Eidgenössische Technische Hochschule Zürich
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Efficient option pricing for time-inhomogeneous processes

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joint work with V. Kazeev and Ch. Schwab
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  $\gamma \in (0,1)$, $M > 1$, $\alpha \in (0,2)$ and $t \in (0,T)$.
- Degenerate diffusion models with generators of the form:

$$\mathcal{A}(t)\phi(x) = t^\gamma b^\top \nabla\phi(x) + t^\gamma \frac{1}{2}\mathrm{tr}[\Sigma D^2\phi(x)]$$

- Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space. An adapted càdlàg stochastic process $(X(t))_{0\leq t\leq T}$ is a time-inhomogeneous Lévy process if
  (i) X has independent increments
  (ii) $\forall t \in [0,T]$, we have

$$\begin{aligned}\mathbb{E}[e^{i(u,X(t))}] &= \exp\Big(\int_0^t (i(u,b(s)) - \frac{1}{2}(u,\Sigma(s)u) \\ &\quad + \int_{\mathbb{R}^d} (e^{i(u,z)} - 1 - i(u,z)\mathbb{1}_{|z|<1})\nu(s,dz)\Big)ds\end{aligned}$$

  - $b(s) \in \mathbb{R}^d$, $\Sigma(s) \in \mathbb{R}^{d\times d}$ symmetric, positive semidefinite
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  - For properties of time-inhomogeneous Lévy processes we refer, for example, to the dissertations of W. Kluge'05 and K.Glau'11.

- Consider the following model problem:

$$\begin{aligned} \partial_t u - t^\gamma L u &= f \text{ on } (0,T] \times D, \\ u(0) &= g \text{ on } \{0\} \times D \text{ and } u|_{\partial D} = 0, \end{aligned}$$

where $\gamma \in (-1,1)$, L is self-adjoint, $L \in \mathcal{L}(V,V^*)$, $V = H_0^1(D)$ such that

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Main Challanges:

- Well-posedness (weighted spaces needed)
- Discretization in time (classical time-marching schemes not applicable)
- Discretization in space due to possibly high-dimensional structure

BB-Conditions I

Let Hilbert spaces X, Y and the bilinear form $B(\cdot,\cdot) : X \times Y \to \mathbb{R}$ be given, then the BB-conditions read

$$\inf_{0\neq u\in X} \sup_{0\neq v\in Y} \frac{B(u,v)}{\|u\|_X \, \|v\|_Y} > 0, \qquad (1)$$

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and

$$\sup_{0\neq u\in X, 0\neq v\in Y} \frac{|B(u,v)|}{\|u\|_X \|v\|_Y} < \infty. \tag{3}$$

BB-Conditions II

Theorem

Let the bilinear form $B(\cdot,\cdot) : X \times Y \to \mathbb{R}$ satisfy (1)-(3), then the problem: find $u \in X$ such that

$$B(u, v) = f(v), \quad \forall v \in Y, \tag{4}$$

admits a unique solution with $\|u\|_X^2 \leq C \|f\|_{Y^*}^2$ for $f \in Y^*$ and $C > 0$.

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In our case

- $B(u,v) = \int_0^T ((\dot{u},v) + t^\gamma a(u,v))\ dt$, $a(u,v) = (Lu,v)$

BB-Conditions III

For $V := H_0^1(D)$, $I = (0,1)$ and

$$L^2_{t^\gamma}(I) \quad := \quad \overline{C^\infty(0,1)}^{\|\cdot\|_{L^2_{t^\gamma}(I)}}, \qquad \|u\|^2_{L^2_{t^\gamma}(I)} := \int_I u^2 t^\gamma \, dt.$$

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We have $B(\cdot,\cdot) : \mathcal{X}_{(0} \times \mathcal{Y} \to \mathbb{R}$ for $\mathcal{X}$ and $\mathcal{Y}$ given as

$$\begin{aligned}
\mathcal{X} &:= H^1_{t^{-\gamma}}(I;V^*) \cap L^2_{t^\gamma}(I;V) \\
&\cong \left(H^1_{t^{-\gamma}}(I) \otimes V^*\right) \cap \left(L^2_{t^\gamma}(I) \otimes V\right), \\
\mathcal{Y} &:= L^2_{t^\gamma}(I;V) \cong L^2_{t^\gamma}(I) \otimes V, \\
\mathcal{X}_{(0} &:= \{w \in \mathcal{X} : w(0,\cdot) = 0 \text{ in } V^*\}.
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Proof see [OR 2012].:

- Eigenfunction expansion of the diffusion operator
- Consideration of the arising systems of ODEs

Discretization

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- We discretize the space-time domain using appropriate tensor products of wavelet functions.
  - Main advantage: They form Riesz bases of the corresponding function spaces allowing for efficient preconditioning.
  - Main drawback: Possibly hard to construct and to implement.
- The temporal basis is given as $\Theta = \{\theta_\lambda : \lambda \in \nabla_\Theta\}$ and spatial basis as $\Sigma = \{\chi_\mu : \mu \in \nabla_\Sigma\} = \bigotimes_{i=1}^{d} \Sigma_i$.

