Convex hedging of non-superreplicable contingent claims in general semimartingale models

Tomasz Tkaliński

Faculty of Mathematics, Informatics and Mechanics
Warsaw University

6th General AMaMeF Conference, 10.06.2013

Overview

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \ldots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t \in [0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \ldots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \dots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \dots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \ldots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \ldots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \ldots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$, $x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \ldots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \dots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t \in [0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \ldots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

Probabilistic setup

Time horizon: $T > 0$

Complete probability space: $(\Omega, \mathcal{F}, \mathbb{P})$,

Filtration: $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying usual conditions,

$\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$.

Consider

- $S = (S^1, \ldots, S^k)$ - nonnegative adapted semimartingale (discounted asset prices),
- Φ - admissible trading strategies, i.e. pairs $\xi = (x, \pi)$,$x \geq 0$, π - predictable $\mathbb{R}^k$-valued process such that **the value process** of ξ given by $V_t(\xi) = x + \int_0^t \pi_u dS_u$ is nonnegative

No arbitrage assumption

Definition 1 (sigma-martingale)

*$\mathbb{R}^k$-valued process Y is a **sigma-martingale** if there exists an $\mathbb{R}^d$-value martingale M and M-integrable predictable $\mathbb{R}_+$-valued process η such that $Y_t = \int_0^t \eta_u dM_u$.*

Let $\mathcal{P}_\sigma$ denote the set of equivalent probability measures $\mathbb{Q}$ such that S is a sigma-martingale under $\mathbb{Q}$.

NFLVR

$\mathcal{P}_\sigma \neq \emptyset$

No arbitrage assumption

Definition 1 (sigma-martingale)

*$\mathbb{R}^k$-valued process Y is a **sigma-martingale** if there exists an $\mathbb{R}^d$-value martingale M and M-integrable predictable $\mathbb{R}_+$-valued process η such that $Y_t = \int_0^t \eta_u dM_u$.*

Let $\mathcal{P}_\sigma$ denote the set of equivalent probability measures $\mathbb{Q}$ such that S is a sigma-martingale under $\mathbb{Q}$.

NFLVR

$\mathcal{P}_\sigma \neq \emptyset$

No arbitrage assumption

Definition 1 (sigma-martingale)

$\mathbb{R}^k$*-valued process* Y *is a* ***sigma-martingale*** *if there exists an* $\mathbb{R}^d$ *-value martingale* M *and* M*-integrable predictable* $\mathbb{R}_+$*-valued process* η *such that* $Y_t = \int_0^t \eta_u dM_u$.

Let $\mathcal{P}_\sigma$ denote the set of equivalent probability measures $\mathbb{Q}$ such that S is a sigma-martingale under $\mathbb{Q}$.

NFLVR

$\mathcal{P}_\sigma \neq \emptyset$

The hedging problem

European contingent claims: $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$

For $\xi \in \Phi$ and a contingent claim H

$$L(\xi, H) := -(V_T(\xi) - H)^-$$

the loss resulting from hedging H with ξ.

Hedging:

minimize $L(\xi, H)$ over ξ satisfying some budget constraint.

The hedging problem

European contingent claims: $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$

For $\xi \in \Phi$ and a contingent claim H

$$L(\xi, H) := -(V_T(\xi) - H)^-$$

the loss resulting from hedging H with ξ.

Hedging:

minimize $L(\xi, H)$ over ξ satisfying some budget constraint.

The hedging problem

European contingent claims: $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$

For $\xi \in \Phi$ and a contingent claim H

$$L(\xi, H) := -(V_T(\xi) - H)^-$$

the loss resulting from hedging H with ξ.

Hedging:

minimize $L(\xi, H)$ over ξ satisfying some budget constraint.

The hedging problem

European contingent claims: $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$

For $\xi \in \Phi$ and a contingent claim H

$$L(\xi, H) := -(V_T(\xi) - H)^-$$

the loss resulting from hedging H with ξ.

Hedging:

minimize $L(\xi, H)$ over ξ satisfying some budget constraint.

The hedging problem

European contingent claims: $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$

For $\xi \in \Phi$ and a contingent claim H

$$L(\xi, H) := -(V_T(\xi) - H)^-$$

the loss resulting from hedging H with ξ.

Hedging:

minimize $L(\xi, H)$ over ξ satisfying some budget constraint.

