Multi-Person Game Options in Discrete and Continuous Time

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Outline

References

This talk is based to the following working papers:

Guo, I. and Rutkowski, M.:
A zero-sum multi-player game. *Demonstratio Mathematica* 45 (2012), 415–433.

Guo, I.:
Unilaterally competitive multi-player stopping games. Working paper, University of Sydney, 2011.

Guo, I. and Rutkowski, M.:
Multi-person game options. Working paper, University of Sydney, 2012.

Nie, T. and Rutkowski, M.:
Multi-dimensional reflected BSDEs for multi-person stopping games. Working paper, 2013.

... and several related papers by other authors, for instance,

Karatzas, I. and Li, Q.:
BSDE approach to non-zero-sum stochastic differential games of control and stopping. Working paper, 2011.

Goals

Our main goals are:

1. to examine equilibria for certain multi-player stochastic games,
2. to find explicit algorithm for finding the value process for a class of multi-player stopping games,
3. to examine the multi-dimensional reflected backward stochastic difference equation for the value process of the multi-player stopping game,
4. to find arbitrage prices and super-hedging strategies for a multi-person game option in discrete time,
5. to propose an extension to the continuous-time setup via multi-dimensional reflected backward stochastic differential equation.

REMINDER: TWO-PERSON GAME OPTIONS

Two-Person Zero-Sum Game Options

Definition

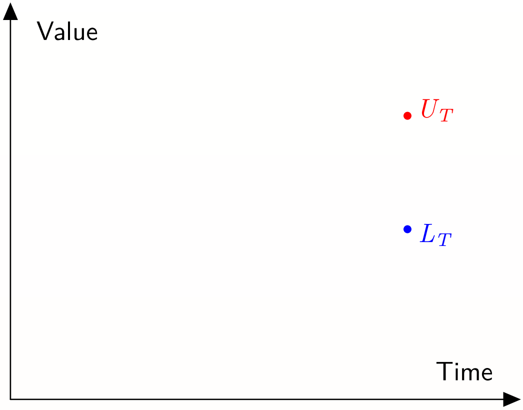
A *game option* is a contract where each party has the right to exercise at any time before expiry T according to the following rules:

- The holder can *exercise* the option at any time $t < T$ for the payoff L_t.
- The isssuer can *cancel* the option at any time $t < T$ for the cancellation fee of U_t.
- If the option is not exercised then it expires at time T and the terminal payoff for the holder equals ξ.
- The assumption that $L_t \leq U_t$ for every t will ensure that the outcome of the contract is always well defined.

We denote by Y_t the *arbitrage price* of the game option at time t.

The game option is closely related to the zero-sum Dynkin (stopping) game.

Valuation Scheme for the Game Option



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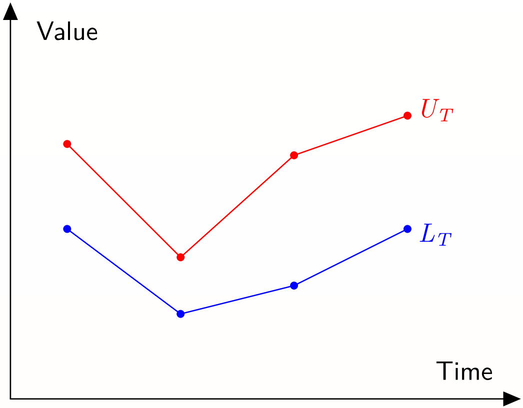

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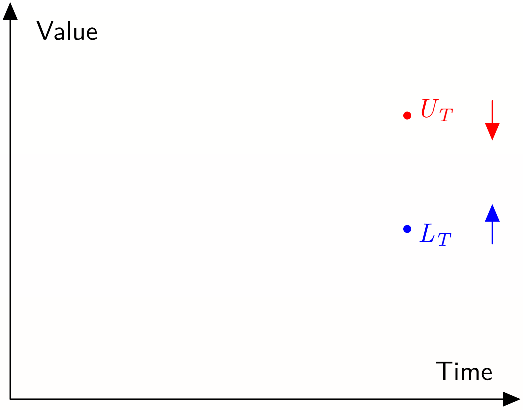

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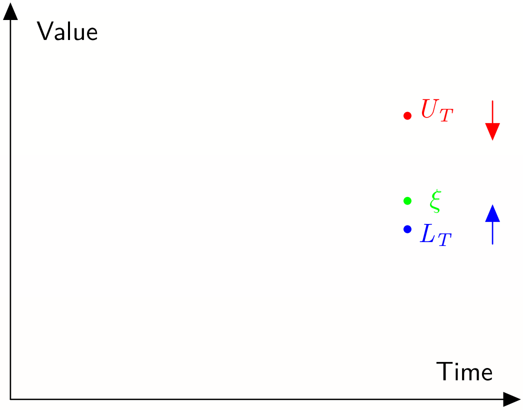

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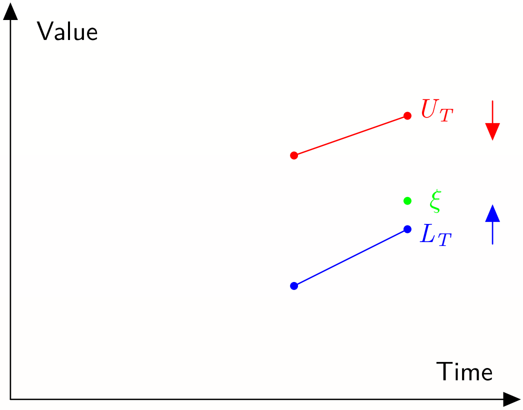

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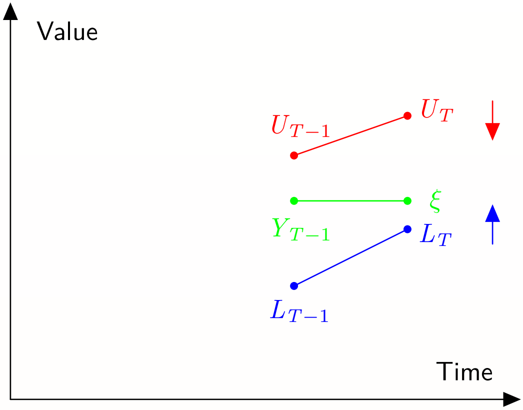

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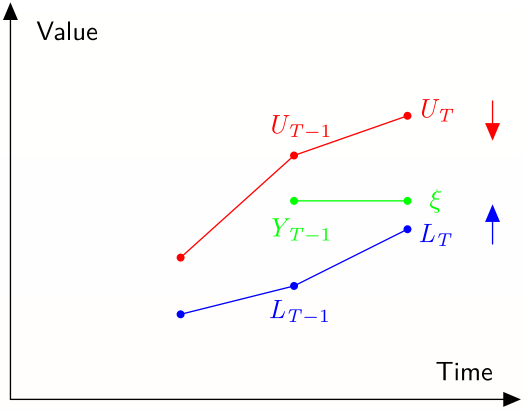

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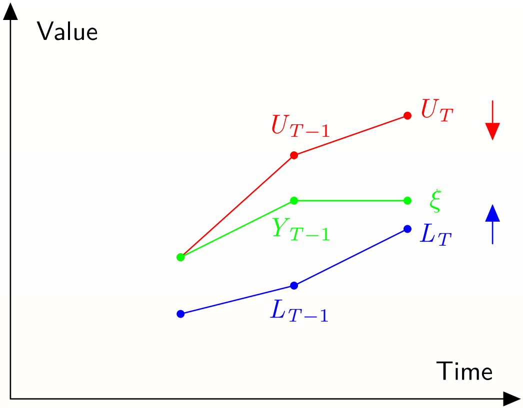

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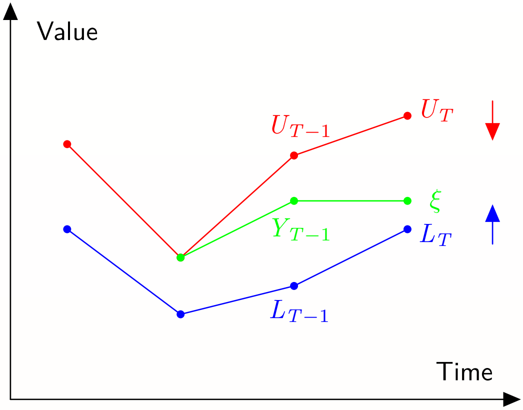

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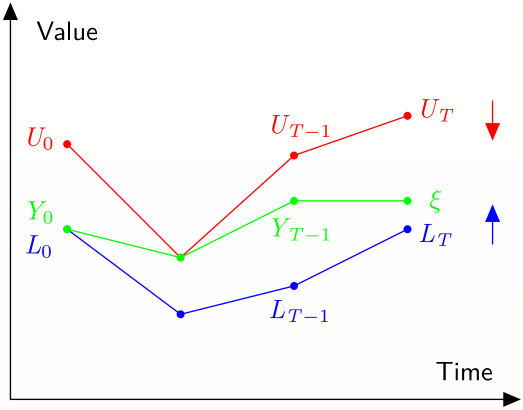


Value Process via Reflected BSDE

Consider a complete and arbitrage-free market model with the unique martingale measure $\mathbb{P}^*$ for the discounted prices S. We denote $\Delta Y_{t+1} = Y_{t+1} - Y_t$.

