- Martingale/super-martingale definition.

Strong duality

Would like to prove that there is no duality gap.

Do this via theorem from Pennanen and Perkkiö which guarantees lower semi-continuity (l.s.c.) of value function.

From duality theorem of Rockafellar (and setup), l.s.c. of value function implies no duality gap (since we chose the perturbation function F convex).

Hence value(primal)=value(dual), so the seller's price of the contingent claim is

$$\sup_{Q \in \bar{\mathcal{M}}_1^a(S,\mathcal{G})} \mathbb{E}_Q[B].$$

Strong duality

Would like to prove that there is no duality gap.

Do this via theorem from Pennanen and Perkkiö which guarantees lower semi-continuity (l.s.c.) of value function.

From duality theorem of Rockafellar (and setup), l.s.c. of value function implies no duality gap (since we chose the perturbation function F convex).

Hence value(primal)=value(dual), so the seller's price of the contingent claim is

$$\sup_{Q \in \bar{\mathcal{M}}_1^a(S,\mathcal{G})} \mathbb{E}_Q[B].$$

Strong duality

Would like to prove that there is no duality gap.

Do this via theorem from Pennanen and Perkkiö which guarantees lower semi-continuity (l.s.c.) of value function.

From duality theorem of Rockafellar (and setup), l.s.c. of value function implies no duality gap (since we chose the perturbation function F convex).

Hence value(primal)=value(dual), so the seller's price of the contingent claim is

$$\sup_{Q \in \bar{\mathcal{M}}_1^a(S,\mathcal{G})} \mathbb{E}_Q[B].$$

Strong duality

Would like to prove that there is no duality gap.

Do this via theorem from Pennanen and Perkkiö which guarantees lower semi-continuity (l.s.c.) of value function.

From duality theorem of Rockafellar (and setup), l.s.c. of value function implies no duality gap (since we chose the perturbation function F convex).

Hence value(primal)=value(dual), so the seller's price of the contingent claim is

$$\sup_{Q \in \bar{\mathcal{M}}_1^a(S,\mathcal{G})} \mathbb{E}_Q[B].$$

The price

The previous derivation goes through similarly if there are short selling constraints on several of the risky assets.

Hence the price of B for a seller who has short selling constraints on risky assets $1, \ldots, k$, where $k \in \{1, \ldots, N\}$ is

$$\beta := \sup_{Q \in \bar{\mathcal{M}}^a_{1,\ldots,k}(S,\mathcal{G})} \mathbb{E}_Q[B].$$

where $\bar{\mathcal{M}}^a_{1,\ldots,k}(S,\mathcal{G})$ is the set of abs. cont. probability measures Q s.t. the prices $S_0, S_{k+1}, \ldots, S_N$ are Q-martingales and $S_1, \ldots, S_k$ are Q-super-martingales (w.r.t. $(\mathcal{G}_t)_t$).

Can also be seen from Kramkov and Föllmer combined with Pulido, but with different approach.

The price

The previous derivation goes through similarly if there are short selling constraints on several of the risky assets.

Hence the price of B for a seller who has short selling constraints on risky assets $1, \ldots, k$, where $k \in \{1, \ldots, N\}$ is

$$\beta := \sup_{Q \in \bar{\mathcal{M}}^a_{1,\ldots,k}(S,\mathcal{G})} \mathbb{E}_Q[B].$$

where $\bar{\mathcal{M}}^a_{1,\ldots,k}(S,\mathcal{G})$ is the set of abs. cont. probability measures Q s.t. the prices $S_0, S_{k+1}, \ldots, S_N$ are Q-martingales and $S_1, \ldots, S_k$ are Q-super-martingales (w.r.t. $(\mathcal{G}_t)_t$).

Can also be seen from Kramkov and Föllmer combined with Pulido, but with different approach.

The price

The previous derivation goes through similarly if there are short selling constraints on several of the risky assets.

Hence the price of B for a seller who has short selling constraints on risky assets $1, \ldots, k$, where $k \in \{1, \ldots, N\}$ is

$$\beta := \sup_{Q \in \bar{\mathcal{M}}^a_{1,\ldots,k}(S,\mathcal{G})} \mathbb{E}_Q[B].$$

where $\bar{\mathcal{M}}^a_{1,\ldots,k}(S,\mathcal{G})$ is the set of abs. cont. probability measures Q s.t. the prices $S_0, S_{k+1}, \ldots, S_N$ are Q-martingales and $S_1, \ldots, S_k$ are Q-super-martingales (w.r.t. $(\mathcal{G}_t)_t$).

Can also be seen from Kramkov and Föllmer combined with Pulido, but with different approach.

Implications

Convex duality and mathematical finance

Introduction

Duality theory

Pricing

Final comments

We compare the price offered by a seller with short selling constraints to that of an unconstrained seller.