The hedging problem

European contingent claims: $L^1_+(\Omega, \mathcal{F}, \mathbb{P})$

For $\xi \in \Phi$ and a contingent claim H

$$L(\xi, H) := -(V_T(\xi) - H)^-$$

the loss resulting from hedging H with ξ.

Hedging:

minimize $L(\xi, H)$ over ξ satisfying some budget constraint.

Superhedging

Consider:

Condition 1 (superhedging)

$U_0 := \sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty.$

Theorem 2.1

Under Condition 1 there exists $\xi \in \Phi$ with $V_0(\xi) = U_0$ for which $L(\xi, H) = 0$ $\mathbb{P}$-a.s.

Brilliant ... but

- this might be unacceptably expensive (Gushchin and Mordecki (2002)),
- what about the case $U_0 = \infty$.

IDEA: Fix $\tilde{V}_0 < U_0$ and (in some sense) minimize $L(\xi, H)$ over ξ satisfying $V_0(\xi) \leq \tilde{V}_0$.

Superhedging

Consider:

Condition 1 (superhedging)

$U_0 := \sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty.$

Theorem 2.1

Under Condition 1 there exists $\xi \in \Phi$ with $V_0(\xi) = U_0$ for which $L(\xi, H) = 0$ $\mathbb{P}$-a.s.

Brilliant ... but

- this might be unacceptably expensive (Gushchin and Mordecki (2002)),
- what about the case $U_0 = \infty$.

IDEA: Fix $\tilde{V}_0 < U_0$ and (in some sense) minimize $L(\xi, H)$ over ξ satisfying $V_0(\xi) \leq \tilde{V}_0$.

Superhedging

Consider:

Condition 1 (superhedging)

$U_0 := \sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty.$

Theorem 2.1

Under Condition 1 there exists $\xi \in \Phi$ with $V_0(\xi) = U_0$ for which $L(\xi, H) = 0$ $\mathbb{P}$-a.s.

Brilliant ... but

- this might be unacceptably expensive (Gushchin and Mordecki (2002)),
- what about the case $U_0 = \infty$.

IDEA: Fix $\tilde{V}_0 < U_0$ and (in some sense) minimize $L(\xi, H)$ over ξ satisfying $V_0(\xi) \le \tilde{V}_0$.

Superhedging

Consider:

Condition 1 (superhedging)

$U_0 := \sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty.$

Theorem 2.1

Under Condition 1 there exists $\xi \in \Phi$ with $V_0(\xi) = U_0$ for which $L(\xi, H) = 0$ $\mathbb{P}$-a.s.

Brilliant ... but

- this might be unacceptably expensive (Gushchin and Mordecki (2002)),
- what about the case $U_0 = \infty$.

IDEA: Fix $\tilde{V}_0 < U_0$ and (in some sense) minimize $L(\xi, H)$ over ξ satisfying $V_0(\xi) \leq \tilde{V}_0$.

Superhedging

Consider:

Condition 1 (superhedging)

$U_0 := \sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty.$

Theorem 2.1

Under Condition 1 there exists $\xi \in \Phi$ with $V_0(\xi) = U_0$ for which $L(\xi, H) = 0$ $\mathbb{P}$-a.s.

Brilliant ... but

- this might be unacceptably expensive (Gushchin and Mordecki (2002)),
- what about the case $U_0 = \infty$.

IDEA: Fix $\tilde{V}_0 < U_0$ and (in some sense) minimize $L(\xi, H)$ over ξ satisfying $V_0(\xi) \leq \tilde{V}_0$.

Superhedging

Consider:

Condition 1 (superhedging)

$U_0 := \sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty.$

Theorem 2.1

Under Condition 1 there exists $\xi \in \Phi$ with $V_0(\xi) = U_0$ for which $L(\xi, H) = 0$ $\mathbb{P}$-a.s.

Brilliant ... but

- this might be unacceptably expensive (Gushchin and Mordecki (2002)),
- what about the case $U_0 = \infty$.

IDEA: Fix $\tilde{V}_0 < U_0$ and (in some sense) minimize $L(\xi, H)$ over ξ satisfying $V_0(\xi) \leq \tilde{V}_0$.

Superhedging

Consider:

Condition 1 (superhedging)

$U_0 := \sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty.$

Theorem 2.1

Under Condition 1 there exists $\xi \in \Phi$ *with* $V_0(\xi) = U_0$ *for which* $L(\xi, H) = 0$ $\mathbb{P}$*-a.s.*

Brilliant ... but

- this might be unacceptably expensive (Gushchin and Mordecki (2002)),
- what about the case $U_0 = \infty$.