Definition

A *solution* to the reflected BSDE (L, U, ξ, S) is a quadruplet (Y, Z, K^1, K^2) of processes that satisfy for $t = 0, 1, \ldots, T$

$$
\begin{aligned}
& Y_t + \sum_{u=t}^{T-1} Z_u \cdot \Delta S_{u+1} - (K^1_T - K^1_t) + (K^2_T - K^2_t) = \xi \\
& L_t \leq Y_t \leq U_t \\
& \sum_{t=0}^{T-1} \mathbb{1}_{\{Y_t > L_t\}} \, \Delta K^1_{t+1} = 0 \\
& \sum_{t=0}^{T-1} \mathbb{1}_{\{Y_t < U_t\}} \, \Delta K^2_{t+1} = 0
\end{aligned}
$$

where K^1 and K^2 are $\mathbb{F}$-predictable and non-decreasing processes.

Value Process via Projection

Proposition

The unique solution to reflected BSDE (L, U, ξ, S) *equals* $Y_T = \xi$ *and*

$$Y_t = \min\Big(U_t,\, \max\big(L_t,\, \mathbb{E}_{\mathbb{P}^*}(Y_{t+1} \,|\, \mathcal{F}_t)\big)\Big).$$

for $t = 0, 1, \dots, T-1$*. Equivalently,* $Y_T = \xi$ *and for* $t = 0, \dots, T-1$

$$Y_t = \pi_{[L_t, U_t]}\left(\mathbb{E}_{\mathbb{P}^*}\left(Y_{t+1} \,\middle|\, \mathcal{F}_t\right)\right).$$

The arbitrage price process of the zero-sum two-person game option equals Y*. The rational exercise time for the buyer equals*

$$\tau_1 = \min\left\{ t \in \{0, \dots, T-1\} \,|\, \Delta K^1_{t+1} > 0 \right\}$$

and the rational exercise time for the seller equals

$$\tau_2 = \min\left\{ t \in \{0, \dots, T-1\} \,|\, \Delta K^2_{t+1} > 0 \right\}.$$

EQUILIBRIA OF MULTI-PLAYER STOPPING GAMES

Equilibria of Multi-Player Stopping Games

Consider an m-person stochastic stopping game in which to goal of each player is to maximise his expected payoff. Let $s = (s^1, \dots, s^m)$ be an m-tuple of exercise times. The expected payoff of the kth player is denoted by $J_k(s^1, \dots, s^m)$ or $J_k(s^k, s^{-k})$ where $s^{-k} = (s^1, \dots, s^{k-1}, s^{k+1}, \dots, s^m)$.

Definition

A family $(\tau^1, \dots, \tau^m)$ of stopping times is said to be a *Nash equilibrium* if

$$J_k(\tau^k, \tau^{-k}) \geq J_k(s_k, \tau^{-k}), \quad \forall\, s^k.$$

A family $(\tau^1, \dots, \tau^m)$ of stopping times is called an *optimal equilibrium* when it is a Nash equilibrium and

$$J_k(\tau^k, s^{-k}) \geq J_k(\tau^k, \tau^{-k}), \quad \forall\, s^{-k}.$$

For zero-sum stopping games any Nash equilibrium is also an optimal equilibrium.

Maximin and Minimax Values

Definition

The *lower value* (or *maximin value*) V_k^l for player k is defined by

$$V_k^l = \sup_{s^k} \inf_{s^{-k}} J_k(s^k, s^{-k}).$$

A *maximin strategy* is any s^k such that $J_k(s^k, s^{-k}) \geq V_k^l$ for all s^{-k}.

Definition

The *upper value* (or *minimax value*) V_k^u for player k is defined by

$$V_k^u = \inf_{s^{-k}} \sup_{s^k} J_k(s^k, s^{-k}).$$

A *minimax strategy* is any s^{-k} such that $J_k(s^k, s^{-k}) \leq V_k^u$ for all s^k.

Value of the Game

Lemma

In any m-person stochastic stopping game the following holds:

1. *if $(\tau^1, \dots, \tau^m)$ is an optimal equilibrium then it is an optimal strategy, in the sense that*

$$\inf_{s^{-k}} J_k(\tau^k, s^{-k}) = J_k(\tau^k, \tau^{-k}) = \sup_{s^k} J_k(s^k, \tau^{-k}),$$

2. *the inequality $V_k^u \geq V_k^l$ is valid,*
3. *If $(\tau^1, \dots, \tau^m)$ is an optimal equilibrium then $V_k^u = V_k^l = J_k(\tau^k, \tau^{-k})$.*

Definition

If $V_k^l = V_k^u$ then $V_k^* := V_k^l = V_k^u$ is called the *value* of the game for player k. The *value* of the game is the vector $(V_1^*, \dots, V_m^*)$, provided that it is well defined.

Weakly Unilaterally Competitive Games

Definition (Kats and Thisse (1992))

An m-player game is said to be *weakly unilaterally competitive* (WUC) if for every $k, l = 1, \dots, m$, $k \neq l$ and all $s^k, \widehat{s}^k, s^{-k}$ the following implications hold

$$J_k(s^k, s^{-k}) > J_k(\widehat{s}^k, s^{-k}) \;\Rightarrow\; J_l(s^k, s^{-k}) \leq J_l(\widehat{s}^k, s^{-k})$$

$$J_k(s^k, s^{-k}) = J_k(\widehat{s}^k, s^{-k}) \;\Rightarrow\; J_l(s^k, s^{-k}) = J_l(\widehat{s}^k, s^{-k}).$$

Proposition (Kats and Thisse (1992), De Wolf (1999))

If $(\tau^1, \dots, \tau^m)$ is a Nash equilibrium for a WUC game then is also an optimal equilibrium and:

1. $\min_{s^{-k}} J_k(\tau^k, s^{-k}) = J_k(\tau^k, \tau^{-k})$ *for every* $k = 1, \dots, m$,
2. *the equality* $J_k(\tau^k, \tau^{-k}) = V_k^u = V_k^l$ *is valid and for every player the strategies* τ^k *and* τ^{-k} *are maximin and minimax strategies, respectively,*
3. (s^k, τ^{-k}) *is an optimal equilibrium if and only if* s^k *is a maximin strategy.*

MULTI-PLAYER SINGLE-PERIOD GAMES

Deterministic Single-Period WUC Game

We first focus on the single-period game where exercise is only allowed at $t = 0$.

- Players: $\mathcal{M} = \{1, 2, \ldots, m\}$.
- Exercise payoff: $\boldsymbol{x} = (x_1, \ldots, x_m)$ where x_k is the amount received by player k if he exercises at time 0.
- Terminal payoff: $\boldsymbol{p} = (p_1, \ldots, p_m)$ where p_k is the amount received by player k if no player exercises at time 0.
- $\sum_{k \in \mathcal{M}} p_k = c$.
- $\sum_{k \in \mathcal{M}} x_k \leq c$.
- c is the total value of the contract.

Redistribution of losses:

- In the two player case, when one player exercises, the payoff, or ‘burden’ of this action is paid entirely by the other player.
- In the multi-player case, when someone exercises, this ‘burden’ should be split among non-exercising players according to some predetermined rule.

Strategies and Exercise

- The *strategy* $s^k \in \mathcal{S}^k = \{0,1\}$ of player k specifies if he will exercise at $t = 0$.
- Then any $s = (s^1, \dots, s^m) \in \{0,1\}^m$ is a *strategy set*.
- Given a strategy set s, the *exercise set* $\mathcal{E}(s)$ is the set of players who exercised at time 0.