Theorem

The difference between the prices offered by the two sellers is

$$\beta - \sup_{Q \in \mathcal{M}^e(S,\mathcal{G})} \mathbb{E}_Q[B] \geq 0,$$

where β is defined as above.

Proof.

Use previous result with no shortselling, or see Delbaen and Schachermayer. □

Implications

We compare the price offered by a seller with short selling constraints to that of an unconstrained seller.

Theorem

The difference between the prices offered by the two sellers is

$$\beta - \sup_{Q \in \mathcal{M}^e(S,\mathcal{G})} \mathbb{E}_Q[B] \geq 0,$$

where β is defined as above.

Proof.

Use previous result with no shortselling, or see Delbaen and Schachermayer. □

Implications (continued)

Convex duality and mathematical finance

Introduction

Duality theory

Pricing

Final comments

Can also compare the prices offered by two sellers with different levels of information.

Theorem

Consider two sellers, both with short selling constraints on risky assets $1, 2, \ldots, k$. Let their filtrations be denoted by $(\mathcal{G}_t)_t$ and $(\mathcal{H}_t)_t$ respectively. Assume one seller has more information than the other seller (so, $\mathcal{H}_t \subseteq \mathcal{G}_t$ for all $0 \leq t \leq T$). Then the difference between the prices offered by the two sellers is

$$\sup_{Q \in \bar{\mathcal{M}}^a_{1,2,\ldots,k}(S,\mathcal{H})} \mathbb{E}_Q[B] - \sup_{Q \in \bar{\mathcal{M}}^a_{1,2,\ldots,k}(S,\mathcal{G})} \mathbb{E}_Q[B] \geq 0.$$

Implications (continued)

Can also compare the prices offered by two sellers with different levels of information.

Theorem

Consider two sellers, both with short selling constraints on risky assets $1, 2, \ldots, k$. Let their filtrations be denoted by $(\mathcal{G}_t)_t$ and $(\mathcal{H}_t)_t$ respectively. Assume one seller has more information than the other seller (so, $\mathcal{H}_t \subseteq \mathcal{G}_t$ for all $0 \leq t \leq T$). Then the difference between the prices offered by the two sellers is

$$\sup_{Q \in \bar{\mathcal{M}}^a_{1,2,\ldots,k}(S,\mathcal{H})} \mathbb{E}_Q[B] - \sup_{Q \in \bar{\mathcal{M}}^a_{1,2,\ldots,k}(S,\mathcal{G})} \mathbb{E}_Q[B] \geq 0.$$

Implications (continued)

Can also compare the prices offered by two sellers with different levels of information.

Theorem

Consider two sellers, both with short selling constraints on risky assets $1, 2, \ldots, k$. Let their filtrations be denoted by $(\mathcal{G}_t)_t$ and $(\mathcal{H}_t)_t$ respectively. Assume one seller has more information than the other seller (so, $\mathcal{H}_t \subseteq \mathcal{G}_t$ for all $0 \leq t \leq T$). Then the difference between the prices offered by the two sellers is

$$\sup_{Q \in \bar{\mathcal{M}}^a_{1,2,\ldots,k}(S,\mathcal{H})} \mathbb{E}_Q[B] - \sup_{Q \in \bar{\mathcal{M}}^a_{1,2,\ldots,k}(S,\mathcal{G})} \mathbb{E}_Q[B] \geq 0.$$

Implications

In particular, the seller with more information will offer B at a lower price than the seller with less information.

Idea of proof:

- Definitions of martingale/super-martingale.
- Double expectation (tower property).

This theorem can give understanding of the origin of price bubbles in financial markets:

- Several buyers believe they have extra information.
- Buyers will offer high prices for B.
- Hence price of claim increases: Bubble.
- If it turns out that the buyers beliefs were wrong, the bubble bursts.

Implications

In particular, the seller with more information will offer B at a lower price than the seller with less information.

Idea of proof:

- Definitions of martingale/super-martingale.
- Double expectation (tower property).

This theorem can give understanding of the origin of price bubbles in financial markets:

- Several buyers believe they have extra information.
- Buyers will offer high prices for B.
- Hence price of claim increases: Bubble.
- If it turns out that the buyers beliefs were wrong, the bubble bursts.

Implications

In particular, the seller with more information will offer B at a lower price than the seller with less information.

Idea of proof:

- Definitions of martingale/super-martingale.
- Double expectation (tower property).

This theorem can give understanding of the origin of price bubbles in financial markets:

- Several buyers believe they have extra information.
- Buyers will offer high prices for B.
- Hence price of claim increases: Bubble.
- If it turns out that the buyers beliefs were wrong, the bubble bursts.

Final comments

Goal of presentation: Study how duality methods can be used to solve a pricing problem.