IDEA: Fix $\tilde{V}_0 < U_0$ and (in some sense) minimize $L(\xi, H)$ over ξ satisfying $V_0(\xi) \leq \tilde{V}_0$.

Superhedging

Consider:

Condition 1 (superhedging)

$U_0 := \sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty.$

Theorem 2.1

Under Condition 1 there exists $\xi \in \Phi$ *with* $V_0(\xi) = U_0$ *for which* $L(\xi, H) = 0$ $\mathbb{P}$*-a.s.*

Brilliant ... but

- this might be unacceptably expensive (Gushchin and Mordecki (2002)),
- what about the case $U_0 = \infty$.

IDEA: Fix $\tilde{V}_0 < U_0$ and (in some sense) minimize $L(\xi, H)$ over ξ satisfying $V_0(\xi) \leq \tilde{V}_0$.

How to quantify risk of a loss $L(\xi, H)$?

- Quantile hedging (Föllmer and Leukert (1999)):

$$\mathbb{P}(L(\xi, H) < 0) \to min.$$

- Efficient hedging (FL (2000), Nakano (2003, 04), Rudloff (2007, 09))

$$\rho(L(\xi, H)) \to min,$$

where ρ is some risk measure (coherent or more generally convex...)

How to quantify risk of a loss $L(\xi, H)$?

- Quantile hedging (Föllmer and Leukert (1999)):

$$\mathbb{P}(L(\xi, H) < 0) \to min.$$

- Efficient hedging (FL (2000), Nakano (2003, 04), Rudloff (2007, 09))

$$\rho(L(\xi, H)) \to min,$$

where ρ is some risk measure (coherent or more generally convex...)

How to quantify risk of a loss $L(\xi, H)$?

- Quantile hedging (Föllmer and Leukert (1999)):

$$\mathbb{P}(L(\xi, H) < 0) \rightarrow min.$$

- Efficient hedging (FL (2000), Nakano (2003, 04), Rudloff (2007, 09))

$$\rho(L(\xi, H)) \rightarrow min,$$

where ρ is some risk measure (coherent or more generally convex...)

How to quantify risk of a loss $L(\xi, H)$?

- Quantile hedging (Föllmer and Leukert (1999)):

$$\mathbb{P}(L(\xi, H) < 0) \to min.$$

- Efficient hedging (FL (2000), Nakano (2003, 04), Rudloff (2007, 09))

$$\rho(L(\xi, H)) \to min,$$

where ρ is some risk measure (coherent or more generally convex...)

How to quantify risk of a loss $L(\xi, H)$?

- Quantile hedging (Föllmer and Leukert (1999)):

$$\mathbb{P}(L(\xi, H) < 0) \to min.$$

- Efficient hedging (FL (2000), Nakano (2003, 04), Rudloff (2007, 09))

$$\rho(L(\xi, H)) \to min,$$

where ρ is some risk measure (coherent or more generally convex...)

How to quantify risk of a loss $L(\xi, H)$?

- Quantile hedging (Föllmer and Leukert (1999)):

$$\mathbb{P}(L(\xi, H) < 0) \to min.$$

- Efficient hedging (FL (2000), Nakano (2003, 04), Rudloff (2007, 09))

$$\rho(L(\xi, H)) \to min,$$

  where ρ is some risk measure (coherent or more generally convex...)

Convex measures of risk

Definition 2

A function $\rho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le p \le \infty$ *is a convex measure of risk if:*

- *it is convex, i.e.*

$$\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y), \qquad \lambda \in [0,1],\ X, Y \in L^p$$

- *monotone, i.e.*

$$X \ge Y \ \Rightarrow \rho(X) \le \rho(Y),\ X, Y \in L^p$$

- *translation invariant , i.e.*

$$\rho(X + c) = \rho(X) - c, \qquad c \in \mathbb{R},\ X \in L^p.$$

Convex measures of risk

Definition 2

A function $\rho : L^p \to \mathbb{R} \cup \{\infty\},\ 1 \leq p \leq \infty$ *is a convex measure of risk if:*

- *it is convex, i.e.*

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \qquad \lambda \in [0,1],\ X, Y \in L^p$$