Solution process

The bi-infinite system corresponding to $B(u, v) = f(v)$ reads

$$\begin{aligned}
\mathbf{B}\mathbf{u} &= \mathbf{f}, \qquad (5)\\
\mathbf{B} &= \left[(\Theta', \Theta) \otimes (\Sigma, \Sigma) + \int_I t^\gamma a(\Theta \otimes \Sigma, \Theta \otimes \Sigma) dt\right] \\
&\quad \times \left(\mathbb{1}_t \otimes \|\Sigma\|_V^{-1}\right) \|\Theta \otimes \Sigma\|_{\mathcal{X}}^{-1} \\
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- Optimal (x,t)-adaptive algorithms for the approximate solution of (5) available, cf. [Ch. Schwab & R. Stevenson 2008], [OR 2012].

Discontinuous Galerkin timestepping I

$\mathcal{M} = \{I_m\}_{m=1}^{M+1}$, $M \in \mathbb{N}$, partition of $(0,T)$, $\underline{r} \in \mathbb{N}_0^{M+1}$ dG orders.
dG-FEM: $U \in \mathcal{V}^{\underline{r}}(\mathcal{M};V) := \{u : J \to V : u|_{I_m} \in \mathcal{P}^{r_m}(I_m, V), m = 1,\dots,M+1\}$, such that for all $v \in \mathcal{V}^{\underline{r}}(\mathcal{M};V)$

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$$
\begin{aligned}
B_{\mathrm{dG}}(U,v) &= F_{\mathrm{dG}}(v), \text{ where} \\
B_{\mathrm{dG}}(U,v) &= \sum_{m=1}^{M} \int_{I_m} (U',v)_{L^2(D)}\,dt + \sum_{m=1}^{M} \int_{I_m} t^\gamma a(U,v)\,dt \\
&+ \sum_{m=2}^{M} ([U]_{m-1}, v_{m-1}^+)_{L^2(D)} + (U_0^+, v_0^+)_{L^2(D)} \\
F_{\mathrm{dG}} &= (u_0, v_0^+)_{L^2(D)} + \sum_{m=1}^{M} \int_{I_m} (f(t),v)_{V^*,V}\,dt.
\end{aligned}
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Geometric Timesteps/ linear order vector

- A geometric time partition $\mathcal{M}_{M,q} = \{I_m\}_{m=1}^{M+1}$ with grading factor $q \in (0,1)$ and $M+1$ time steps I_m, $m = 1, \ldots, M+1$ is given by the nodes

$$t_0 = 0, \quad t_m = Tq^{M+1-m}, \quad 1 \leq m \leq M+1.$$

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- A polynomial degreee vector $\underline{r} = \{r_m\}_{m=1}^{M+1}$ is called linear with slope $\nu > 0$ on the geometric partition $\mathcal{M}_{M,q}$ on $(0,T)$ of

$$r_1 = 0 \text{ and } r_m = \lfloor \nu m \rfloor \text{ for } 2 \leq m \leq M+1.$$

Discontinuous Galerkin timestepping II

Theorem (V. Kazeev, OR, Ch. Schwab 2012)

Consider the time-inhomogeneous forward problem on $J = (0, 1)$ with initial data $u_0 \in H_\theta$ for some $\theta \in (0, 1]$ and right hand side f. Discretize in time using dGFEM on a geometric partition $\mathcal{M}_{M,q}$. Then for all degree vectors $\underline{r} = (r_1, \dots, r_M)$ with slope $\nu \geq \nu_0 > 0$ the semidiscrete dGFEM solution U obtained in $\mathcal{V}^{\underline{r}}(\mathcal{M}_{M,q}, V)$ converges exponentially w.r.t. to N, No. of "time-DOFs":

$$\|u - U\|_{L^2_{t^{\gamma/2}}(J;V)} \leq C(q, \nu_0) \exp(-bN^{-1/2}).$$

Curse of dimension

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- Spatial discretization using finite elements or finite differences suffers from the "curse of dimension"
- Sparse grids can be used

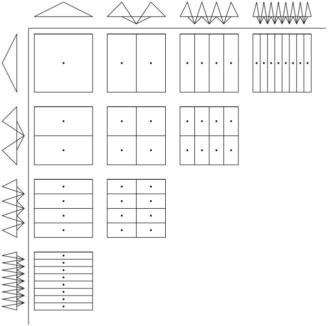

Figure: Sparse grid in two dimensions

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- We say that $A \in \mathbb{R}^{n_1 \times \ldots \times n_d}$, $n_k \geq 1$, $k \in \{1, \ldots, d\}$, $d \geq 1$ is represented in the TT-format if

$$A(i_1, \ldots, i_d) = G_1(i_1)G_2(i_2)\ldots G_d(i_d),$$

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- Storage cost: $\mathcal{O}(dr^2 n)$, where $n_k \leq n$ and $r_k \leq r$, $k = 1, \ldots, d$
- Operations such as addition, matrix-matrix multiplication, matrix-vector multiplication available in the format
- Solver for linear equations available based on alternating least squares

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- Non-uniform meshes are required in order to resolve incompatible initial data appropriately, the solution has boundary layers for small times t.
- Shishkin meshes employed here.