Definition

For a strategy set s, the *modified payoff* $\boldsymbol{v}(s) = (v_1(s), \dots, v_m(s))$ is the actual payoff received by the players if a strategy set s is carried out. We set

$$v_k(s) = \begin{cases} x_k & k \in \mathcal{E}(s), \\ p_k - w_k(s) \sum_{j \in \mathcal{E}(s)} (x_j - p_j) & k \in \mathcal{M} \setminus \mathcal{E}(s). \end{cases}$$

This means that

- exercising players receive their exercise payoffs,
- non-exercising payoffs receive their terminal payoffs diminished by their allocated 'burdens'.

Weights of Strategy Sets

- This is a *constant-sum game*: $\sum_{k\in\mathcal{M}} v_k(s) = \sum_{k\in\mathcal{M}} p_k = c$, except when all players exercise.
- Weights are used to determine how the burden of exercising is split between the non-exercising players. They depend on strategy sets.
- For any strategy set s, $w_k(s)$ is defined for all non-exercising players, that is, for all $k \in \mathcal{M} \setminus \mathcal{E}(s)$.
- We assume $w_k(\mathcal{E}) \neq 0$ for any non-empty subset $\mathcal{E} \subset \mathcal{M}$, $\mathcal{E} \neq \mathcal{M}$ and $k \notin \mathcal{E}$.

Proposition

The game $\mathcal{G}$ is WUC for all choices $\boldsymbol{x}$ and $\boldsymbol{p}$ if and only if the weights can be written in the following form:

$$w_k(\mathcal{E}) = \frac{\alpha_k}{1 - \sum_{i\in\mathcal{E}} \alpha_i}$$

where $\alpha_k > 0$ and $\sum_{i\neq k} \alpha_i < 1$ for all k.

Vector Space and Projection

Definition

The modified payoff $\boldsymbol{v}(s^*)$ corresponding to an optimal equilibrium s^* is called the *value* of the game.

The value $\boldsymbol{v}^* = \boldsymbol{v}(s^*)$ is unique. We will now address the following question: how to express the value $\boldsymbol{v}^*$ in terms of vectors $\boldsymbol{v}$ and $\boldsymbol{x}$?

Proposition

If $\sum_{k\in\mathcal{M}} x_k = c$ *then the unique value satisfies* $\boldsymbol{v}^* = \boldsymbol{x}$. *Moreover, the strategy set* $s^* = (0, \ldots, 0)$ *is an optimal equilibrium.*

We endow the space $\mathbb{R}^m$ with the norm $\|\cdot\|$ generated by the inner product

$$\langle \boldsymbol{y}, \boldsymbol{z} \rangle = \sum_{k=1}^{m} \left(\frac{y_k z_k}{\alpha_k} \right).$$

Hyperplanes and Modified Payoffs

For any vector $\boldsymbol{p}$ and any closed convex set $\mathbb{K}$ in $\mathbb{R}^m$, there exists a unique projection $\pi_{\mathbb{K}}(\boldsymbol{p})$ of $\boldsymbol{p}$ onto $\mathbb{K}$ such that: $\pi_{\mathbb{K}}(\boldsymbol{p}) \in \mathbb{K}$ and

$$\|\pi_{\mathbb{K}}(\boldsymbol{p}) - \boldsymbol{p}\| \leq \|\boldsymbol{q} - \boldsymbol{p}\| \quad \forall\, \boldsymbol{q} \in \mathbb{K}.$$

For any proper subset $\mathcal{E} \subset \mathcal{M}$, we define the hyperplane

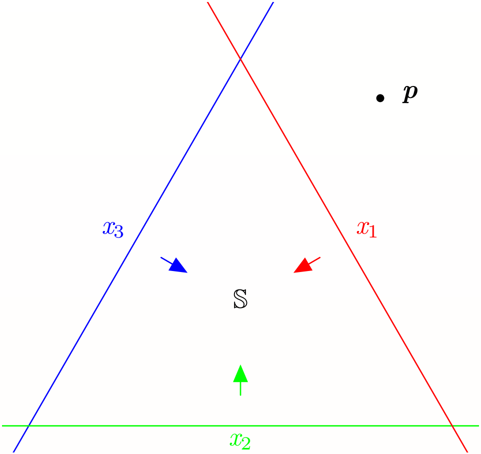
$$\mathcal{H}_{\mathcal{E}} = \left\{ \boldsymbol{y} \in \mathbb{R}^m : \; y_i = x_i \text{ for all } i \in \mathcal{E} \text{ and } \sum_{k=1}^{m} y_k = c \right\}.$$

Lemma

Let s be any strategy set such that $\mathcal{E}(s)$ is a proper subset of $\mathcal{M}$. Then the vector $\boldsymbol{v}(s)$ of modified payoffs equals

$$\boldsymbol{v}(s) = \pi_{\mathcal{H}_{\mathcal{E}(s)}}(\boldsymbol{p}).$$

Modified Payoff as Projection: Suboptimal



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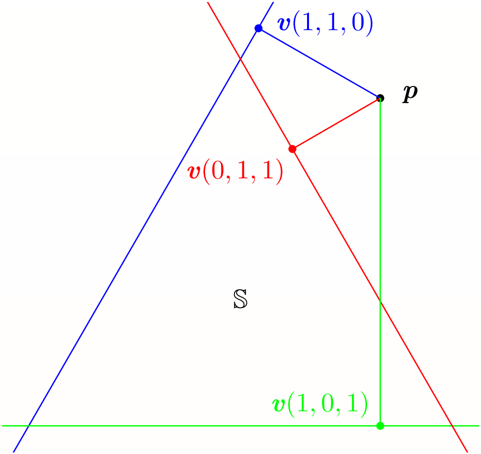

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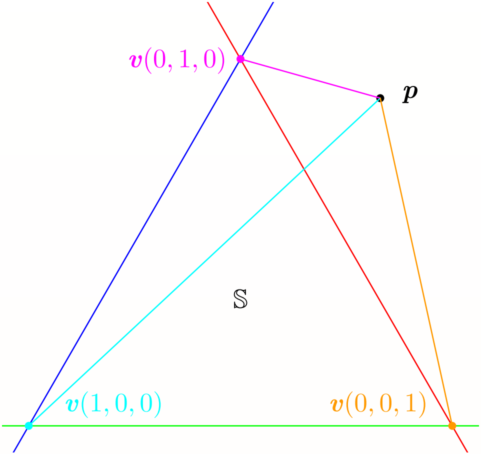


Existence and Uniqueness of the Value

Consider the simplex $\mathbb{S}$ given by

$$\mathbb{S} = \left\{ \boldsymbol{y} \in \mathbb{R}^m : \; y_k \geq x_k, \, 1 \leq k \leq m \text{ and } \sum_{k=1}^{m} y_k = c \right\}.$$

Proposition

Assume that $\sum_{k=1}^{m} x_k < c$. *Then:*

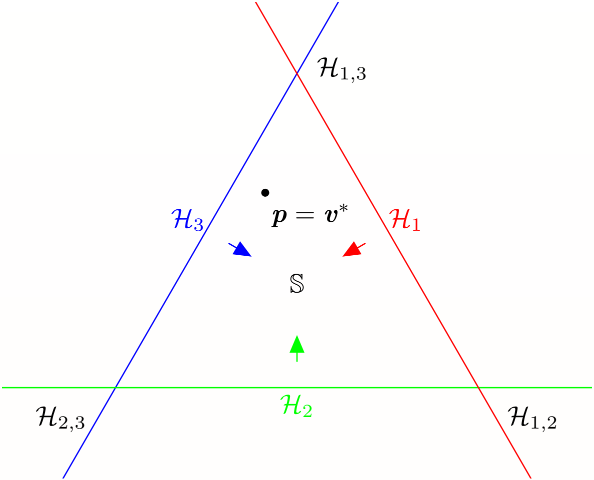
1. *a strategy set* s^* *is an optimal equilibrium for the game if and only if the set of exercising players* $\mathcal{E}(s^*)$ *is such that*

$$\pi_{\mathcal{H}_{\mathcal{E}(s^*)}}(\boldsymbol{p}) = \pi_{\mathbb{S}}(\boldsymbol{p}), \qquad (*)$$