- *monotone, i.e.*

$$X \geq Y \ \Rightarrow \rho(X) \leq \rho(Y),\ X, Y \in L^p$$

- *translation invariant , i.e.*

$$\rho(X + c) = \rho(X) - c, \qquad c \in \mathbb{R},\ X \in L^p.$$

Convex measures of risk

Definition 2

A function $\rho : L^p \to \mathbb{R} \cup \{\infty\}$, $1 \le p \le \infty$ *is a convex measure of risk if:*

- *it is convex, i.e.*

$$\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y), \qquad \lambda \in [0,1],\ X, Y \in L^p$$

- *monotone, i.e.*

$$X \ge Y \ \Rightarrow \rho(X) \le \rho(Y),\ X, Y \in L^p$$

- *translation invariant , i.e.*

$$\rho(X + c) = \rho(X) - c, \qquad c \in \mathbb{R},\ X \in L^p.$$

Convex measures of risk

Definition 2

A function $\rho : L^p \to \mathbb{R} \cup \{\infty\},\ 1 \le p \le \infty$ *is a convex measure of risk if:*

- *it is convex, i.e.*

$$\rho(\lambda X + (1-\lambda)Y) \le \lambda\rho(X) + (1-\lambda)\rho(Y), \qquad \lambda \in [0,1],\ X, Y \in L^p$$

- *monotone, i.e.*

$$X \ge Y \ \Rightarrow \rho(X) \le \rho(Y),\ X, Y \in L^p$$

- *translation invariant , i.e.*

$$\rho(X + c) = \rho(X) - c, \qquad c \in \mathbb{R},\ X \in L^p.$$

Convex measures of risk

Definition 2

A function $\rho : L^p \to \mathbb{R} \cup \{\infty\},\ 1 \leq p \leq \infty$ *is a convex measure of risk if:*

- *it is convex, i.e.*

$$\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y), \qquad \lambda \in [0,1],\ X, Y \in L^p$$

- *monotone, i.e.*

$$X \geq Y \ \Rightarrow \rho(X) \leq \rho(Y),\ X, Y \in L^p$$

- *translation invariant , i.e.*

$$\rho(X + c) = \rho(X) - c, \qquad c \in \mathbb{R},\ X \in L^p.$$

Convex hedging problem formulation

Let

- ρ be a convex measure of risk on L^1,
- $0 \leq H \in L^1(\mathbb{P})$
- $\tilde{V}_0 > 0$
- $\mathcal{V}_{\tilde{V}_0} = \{\xi \in \Phi \ : \ V(\xi) \geq 0, \ V_0(\xi) \leq \tilde{V}_0\}$

The convex hedging problem:

$$\inf_{\xi \in \mathcal{V}_{\tilde{V}_0}} \rho(L(\xi, H)) \tag{1}$$

Convex hedging problem formulation

Let

- ρ be a convex measure of risk on L^1,
- $0 \leq H \in L^1(\mathbb{P})$
- $\tilde{V}_0 > 0$
- $\mathcal{V}_{\tilde{V}_0} = \{\xi \in \Phi \;:\; V(\xi) \geq 0,\; V_0(\xi) \leq \tilde{V}_0\}$

The convex hedging problem:

$$\inf_{\xi \in \mathcal{V}_{\tilde{V}_0}} \rho(L(\xi, H)) \tag{1}$$

Convex hedging problem formulation

Let

- ρ be a convex measure of risk on L^1,
- $0 \leq H \in L^1(\mathbb{P})$
- $\tilde{V}_0 > 0$
- $\mathcal{V}_{\tilde{V}_0} = \{\xi \in \Phi \ : \ V(\xi) \geq 0, \ V_0(\xi) \leq \tilde{V}_0\}$

The convex hedging problem:

$$\inf_{\xi \in \mathcal{V}_{\tilde{V}_0}} \rho(L(\xi, H)) \tag{1}$$

Convex hedging problem formulation

Let

- ρ be a convex measure of risk on L^1,
- $0 \leq H \in L^1(\mathbb{P})$
- $\tilde{V}_0 > 0$
- $\mathcal{V}_{\tilde{V}_0} = \{\xi \in \Phi \ : \ V(\xi) \geq 0, \ V_0(\xi) \leq \tilde{V}_0\}$

The convex hedging problem:

$$\inf_{\xi \in \mathcal{V}_{\tilde{V}_0}} \rho(L(\xi, H)) \tag{1}$$

Convex hedging problem formulation

Let

- ρ be a convex measure of risk on L^1,
- $0 \leq H \in L^1(\mathbb{P})$
- $\tilde{V}_0 > 0$
- $\mathcal{V}_{\tilde{V}_0} = \{\xi \in \Phi \; : \; V(\xi) \geq 0, \; V_0(\xi) \leq \tilde{V}_0\}$