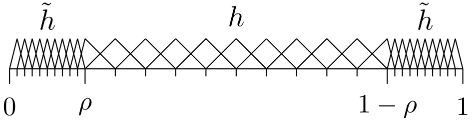

Figure: A Shishkin mesh in 1D. The meshwidths are h and $\tilde{h}$, the width of the boundary zone is ρ.

Test problem

Find $u \in \mathcal{X}$ such that for all $v \in \mathcal{Y}$,

$$\int_J \left[\langle \dot{u}(t), v(t) \rangle_{L^2(D)} + t^\gamma \langle \nabla u(t), \nabla v(t) \rangle_{L^2(D)} \right] \mathrm{d}t = 0$$

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- Compatible initial data:

$$u_0(x_1, \ldots, x_d) = \prod_{k=1}^{d} \sin \pi x_k \quad \text{for} \quad x_k \in (0,1), \quad 1 \le k \le d.$$

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- Incompatible initial data:

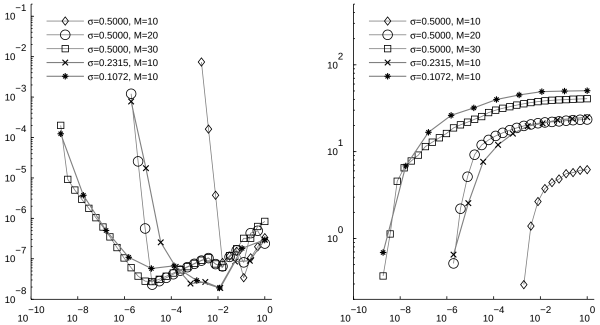
$$u_0(x_1, \ldots, x_d) = 1 \quad \text{for} \quad x_k \in (0,1), \quad 1 \le k \le d.$$

Compatible initial conditions

	$\gamma = -\frac{1}{2}$, $M = 30$		$\gamma = 0$, $M = 10$		$\gamma = \frac{1}{2}$, $M = 10$	
d	$\texttt{err}\left[u_M^\delta\right]$	time	$\texttt{err}\left[u_M^\delta\right]$	time	$\texttt{err}\left[u_M^\delta\right]$	time
5	$1.1 \cdot 10^{-8}$	12.2	$8.8 \cdot 10^{-10}$	3.9	$1.0 \cdot 10^{-11}$	4.1
10	$3.1 \cdot 10^{-8}$	24.2	$1.4 \cdot 10^{-9}$	7.5	$6.9 \cdot 10^{-11}$	7.5
20	$5.6 \cdot 10^{-8}$	47.4	$2.4 \cdot 10^{-9}$	15.2	$1.7 \cdot 10^{-10}$	14.6
30	$9.0 \cdot 10^{-8}$	71.8	$3.1 \cdot 10^{-9}$	23.1	$1.9 \cdot 10^{-10}$	21.6
40	$1.9 \cdot 10^{-7}$	96.4	$3.7 \cdot 10^{-9}$	31.6	$2.8 \cdot 10^{-10}$	29.3

Table: Compatible initial data in d dimensions: relative L^2-error ($\texttt{err}\left[u_M^\delta\right]$) at $t = T$ and computation times in seconds for $q = 0.5$.

Incompatible initial conditions: time discretization

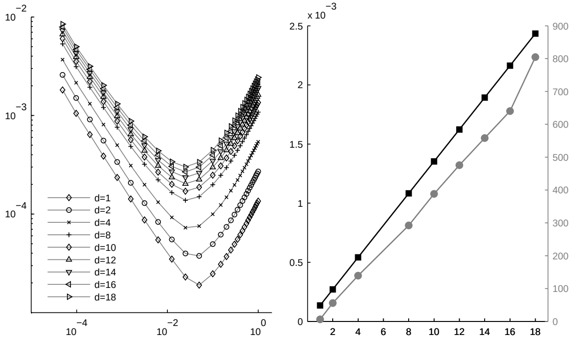


(a) Relative L^2-error vs. t_m

(b) Computation time vs. t_m

Figure: Comparison of DG-discretizations in time

Incompatible initial conditions: space discretization



(a) Relative L^2-error vs. t_m

(b) relative L^2-error (black) and total computation time (gray) vs. d

Figure: Multivariate problem with incompatible initial data

Conclusion

- Time-degenerate models using weighted spaces in time and a space-time approach were considered.
- CG discretizations in space-time were analyzed.
- DG in time for time-inhomogeneous models was discussed.
- Spatial discretization using the TT-format was outlined.
- Shishkin meshes for the resolution of boundary layers were used.

References:

- Low-rank tensor structure of linear diffusion operators in the TT and QTT formats, with V. Kazeev and Ch. Schwab, LAA, 2013.
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Thank you very much!!