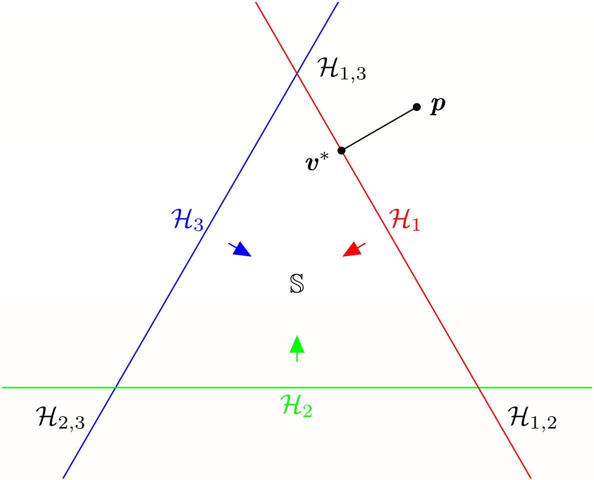
2. *a strategy set* s^* *satisfying* $(*)$ *always exists and the unique value of the game equals*

$$\boldsymbol{v}^* = \boldsymbol{v}(s^*) = (v_1(s^*), \ldots, v_m(s^*)) = \pi_{\mathbb{S}}(\boldsymbol{p}).$$

Value of the Game: Optimal Equilibrium



Value of the Game: Optimal Equilibrium



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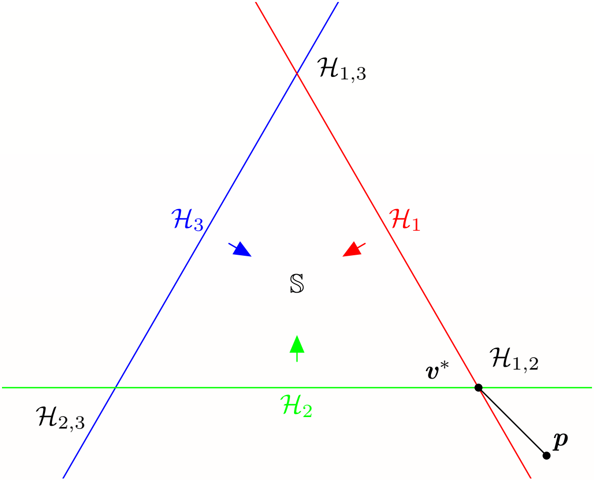

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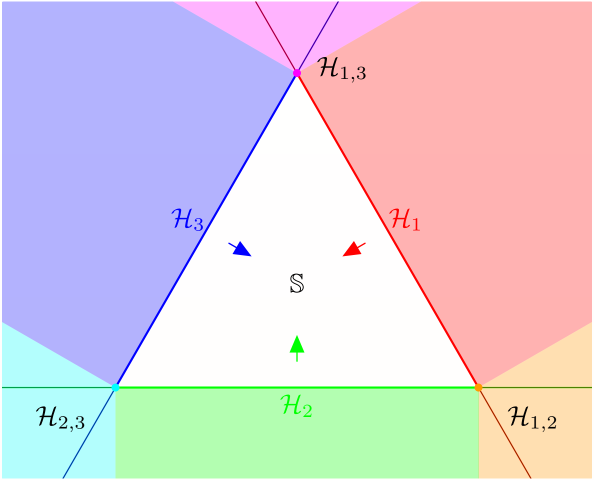


Multi-Period Zero-Sum Extension

One possible formulation is the *compound game* approach: for $t = T$ we set $V^*(T) = X_T$. For each $t = 0, \dots, T-1$, we consider the game with modified payoffs:

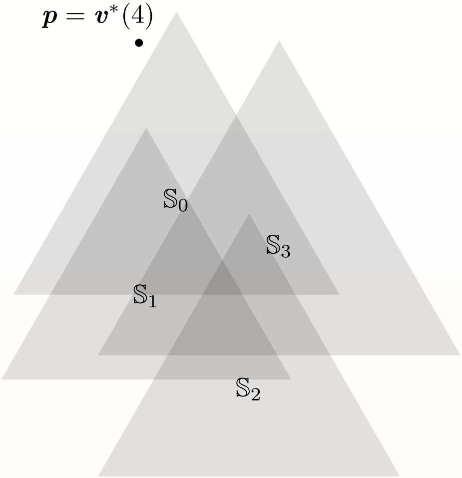
$$V_k(t) = \begin{cases} X_k(t), & k \in \mathcal{E}_t, \\ V_k^*(t+1) - w_k(\mathcal{E}_t) \sum_{j \in \mathcal{E}_t} (X_j(t) - V_j^*(t+1)), & k \notin \mathcal{E}_t. \end{cases}$$

According to this specification of a multi-period game at each time t player k can either

- stop (or exercise) the game for $X_{k,t}$ or
- receive a suitably adjusted amount based on the value of the subgame starting at time $t+1$.

Let us first assume that the multi-period game happens to be a zero-sum game at each stage. Then it can be solved using the method developed for the single-period game.

Multi-Period Zero-Sum Extension: $T = 4$



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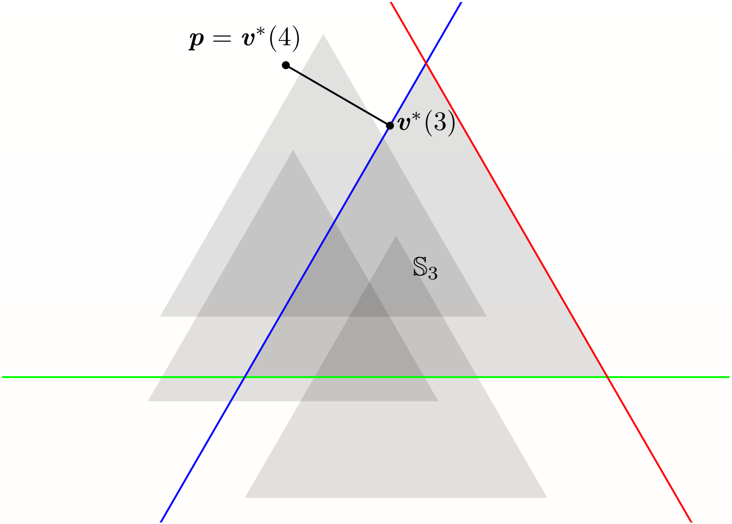

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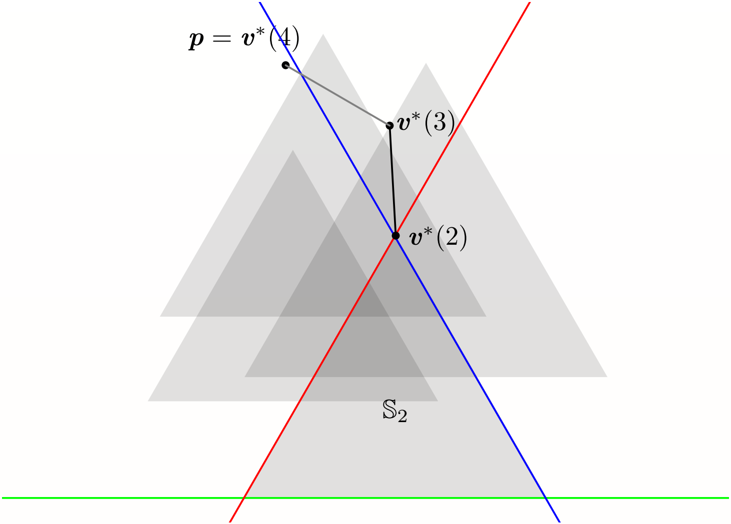