The convex hedging problem:

$$\inf_{\xi \in \mathcal{V}_{\tilde{V}_0}} \rho(L(\xi, H)) \tag{1}$$

Convex hedging problem formulation

Let

- ρ be a convex measure of risk on L^1,
- $0 \leq H \in L^1(\mathbb{P})$
- $\tilde{V}_0 > 0$
- $\mathcal{V}_{\tilde{V}_0} = \{\xi \in \Phi \ : \ V(\xi) \geq 0, \ V_0(\xi) \leq \tilde{V}_0\}$

The convex hedging problem:

$$\inf_{\xi \in \mathcal{V}_{\tilde{V}_0}} \rho(L(\xi, H)) \tag{1}$$

Key concept

Let

- $\mathcal{R} = \{\phi : \Omega \to [0,1] \ : \ \phi - \mathcal{F} - \textit{measurable}\}$,
- $\mathcal{R}_0 = \{\phi \in \mathcal{R} \ : \ sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} \phi H \leq \tilde{V}_0\}$,

Theorem 3.1

*If $\tilde{\phi} \in \mathcal{R}_0$ solves **the static problem***

$$\inf_{\phi \in \mathcal{R}_0} \rho(H(\phi - 1)), \tag{2}$$

the strategy $(\tilde{V}_0, \tilde{\xi})$ superreplicating $\tilde{\phi}H$ solves the efficient hedging problem (1).

Key concept

Let

- $\mathcal{R} = \{\phi : \Omega \to [0,1] \ : \ \phi - \mathcal{F} - measurable\}$,
- $\mathcal{R}_0 = \{\phi \in \mathcal{R} \ : \ sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} \phi H \leq \tilde{V}_0\}$,

Theorem 3.1

If $\tilde{\phi} \in \mathcal{R}_0$ solves ***the static problem***

$$\inf_{\phi \in \mathcal{R}_0} \rho(H(\phi - 1)), \tag{2}$$

the strategy $(\tilde{V}_0, \tilde{\xi})$ superreplicating $\tilde{\phi}H$ solves the efficient hedging problem (1).

Selected results obtained so far (1)

1. (Föllmer, Leukert (2000)), ρ = expectation - existence and structure of the solution,
2. (Nakano (2004)), ρ - coherent measure of risk on L^1 - existence, structure in particular cases,
3. (Rudloff (2007)), ρ - convex, l.s.c. measure of risk on L^1 satisfying some continuity assumption - existence and structure of the solution.

Selected results obtained so far (1)

1. (Föllmer, Leukert (2000)), ρ = expectation - existence and structure of the solution,
2. (Nakano (2004)), ρ - coherent measure of risk on L^1 - existence, structure in particular cases,
3. (Rudloff (2007)), ρ - convex, l.s.c. measure of risk on L^1 satisfying some continuity assumption - existence and structure of the solution.

Selected results obtained so far (1)

1. (Föllmer, Leukert (2000)), ρ = expectation - existence and structure of the solution,
2. (Nakano (2004)), ρ - coherent measure of risk on L^1 - existence, structure in particular cases,
3. (Rudloff (2007)), ρ - convex, l.s.c. measure of risk on L^1 satisfying some continuity assumption - existence and structure of the solution.

Selected results obtained so far (2)

Standard assumptions:

Assumption 1

$sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty$

For a convex measures of risk ρ:

Assumption 2

$\rho : L^1 \to \mathbb{R} \cup \{\infty\}$ *finite and continuous at* $H(\phi_0 - 1)$ *with some* $\phi_0 \in \mathcal{R}_0$.

Selected results obtained so far (2)

Standard assumptions:

Assumption 1

$sup_{\mathbb{P}^* \in \mathcal{P}_\sigma} E_{\mathbb{P}^*} H < \infty$

For a convex measures of risk ρ:

Assumption 2

$\rho : L^1 \to \mathbb{R} \cup \{\infty\}$ *finite and continuous at* $H(\phi_0 - 1)$ *with some* $\phi_0 \in \mathcal{R}_0$.