Conjugate duality method:

- Have primal optimization problem: Seller's pricing problem.
- Define suitable perturbation function F.
- Find corresponding Lagrange function K.
- Find corresponding dual problem, which gives lower bound of the primal problem.
- Rewrite dual problem so that it can be interpreted.
- Show that there is no duality gap (using theorems by Rockafellar and Pennanen and Perkkiö).

Hope this presentation has illustrated potential for exploiting duality theory in many other problems in mathematical finance!

Final comments

Goal of presentation: Study how duality methods can be used to solve a pricing problem.

Conjugate duality method:

- Have primal optimization problem: Seller's pricing problem.
- Define suitable perturbation function F.
- Find corresponding Lagrange function K.
- Find corresponding dual problem, which gives lower bound of the primal problem.
- Rewrite dual problem so that it can be interpreted.
- Show that there is no duality gap (using theorems by Rockafellar and Pennanen and Perkkiö).

Hope this presentation has illustrated potential for exploiting duality theory in many other problems in mathematical finance!

Final comments

Goal of presentation: Study how duality methods can be used to solve a pricing problem.

Conjugate duality method:

- Have primal optimization problem: Seller's pricing problem.
- Define suitable perturbation function F.
- Find corresponding Lagrange function K.
- Find corresponding dual problem, which gives lower bound of the primal problem.
- Rewrite dual problem so that it can be interpreted.
- Show that there is no duality gap (using theorems by Rockafellar and Pennanen and Perkkiö).

Hope this presentation has illustrated potential for exploiting duality theory in many other problems in mathematical finance!

Final comments

Goal of presentation: Study how duality methods can be used to solve a pricing problem.

Conjugate duality method:

- Have primal optimization problem: Seller's pricing problem.
- Define suitable perturbation function F.
- Find corresponding Lagrange function K.
- Find corresponding dual problem, which gives lower bound of the primal problem.
- Rewrite dual problem so that it can be interpreted.
- Show that there is no duality gap (using theorems by Rockafellar and Pennanen and Perkkiö).

Hope this presentation has illustrated potential for exploiting duality theory in many other problems in mathematical finance!

Final comments

Goal of presentation: Study how duality methods can be used to solve a pricing problem.

Conjugate duality method:

- Have primal optimization problem: Seller's pricing problem.
- Define suitable perturbation function F.
- Find corresponding Lagrange function K.
- Find corresponding dual problem, which gives lower bound of the primal problem.
- Rewrite dual problem so that it can be interpreted.
- Show that there is no duality gap (using theorems by Rockafellar and Pennanen and Perkkiö).

Hope this presentation has illustrated potential for exploiting duality theory in many other problems in mathematical finance!

Final comments

Goal of presentation: Study how duality methods can be used to solve a pricing problem.

Conjugate duality method:

- Have primal optimization problem: Seller's pricing problem.
- Define suitable perturbation function F.
- Find corresponding Lagrange function K.
- Find corresponding dual problem, which gives lower bound of the primal problem.
- Rewrite dual problem so that it can be interpreted.
- Show that there is no duality gap (using theorems by Rockafellar and Pennanen and Perkkiö).

Hope this presentation has illustrated potential for exploiting duality theory in many other problems in mathematical finance!

Final comments

Goal of presentation: Study how duality methods can be used to solve a pricing problem.

Conjugate duality method:

- Have primal optimization problem: Seller's pricing problem.
- Define suitable perturbation function F.
- Find corresponding Lagrange function K.
- Find corresponding dual problem, which gives lower bound of the primal problem.
- Rewrite dual problem so that it can be interpreted.
- Show that there is no duality gap (using theorems by Rockafellar and Pennanen and Perkkiö).

Hope this presentation has illustrated potential for exploiting duality theory in many other problems in mathematical finance!

Final comments

Goal of presentation: Study how duality methods can be used to solve a pricing problem.

Conjugate duality method:

- Have primal optimization problem: Seller's pricing problem.
- Define suitable perturbation function F.
- Find corresponding Lagrange function K.
- Find corresponding dual problem, which gives lower bound of the primal problem.
- Rewrite dual problem so that it can be interpreted.
- Show that there is no duality gap (using theorems by Rockafellar and Pennanen and Perkkiö).

Hope this presentation has illustrated potential for exploiting duality theory in many other problems in mathematical finance!

Thank you for your attention! :-)

Some key references

- Pennanen, T.: Convex duality in stochastic optimization and mathematical finance. Mathematics of Opererations Research. **36**, 340–362 (2011)
- Pennanen, T., Perkkiö, A.P.: Stochastic programs without duality gaps. Mathematical Programming. **136**, 91–110 (2012)
- Rockafellar, R. T.: Conjugate Duality and Optimization. Society for Industrial and Applied Mathematics, Philadelphia (1974)
- Rockafellar, R. T.: Convex Analysis. Princeton University Press, Princeton, (1972)