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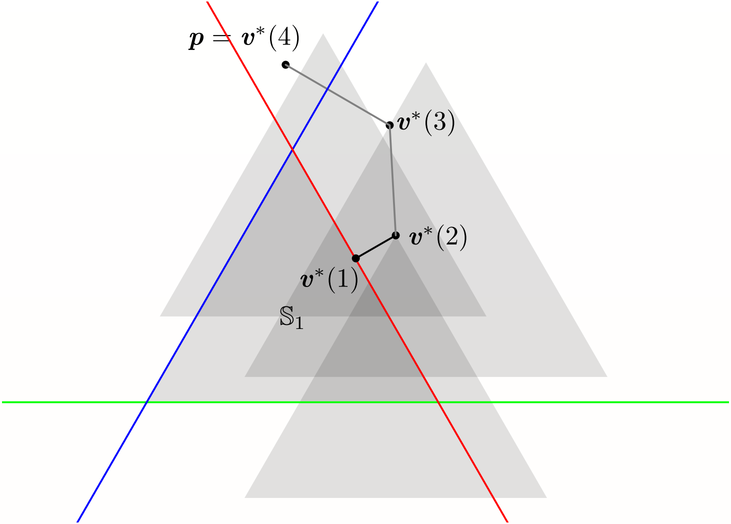
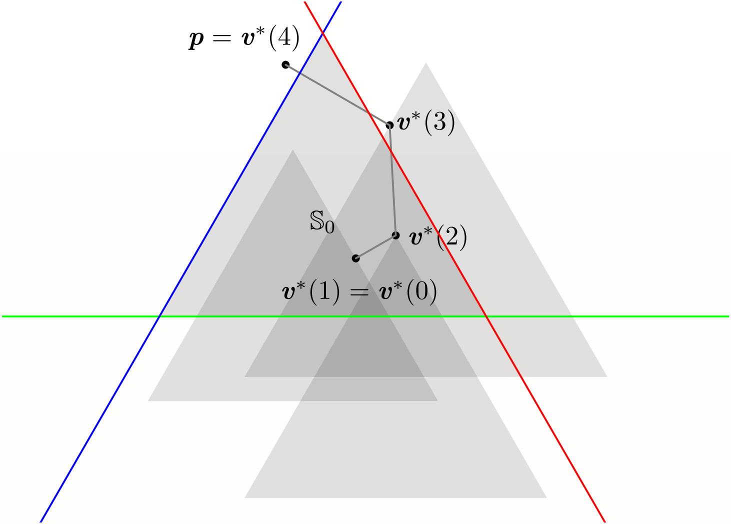


Multi-Period Zero-Sum Extension: $T = 4$



Non-Zero-Sum Multi-Period Stopping Game

The assumption that the game is zero-sum has essential drawbacks:

- It is not suitable to impose this condition in the multi-period stochastic case,
- One has to decide how the game is settled when everyone decides to exercise prematurely.

To overcome this difficulty, we propose to introduce a *dummy* player $m+1$ who

- does not has the right to exercise the game,
- covers a possible shortfall when all other players decide to exercise simultaneously.

Then the non-zero-sum game can be solved using similar techniques as for the zero-sum case.

MULTI-PLAYER STOCHASTIC STOPPING GAMES

Multi-Player Stochastic Stopping Game

The following building blocks are used to construct the multi-period stochastic stopping game:

- The set $\mathcal{M} = \{1, 2, \ldots, m\}$ of players.
- The probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^{T}$ representing the information flow available to all players.
- The class $\mathcal{S}_t$ of all $\mathbb{F}$-stopping times taking values in $\{t, \ldots, T\}$.
- The $\mathbb{F}$-adapted *exercise payoff* $X_t = (X_t^1, \ldots, X_t^m)$ for $t = 0, 1, \ldots, T$.
- The random subsets $\mathcal{E}_t \subset \mathcal{M}$ of exercising players.
- For every $k \in \mathcal{M}$ and every non-empty subset $\mathcal{E} \subset \mathcal{M}$ such that $k \notin \mathcal{E}$ the real-valued, $\mathbb{F}$-adapted *non-exercise payoff* process

$$\widetilde{X}_t^k = \widetilde{X}_t^k(\mathcal{E}), \quad t = 0, 1, \ldots, T-1.$$

- The random variable $\widetilde{X}_t^k$ is the payoff received by player k when all players from $\mathcal{E}$ exercise at t assuming that the game was not yet stopped.

Multi-Player Stochastic Stopping Game

The m*-player stochastic stopping game* $\mathcal{G} = (\mathcal{G}_0, \dots, \mathcal{G}_T)$ is defined recursively:

- All players are assumed to exercise at time T. The game $\mathcal{G}_T$ is trivial with the value $V_T^* = X_T$.
- Assuming that the games $\mathcal{G}_{t+1}, \dots, \mathcal{G}_T$ were already defined, the game $\mathcal{G}_t$ is specified as follows.
- The game starts at time t and each player can exercise at any time in the interval $[t, T]$. The game stops as soon as anyone exercises.
- The *strategy* s_t^k of player k is a stopping time from the space $\mathcal{S}_t$, so that the *strategy profile* $s_t = (s_t^1, \dots, s_t^m) \in \mathcal{S}_t^m$.
- Let $\widehat{s}_t = s_t^1 \wedge \dots \wedge s_t^m \in \mathcal{S}_t$. The *exercise set*

$$\mathcal{E}(s_t) = \{i \in \mathcal{M} : s_t^i = \widehat{s}_t\}$$

  is the $\mathcal{F}_T$-measurable random set of earliest exercising players.

Multi-Player Stochastic Stopping Game

- For each strategy profile s_t, the *expected payoff* at time t

$$V_t(s_t) = (V_t^1(s_t), \ldots, V_t^m(s_t))$$

is defined by

$$V_t^k(s_t) = \mathbb{E}_{\mathbb{P}}\big(X^k_{\widehat{s}_t} \mathbb{1}_{\{k \in \mathcal{E}(s_t)\}} + \widetilde{X}^k_{\widehat{s}_t} \mathbb{1}_{\{k \notin \mathcal{E}(s_t)\}} \,\big|\, \mathcal{F}_t \big).$$

In general, the non-exercise payoffs are given by

$$\widetilde{X}^k_{\widehat{s}_t} = g^k_{\mathcal{E}(s_t)}(X_{\widehat{s}_t}, V^*_{\widehat{s}_t+1}, \widehat{s}_t) \mathbb{1}_{\{\widehat{s}_t < T\}}$$

for a family of functions $g^k_{\mathcal{E}} : \mathbb{R}^{2m} \times [0, T] \to \mathbb{R}$ where we denote by $V^*_u = (V_u^{1*}, \ldots, V_u^{m*})$ the value of the game $\mathcal{G}_u$ for $u = t+1, \ldots, T$.

Multi-Player Stochastic Stopping Game

- To summarize, for any strategy profile s_t

$$V_t^k(s_t) = \mathbb{E}_{\mathbb{P}}\bigg(\sum_{u=t}^{T} X_u^k \mathbb{1}_{\{k \in \mathcal{E}_u(s_t)\}} \mathbb{1}_{\{\widehat{s}_t = u\}} \,\bigg|\, \mathcal{F}_t \bigg) + \mathbb{E}_{\mathbb{P}}\bigg(\sum_{u=t}^{T-1} \widetilde{X}_u^k(\mathcal{E}_u(s_t)) \mathbb{1}_{\{k \notin \mathcal{E}_u(s_t)\}} \mathbb{1}_{\{\widehat{s}_t = u\}} \,\bigg|\, \mathcal{F}_t \bigg).$$

where

$$\mathcal{E}_u(s_t) = \{ i \in \mathcal{M} : s_t^i = \widehat{s}_t = u \}$$

is the $\mathcal{F}_u$-measurable random subset of earliest exercising players who decide to exercise at time u.

Candidate for the Value Process

- We will now search for the candidate for the value process of the game.
- Let $U = (U^1, \dots, U^m)$ be an arbitrary $\mathbb{F}$-adapted, $\mathbb{R}^m$-valued process such that $U_T = X_T$.
- We define the family $\tau_t = (\tau_t^1, \dots, \tau_t^m) \in \mathcal{S}_t^m$ of stopping times

$$\tau_t^k := \inf \big\{ u \geq t : U_u^k = X_u^k \big\}.$$

- Let $\mathcal{E}(\tau_t)$ stand for the following random set

$$\mathcal{E}(\tau_t) := \{k \in \mathcal{M} : U_t^k = X_t^k\} = \{k \in \mathcal{M} : \tau_t^k = t\} = \{i \in \mathcal{M} : \tau_t^k = \widehat{\tau}_t\}$$

where $\widehat{\tau}_t := \tau_t^1 \wedge \cdots \wedge \tau_t^m$. We write

$$\widehat{\tau}_t^{-k} := \tau_t^1 \wedge \cdots \wedge \tau_t^{k-1} \wedge \tau_t^{k+1} \wedge \cdots \wedge \tau_t^m.$$

- For brevity, we denote $P_t = \mathbb{E}_{\mathbb{P}}\big(U_{t+1} \,\big|\, \mathcal{F}_t\big)$.