Call on the non-traded securities/ risk factors(1)

Consider standard BS model:

$$S_t = exp\left(W_t - \frac{1}{2}\right), \qquad B_t = 1 \qquad t \in [0,1]$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports infinite iid sequence of standard Gaussian random variables $X_1, X_2, \ldots$ (independent of W) and an independent $U \sim U[0,1]$.

For $n = 1, 2, \ldots$ define R^n (quoted at discrete times: $t = 0, 1$):

$$R_0^n = 1, \quad R_1^n = \exp\left[\rho_n W_1 + \sqrt{1-\rho_n^2} n X_n - \frac{1}{2}(\rho_n^2 + n^2(1-\rho_n^2))\right].$$

Assume correlation $\rho_n \in (-1,1)$ decays to 0 with n, i.e. $\lim_{n\to\infty} \rho_n = 0$. Now consider a hedging problem of a call option with the payoff

$$H = \left(\sum_{n=1}^{\infty} \frac{1}{2^n} R_1^n - K\right)^+, \tag{3}$$

for some $K > 0$.

Call on the non-traded securities/ risk factors(1)

Consider standard BS model:

$$S_t = exp\left(W_t - \frac{1}{2}\right), \qquad B_t = 1 \qquad t \in [0,1]$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports infinite iid sequence of standard Gaussian random variables $X_1, X_2, \dots$ (independent of W) and an independent $U \sim U[0,1]$.
For $n = 1, 2, \dots$ define R^n (quoted at discrete times: $t = 0, 1$):

$$R_0^n = 1, \quad R_1^n = \exp\left[\rho_n W_1 + \sqrt{1-\rho_n^2} n X_n - \frac{1}{2}(\rho_n^2 + n^2(1-\rho_n^2))\right].$$

Assume correlation $\rho_n \in (-1,1)$ decays to 0 with n, i.e. $\lim_{n\to\infty} \rho_n = 0$.
Now consider a hedging problem of a call option with the payoff

$$H = \left(\sum_{n=1}^{\infty} \frac{1}{2^n} R_1^n - K\right)^+, \qquad (3)$$

for some $K > 0$.

Call on the non-traded securities/ risk factors(1)

Consider standard BS model:

$$S_t = exp\left(W_t - \frac{1}{2}\right), \qquad B_t = 1 \qquad t \in [0,1]$$

on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which supports infinite iid sequence of standard Gaussian random variables $X_1, X_2, \ldots$ (independent of W) and an independent $U \sim U[0,1]$.
For $n = 1, 2, \ldots$ define R^n (quoted at discrete times: $t = 0, 1$):

$$R_0^n = 1, \quad R_1^n = \exp\left[\rho_n W_1 + \sqrt{1-\rho_n^2} n X_n - \frac{1}{2}(\rho_n^2 + n^2(1-\rho_n^2))\right].$$

Assume correlation $\rho_n \in (-1, 1)$ decays to 0 with n, i.e. $\lim_{n\to\infty} \rho_n = 0$.
Now consider a hedging problem of a call option with the payoff

$$H = \left(\sum_{n=1}^{\infty} \frac{1}{2^n} R_1^n - K\right)^+, \tag{3}$$

for some $K > 0$.

Call on the non-traded securities/ risk factors(2)

Proposition 1

1. *H is a well-defined and integrable contingent claim.*
2. *For $n \in \mathbb{N}$*

$$\sqrt{2\pi} \cdot (2^n)^2 \cdot \exp\left(\frac{X_n^2}{2}\right) \cdot \mathbf{1}_{\{|X_n - \sqrt{1-\rho_n^2}n| \leq 2^{-2n-1}\}}$$

is a density of a martingale measure $\mathbb{Q}_n$.

3. $\sup_{n\in\mathbb{N}} E_{\mathbb{Q}_n} H = \infty$.

Convex risk measure violating continuity assumption(1)

Define $g : [0, 1] \to \mathbb{R}$:

$$g(x) = \begin{cases} \frac{3}{4}x^{-\frac{1}{4}} & \text{for } x \in (0, 1], \\ 0 & \text{gdy } x = 0. \end{cases}$$

For $n \in \mathbb{N}$ define $g_n : [0, 1] \to \mathbb{R}$:

$$g_n(x) = c_n \mathbf{1}_{[\frac{1}{n+1}, 1]}(x) g(x), \qquad x \in [0, 1],$$

where $c_n^{-1} := 1 - (\frac{1}{n+1})^{\frac{3}{4}}$ is a normalizing constant (such that $E[g_n(U)] = 1$ for $U \sim U[0, 1]$) .