Value Process: Sufficient Conditions

Proposition

Let $U = (U^1, \dots, U^m)$ *be an arbitrary* $\mathbb{F}$*-adapted,* $\mathbb{R}^m$*-valued process such that* $U_T = X_T$*. Assume that for all* $k \in \mathcal{M}$ *and* $t = 0, 1, \dots, T-1$*,*

1. $U^k_t \geq X^k_t$,
2. $U^k_t \leq P^k_t$ *on the event* $\{\tau^k_t > t\}$,
3. $U^k_t \geq P^k_t$ *on the event* $\{\widehat{\tau}^{-k}_t > t\}$,
4. $U^k_t \geq \widetilde{X}^k_t$ *on the event* $\{\widehat{\tau}^{-k}_t = t < s^k_t\}$ *for every* $s^k_t \in \mathcal{S}_t$,
5. $U^k_t \leq \widetilde{X}^k_t$ *on the event* $\{\widehat{s}^{-k}_t = t < \tau^k_t\}$ *for every* $s^{-k}_t \in \mathcal{S}^{m-1}_t$.

Then, for every $k \in \mathcal{M}$, $t = 0, 1, \dots, T-1$, *and* $s^1_t, \dots, s^m_t$ *in* $\mathcal{S}_t$*,*

$$\mathbb{E}_{\mathbb{P}}\big(Z^k(s^k_t, \tau^{-k}_t) \,|\, \mathcal{F}_t\big) \leq U^k_t \leq \mathbb{E}_{\mathbb{P}}\big(Z^k(\tau^k_t, s^{-k}_t) \,|\, \mathcal{F}_t\big)$$

and thus

$$\mathbb{E}_{\mathbb{P}}\big(Z^k(s^k_t, \tau^{-k}_t) \,|\, \mathcal{F}_t\big) \leq \mathbb{E}_{\mathbb{P}}\big(Z^k(\tau^k_t, \tau^{-k}_t) \,|\, \mathcal{F}_t\big) \leq \mathbb{E}_{\mathbb{P}}\big(Z^k(\tau^k_t, s^{-k}_t) \,|\, \mathcal{F}_t\big).$$

Value Process: Sufficient Conditions

Proposition

Consequently:

1. *The process U is the value process of the m-player stopping game, that is, for all $k \in \mathcal{M}$ and $t = 0, 1, \ldots, T$,*

$$\begin{aligned} U_t^k &= \inf_{s_t^{-k} \in \mathcal{S}_t^{m-1}} \sup_{s_t^k \in \mathcal{S}_t} \mathbb{E}_{\mathbb{P}}\big(Z^k(s_t^k, s_t^{-k}) \,|\, \mathcal{F}_t\big) \\ &= \mathbb{E}_{\mathbb{P}}\big(Z^k(\tau_t^k, \tau_t^{-k}) \,|\, \mathcal{F}_t\big) \\ &= \sup_{s_t^k \in \mathcal{S}_t} \inf_{s_t^{-k} \in \mathcal{S}_t^{m-1}} \mathbb{E}_{\mathbb{P}}\big(Z^k(s_t^k, s_t^{-k}) \,|\, \mathcal{F}_t\big) = V_t^{k*}. \end{aligned}$$

2. *For every $t = 0, 1, \ldots, T$, the family $\tau_t = (\tau_t^1, \ldots, \tau_t^m) \in \mathcal{S}_t^m$ is an optimal equilibrium for the game $\mathcal{G}_t$.*
3. *For all $t = 0, 1, \ldots, T-1$, the stopped process $(U_u^{\widehat{\tau_t}})_{u=t}^T$ is an $\mathbb{F}$-martingale.*

Affine Stopping Games

Definition

The m-player stochastic stopping game $\mathcal{G} = (\mathcal{G}_0, \dots, \mathcal{G}_T)$ is said to be *affine* whenever:

1. For any $\mathcal{E} \subset \mathcal{M}$, we are given the set of *weights*

$$w_k(\mathcal{E}) = \frac{\alpha_k}{1 - \sum_{i \in \mathcal{E}} \alpha_i}$$

   for $k \in \mathcal{M} \setminus \mathcal{E}$ where $\alpha_i > 0$ and $\sum_{i \in \mathcal{M}} \alpha_i < 1$.

2. The non-exercise payoff on the event $\{\widehat{s}_t < T\}$ is given by

$$\widetilde{X}^k_{\widehat{s}_t} = V^{k*}_{\widehat{s}_t+1} - w_k(\mathcal{E}(s_t)) \sum_{i \in \mathcal{E}(s_t)} \left(X^i_{\widehat{s}_t} - V^{i*}_{\widehat{s}_t+1} \right)$$

   where $V^*_u = (V^{1*}_u, \dots, V^{m*}_u)$ is the value of the game $\mathcal{G}_u$.

Expected Payoff as Projection

Given the vector $\alpha = (\alpha_1, \ldots, \alpha_m)$, we endow $\mathbb{R}^m$ with the inner product $\langle \cdot, \cdot \rangle_a$

$$\langle x, y \rangle_\alpha = \sum_{i=1}^m \frac{x_i y_i}{\alpha_i} + \frac{\left(\sum_{i=1}^m x_i\right)\left(\sum_{i=1}^m y_i\right)}{1 - \sum_{i=1}^m \alpha_i}.$$

Proposition

The expected payoff $V_t(s_t) = \left(V_t^1(s_t), \ldots, V_t^m(s_t)\right)$ can be represented as follows

$$V_t(s_t) = \mathbb{E}_{\mathbb{P}}\left(\mathbb{1}_{\{\widehat{s}_t < T\}} \pi_{\mathcal{H}_{\mathcal{E}(s_t)}}\left(V^*_{\widehat{s}_t+1}\right) + \mathbb{1}_{\{\widehat{s}_t = T\}} X_T \,\middle|\, \mathcal{F}_t \right)$$

where $\mathcal{H}_{\mathcal{E}(s_t)}$ is the $\mathcal{F}_{\widehat{s}_t}$-measurable random hyperplane

$$\mathcal{H}_{\mathcal{E}(s_t)} := \left\{ y \in \mathbb{R}^m : \; y_i = X^i_{\widehat{s}_t}, \; \forall\, i \in \mathcal{E}(s_t) \right\}.$$

Value Process via Projection

Definition

Let the $\mathbb{F}$-adapted payoff processes be given. The $\mathbb{F}$-adapted, $\mathbb{R}^m$-valued process $U = (U^1, \dots, U^m)$ is defined by setting $U_T := X_T$ and for $t = 0, 1, \dots, T-1$

$$U_t := \pi_{\mathbb{O}(X_t)} \left(\mathbb{E}_{\mathbb{P}} \left(U_{t+1} \,\middle|\, \mathcal{F}_t \right) \right)$$

where $\mathbb{O}(X_t)$ is the $\mathcal{F}_t$-measurable orthant

$$\mathbb{O}(X_t(\omega)) := \left\{ y \in \mathbb{R}^m : \, y_i \geq X_t^i(\omega), \, \forall \, i \in \mathcal{M} \right\}.$$

We define the strategy set $\tau_t = (\tau_t^1, \dots, \tau_t^m) \in \mathcal{S}_t^m$ by setting

$$\tau_t^k := \inf \left\{ u \geq t : \, U_u^k = X_u^k \right\}.$$

Value Process via Projection

Lemma

Recall that we set $U_T = X_T$

$$U_t = \pi_{\mathbb{O}(X_t)}\left(\mathbb{E}_{\mathbb{P}}\left(U_{t+1} \,\middle|\, \mathcal{F}_t\right)\right), \quad t = 0, 1, \ldots, T-1,$$

and

$$\tau_t^k := \inf\left\{u \geq t : U_u^k = X_u^k\right\}.$$

Then for every $k \in \mathcal{M}$ *and* $t = 0, 1, \ldots, T-1$*:*

1. $U_t^k \geq X_t^k$*; moreover,* $U_t^k = P_t^k$ *on the event* $\{\widehat{\tau}_t > t\}$*,*
2. $U_t^k \leq P_t^k$ *on the event* $\{\tau_t^k > t\}$*,*
3. $U_t^k \geq P_t^k$ *on the event* $\{\widehat{\tau}_t^{-k} > t\}$*,*
4. $U_t^k \geq \widetilde{X}_t^k$ *on the event* $\{\widehat{\tau}_t^{-k} = t < s_t^k\}$ *for every* $s_t^k \in \mathcal{S}_t$*,*
5. $U_t^k \leq \widetilde{X}_t^k$ *on the event* $\{\widehat{s}_t^{-k} = t < \tau_t^k\}$ *for every* $s_t^{-k} \in \mathcal{S}_t^{m-1}$.

Value Process via Projection

The main result for the affine stopping game is the following corollary.

Corollary

Consider the m-person affine stopping game $\mathcal{G} = (\mathcal{G}_0, \dots, \mathcal{G}_T)$ with the vector of powers $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $\sum_{i=1}^m \alpha_i < 1$. The game is solvable with the value process V^* given by the recursive formula: $V_T^* = X_T$ and

$$V_t^* := \pi_{\mathbb{O}(X_t)} \left(\mathbb{E}_{\mathbb{P}} \left(V_{t+1}^* \,\middle|\, \mathcal{F}_t \right) \right) = \pi_{\mathbb{O}(X_t)} \left(P_t \right).$$

The sequence of optimal equilibria $(\tau_0, \dots, \tau_T)$ is given by

$$\tau_t^k := \inf \left\{ u \geq t : V_u^{k*} = X_u^k \right\}.$$

Value Process via Reflected BSDE

Assume that $\sum_{i=1}^m \alpha_i < 1$. Recall that we endowed $\mathbb{R}^m$ with the following inner product

$$\langle y, z\rangle = \sum_{i=1}^m \left(\frac{y_i z_i}{\alpha_i}\right) + \frac{\left(\sum_{i=1}^m y_i\right)\left(\sum_{i=1}^m z_i\right)}{1 - \sum_{i=1}^m \alpha_i} =: y^T Dz.$$

It can be shown that $\widehat{D} := D^{-1}$ equals

$$\widehat{D} = \begin{pmatrix} \alpha_1 - \alpha_1^2 & -\alpha_1\alpha_2 & \dots & -\alpha_1\alpha_m \\ -\alpha_2\alpha_1 & \alpha_2 - \alpha_2^2 & \dots & -\alpha_2\alpha_m \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_m\alpha_1 & -\alpha_m\alpha_2 & \dots & \alpha_m - \alpha_m^2 \end{pmatrix}.$$

The matrix $\widehat{D}$ will be used to derive the reflected BSDE.

Affine Variational Inequality

Lemma

A vector $v^* = \Pi_{\mathbb{O}(x)}(p)$ *if there exists a vector* μ^* *such that* (v^*, μ^*) *is a solution to the following affine variational inequality (AVI)*

$$\begin{aligned} & v^* - \widehat{D}\mu^* = p, \\ & v^* \geq x, \quad \mu^* \geq 0, \\ & \langle v^* - x, \mu^* \rangle = 0, \end{aligned}$$

or, more explicitly, for all $i = 1, \ldots, m$,

$$\begin{aligned} & v_i^* = p_i + \sum_{j=1}^{m} \widehat{D}_{ij}\mu_j^*, \\ & v_i^* \geq x_i, \quad \mu_i^* \geq 0, \quad (v_i^* - x_i)\mu_i^* = 0, \end{aligned}$$

where $\widehat{D}_i = (\widehat{D}_{i1}, \ldots, \widehat{D}_{im})$ *is the* i*th row of the matrix* $\widehat{D}$.

Value Process via Reflected Backward Equation

Corollary

Assume that the pair (v^*, μ^*) solves the AVI. Then (v^*, μ^*) solves the following reflected backward equation (RBE)

$$v_i^* + \alpha_i \sum_{l=1,\, l\neq i}^{m} \alpha_l \mu_l^* \mathbb{1}_{\{v_l^*=x_l\}} - \alpha_i(1-\alpha_i)\mu_i^* \mathbb{1}_{\{v_i^*=x_i\}} = p_i,$$

$$v_i^* - x_i \geq 0, \quad \mu_i^* \geq 0,$$

or, equivalently,

$$v_i^* + \alpha_i \sum_{l=1,\, l\neq i}^{m} \alpha_l \mu_l^* - \alpha_i(1-\alpha_i)\mu_i^* = p_i,$$

$$v_i^* - x_i \geq 0, \quad \mu_i^* \geq 0, \quad (v_i^* - x_i)\mu_i^* = 0.$$

Classes of Players

We can identify three classes of players:

1. Players for whom it is optimal to exercise since their continuation payoff is strictly below their exercise payoff: $p_i < x_i = v_i^*$ and $\mu_i^* \geq x_i - p_i > 0$,
2. Players who are forced to exercise: $p_i \geq x_i = v_i^*$ and $\mu_i^* > 0$,
3. Players who do not exercise: $p_i \geq x_i$ and $\mu_i^* = 0$.

To simplify the reflected backward equation, we denote $k^l := \alpha_l \mu_l^*$.

Then we obtain the following equation for vectors $v = (v_1, \dots, v_m)^T \in \mathbb{R}^m$ and $k = (k_1, \dots, k_m)^T \in \mathbb{R}_+^m$

$$
\begin{aligned}
& v_i + \alpha_i \sum_{l=1,\, l \neq i}^{m} k_l \mathbb{1}_{\{v_l = x_l\}} - (1 - \alpha_i) k_i \mathbb{1}_{\{v_i = x_i\}} = p_i, \\
& v_i \geq x_i, \quad k_i \geq 0.
\end{aligned}
$$

CONTINUOUS-TIME MULTI-PERSON STOPPING GAMES

Continuous-Time Multi-Person Stopping Game

The continuous-time multi-person stopping game is given by its terminal value ξ, the exercise payoffs X^i and the redistribution rule $(\alpha^1, \ldots, \alpha^m)$ upon stopping. The randomness is introduced via the Brownian motion $B = (B^1, \ldots, B^d)$.

Definition

The m-dimensional RBSDE corresponding to the *continuous-time multi-person stopping game* reads: for all $t \in [0, T]$,

$$\begin{cases} Y_t^i = \xi^i - \sum_{j \neq i, j=1}^m r_{i,j}(K_T^j - K_t^j) - (K_T^i - K_t^i) - \displaystyle\int_t^T \sum_{l=1}^d Z_s^{i,l} dB_s^l, \\ Y_t^i \geq X_t^i, \\ \int_0^t \mathbb{1}_{\{Y_s^i > X_s^i\}} \, dK_s^i = 0, \quad 1 \leq i \leq m, \end{cases}$$

where $r_{i,j} = \frac{\alpha_i}{1-\alpha_j}$ for $i \neq j$, and $\alpha_i > 0$ are such that $\sum_{i=1}^m \alpha_i < 1$.

Multi-Reflected BSDE

In general, we consider the following multi-reflected BSDE (ξ, X, f, R)

$$\begin{cases} Y_t^i = \xi^i + \int_t^T f_i(s, Y_s)\, ds + \sum_{j\neq i, j=1}^m \int_t^T r_{i,j}(s, Y_s)\, dK_s^j + K_T^i - K_t^i \\ \qquad - \int_t^T \sum_{l=1}^d Z_s^{i,l}\, dB_s^l, \\ Y_t^i \geq X_t^i \quad \text{and} \quad K_t^i = \int_0^t \mathbb{1}_{\{Y_s^i = X_s^i\}}\, dK_s^i, \quad 1 \leq i \leq m. \end{cases}$$

where

- $\xi = (\xi_1, \ldots, \xi_m)$ is an $\mathcal{F}_T$-measurable bounded random variable such that $\xi_i \geq X_T^i$, for each $1 \leq i \leq m$,
- the process $X = (X^1, \ldots, X^m)$ is a continuous semimartingale,
- the map $f = (f_1, \ldots, f_m) : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ and the map $R = (r_{i,j})_{1\leq i,j\leq m} : \Omega \times [0, T] \times \mathbb{R}^m \to \mathbb{M}_m(\mathbb{R})$ are both bounded measurable functions,
- $\mathbb{M}_m(\mathbb{R})$ denotes the class of $m \times m$ matrices with real entries.

Solution to Multi-Reflected BSDE

Definition

A pair (Y, K) of $\mathbb{F}$-progressively measurable and continuous processes is a *solution* to RBSDE (ξ, X, f, R) if there exists an $\mathbb{F}$-progressively measurable, square-integrable process $Z_t = (Z_t^{i,j})_{1 \leq i,j \leq m}$ such that:

- the following equality is satisfied, for all $1 \leq i \leq m$ and $0 \leq t \leq T$,

$$
\begin{aligned}
Y_t^i = & \xi^i + \int_t^T f_i(s, Y_s)\, ds + \sum_{j \neq i, j=1}^{m} \int_t^T r_{i,j}(s, Y_s)\, dK_s^j + K_T^i - K_t^i \\
& - \int_t^T \sum_{l=1}^{d} Z_s^{i,l}\, dB_s^l,
\end{aligned}
$$

- the inequality $Y_t^i \geq X_t^i$ holds for all $1 \leq i \leq m$ and $t \in [0, T]$,
- for every $1 \leq i \leq m$, the process K^i is continuous, non-decreasing, with $K_0^i = 0$ and $K_t^i = \int_0^t \mathbb{1}_{\{Y_s^i = X_s^i\}}\, dK_s^i$.

Assumptions

(H_1) The $\mathbb{R}^m$-valued random variable ξ is $\mathcal{F}_T$-measurable and bounded.

(H_2) For $1 \le i \le m$, the mapping $y \mapsto f_i(\omega, t, y) : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz continuous, uniformly with respect to (ω, t) and $f_i(\cdot, \cdot, y)$ is an $\mathbb{F}$-predictable process bounded by β_i for all fixed $y \in \mathbb{R}^m$.

(H_3) For $i \neq j$, the map $y \mapsto r_{i,j}(\omega, t, y) : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz-continuous, uniformly with respect to (ω, t) and $r_{i,j}(\cdot, \cdot, y)$ is an $\mathbb{F}$-predictable process.

(H_4) For $i \neq j$, there exists a constant $\lambda_{i,j} \ge 0$ such that $|r_{i,j}(\omega, t, y)| \le \lambda_{i,j}$ for all (ω, t, y). Setting $\Lambda = (\lambda_{i,j})_{1 \le i,j \le m}$ with $\lambda_{i,i} = 0$, we assume that the spectral radius $\rho(\Lambda) < 1$.

(H_5) For $1 \le i \le m$, the process X^i satisfies

$$X_t^i = X_0^i + \int_0^t G_s^i \, ds + \int_0^t \sum_{l=1}^d H_s^{i,l} \, dB_s^l,$$

where G^i and $H^{i,l}$ are processes such that there exists a constant $L_i \ge 0$ such that $|G_t^i| \le L_i$ for all (ω, t), and $\int_0^T |H_s^{i,l}|^2 \, ds < \infty$. Finally, $\xi_i \ge X_T^i$ for $1 \le i \le m$.

Alternative Assumption

The following alternative assumption, weaker than (H_4), will be sufficient:

(H_4') For $i \neq j$, there exists a constant $\lambda_{i,j} \geq 0$ such that for all (ω, t, y)

$$|r_{i,j}(\omega, t, y)| \leq \lambda_{i,j}.$$

We set $\Lambda = (\lambda_{i,j})_{1 \leq i,j \leq d}$ with $\lambda_{i,i} = 0$ and we assume that $(I - \Lambda)^{-1}$ is a matrix with nonnegative entries. Moreover, there are constants $a_j > 0$, $1 \leq j \leq d$ and $0 < \delta < 1$ such that

$$\sum_{i \neq j,\, i=1}^{m} a_i |r_{i,j}(\omega, t, y)| \leq \sum_{i \neq j,\, i=1}^{m} a_i \lambda_{i,j} \leq \delta$$

for all $1 \leq j \leq d$ and $(\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}^m$.

An analysis of the proof of the main result in Ramasubramanian (2002) shows that if we replace (H_4) by the weaker condition (H_4') then the assertion of the theorem is still valid.

Space of Solutions

Using the vector $(a_1, \ldots, a_m)$ in assumption (H_4'), we introduce the space $\mathcal{H}_X$ associated with the semimartingale X as the space of all $\mathbb{F}$-progressively measurable processes (Y, K) such that:

- the inequality $Y_t^i \geq X_t^i$ holds for all $1 \leq i \leq d$ and $t \in [0, T]$,
- for all $1 \leq i \leq m$, the process K^i is nondecreasing with $K_0^i = 0$,
- $\mathbb{E}_{\mathbb{P}}\Big(\sum_{i=1}^m \int_0^T e^{\theta t} a_i |Y_t^i| \, dt \Big) < \infty$,
- $\mathbb{E}_{\mathbb{P}}\Big(\sum_{i=1}^m \int_0^T e^{\theta t} a_i \| K^i \|_{[t,T]} \, dt \Big) < \infty$,

where θ is a constant and $\| K^i \|_{[t,T]}$ denotes the total variation of the process K^i over $[t, T]$, that is, $\| K^i \|_{[t,T]} = \int_t^T |dK_s^i|$. If we define the metric on $\mathcal{H}_X$

$$d((Y,K),(\widehat{Y},\widehat{K})) := \mathbb{E}_{\mathbb{P}}\bigg(\sum_{i=1}^m \int_0^T e^{\theta t} a_i |Y_t^i - \widehat{Y}_t^i| \, dt \bigg) + \mathbb{E}_{\mathbb{P}}\bigg(\sum_{i=1}^m \int_0^T e^{\theta t} a_i \| K^i - \widehat{K}^i \|_{[t,T]} \, dt \bigg)$$

then $(\mathcal{H}_X, d)$ is a complete metric space.

Theorem of Ramasubramanian (2002)

Theorem (Ramasubramanian (2002))

Let the assumptions (H_1)–(H_4) *hold. If* $\xi_i \geq 0$ *for* $1 \leq i \leq m$ *then there exists a unique solution* $(Y, K) \in \mathcal{H}_0$ *to the RBSDE* $(\xi, 0, f, R)$

$$\begin{cases} Y_t^i = \xi_i + \int_t^T f_i(s, Y_s)\, ds + \sum_{j \neq i, j=1}^m \int_t^T r_{i,j}(s, Y_s)\, dK_s^j + K_T^i - K_t^i \\ \qquad - \int_t^T \sum_{l=1}^d Z_s^{i,l}\, dB_s^l, \\ Y_t^i \geq 0, \quad 1 \leq i \leq m. \end{cases}$$

Moreover,

$$0 \leq dK_t^i \leq ((I - \Lambda)^{-1}\beta)_i\, dt$$

for all $t \in [0, T]$ *and* $1 \leq i \leq m$*, where* $\beta = (\beta_1, \ldots, \beta_m)$ *satisfies* (H_3).

Multi-Reflected BSDE for Affine Stopping Game

Recall that the Multi-Reflected BSDE corresponding to the continuous-time affine stopping game reads: for all $t \in [0, T]$

$$\begin{cases} Y_t^i = \xi_i - \sum_{j \neq i, j=1}^m r_{i,j}(K_T^j - K_t^j) - (K_T^i - K_t^i) - \int_t^T \sum_{l=1}^d Z_s^{i,l} dB_s^l, \\ Y_t^i \geq X_t^i, \\ \int_0^t \mathbb{1}_{\{Y_s^i > X_s^i\}} \, dK_s^i = 0, \quad 1 \leq i \leq m, \end{cases}$$

where $r_{i,j} = \frac{\alpha_i}{1-\alpha_j}$ for $i \neq j$, and $\alpha_i > 0$, $\sum_{i=1}^m \alpha_i < 1$. According to assumption (H_4'), we can set $\lambda_{i,j} = r_{i,j}$, for $i \neq j$, $1 \leq i, j \leq m$ and $\lambda_{i,i} = 0$. This means that

$$\Lambda = \begin{pmatrix} 0 & \frac{\alpha_1}{1-\alpha_2} & \cdots & \frac{\alpha_1}{1-\alpha_m} \\ \frac{\alpha_2}{1-\alpha_1} & 0 & \cdots & \frac{\alpha_2}{1-\alpha_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_m}{1-\alpha_1} & \frac{\alpha_m}{1-\alpha_2} & \cdots & 0 \end{pmatrix}.$$

Value Process for Continuous-Time Multi-Person Game

Lemma

Assume that $\alpha_i > 0$ *and* $\sum_{i=1}^{m} \alpha_i < 1$. *Then* Λ *satisfies condition* (H_4').

The following result shows that the continuous-time multi-person stopping game has the unique value process.

Theorem

Under assumptions (H_1) *and* (H_5), *the Multi-Reflected BSDE associated with the multi-person game has a unique solution* $(Y, K) \in \mathcal{H}_X$. *Moreover,*

$$0 \leq dK_t^i \leq ((I - \Lambda)^{-1} L)_i \, dt$$

for all $i = 1, \ldots, m$ *and* $t \in [0, T]$, *where* $L = (L_1, \ldots, L_m)$.

References

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