Fix $U \sim U[0, 1]$ and let $\mathcal{Q} = \{g_n(U) : n \in \mathbb{N}\}$.

Convex risk measure violating continuity assumption(1)

Define $g : [0,1] \to \mathbb{R}$:

$$g(x) = \begin{cases} \frac{3}{4}x^{-\frac{1}{4}} & \textit{for } x \in (0,1], \\ 0 & \textit{gdy } x = 0. \end{cases}$$

For $n \in \mathbb{N}$ define $g_n : [0,1] \to \mathbb{R}$:

$$g_n(x) = c_n \mathbf{1}_{[\frac{1}{n+1},1]}(x) g(x), \qquad x \in [0,1],$$

where $c_n^{-1} := 1 - (\frac{1}{n+1})^{\frac{3}{4}}$ is a normalizing constant (such that $E[g_n(U)] = 1$ for $U \sim U[0,1]$) .

Fix $U \sim U[0,1]$ and let $\mathcal{Q} = \{g_n(U) : n \in \mathbb{N}\}$.

Convex risk measure violating continuity assumption(1)

Define $g : [0,1] \to \mathbb{R}$:

$$g(x) = \begin{cases} \frac{3}{4}x^{-\frac{1}{4}} & \textit{for } x \in (0,1], \\ 0 & \textit{gdy } x = 0. \end{cases}$$

For $n \in \mathbb{N}$ define $g_n : [0,1] \to \mathbb{R}$:

$$g_n(x) = c_n \mathbf{1}_{[\frac{1}{n+1},1]}(x) g(x), \qquad x \in [0,1],$$

where $c_n^{-1} := 1 - (\frac{1}{n+1})^{\frac{3}{4}}$ is a normalizing constant (such that $E[g_n(U)] = 1$ for $U \sim U[0,1]$) .

Fix $U \sim U[0,1]$ and let $\mathcal{Q} = \{g_n(U) : n \in \mathbb{N}\}$.

Convex risk measure violating continuity assumption(2)

Define $\bar{\rho} : L^1 \to \mathbb{R} \cup \{\infty\}$: $\bar{\rho}(Y) := \sup_{\mathbb{Q} \in \mathcal{Q}} E_{\mathbb{Q}}(-Y)$.

Proposition 2

$\bar{\rho}$ is a $\sigma(L^1, L^\infty)$-l.s.c, coherent measure of risk.

Consider the static problem:

$$\inf_{\phi \in \mathcal{R}_0} \bar{\rho}((\phi - 1)H),$$

Proposition 3

Let H be and arbitrary contingent claim for which $\bar{\rho}((\widehat{\phi} - 1)H) < \infty$ with some $\widehat{\phi} \in \mathcal{R}_0$.
1. The function $\bar{\rho} : L^1 \to \mathbb{R} \cup \{\infty\}$ is not continuous at $(\phi_0 - 1)H$ for any $\phi_0 \in \mathcal{R}_0$ satisfying $\bar{\rho}((\phi_0 - 1)H) < \infty$.
2. $\bar{\rho}|_{L^2} < \infty$

Convex risk measure violating continuity assumption(2)

Define $\bar{\rho} : L^1 \to \mathbb{R} \cup \{\infty\}$: $\bar{\rho}(Y) := \sup_{\mathbb{Q} \in \mathcal{Q}} E_{\mathbb{Q}}(-Y)$.

Proposition 2

$\bar{\rho}$ is a $\sigma(L^1, L^\infty)$-l.s.c, coherent measure of risk.

Consider the static problem:

$$\inf_{\phi \in \mathcal{R}_0} \bar{\rho}((\phi - 1)H),$$

Proposition 3

Let H be and arbitrary contingent claim for which $\bar{\rho}((\widehat{\phi} - 1)H) < \infty$ with some $\widehat{\phi} \in \mathcal{R}_0$.

1. The function $\bar{\rho} : L^1 \to \mathbb{R} \cup \{\infty\}$ is not continuous at $(\phi_0 - 1)H$ for any $\phi_0 \in \mathcal{R}_0$ satisfying $\bar{\rho}((\phi_0 - 1)H) < \infty$.

2. $\bar{\rho}|_{L^2} < \infty$

Convex risk measure violating continuity assumption(2)

Define $\bar{\rho} : L^1 \to \mathbb{R} \cup \{\infty\}$: $\bar{\rho}(Y) := \sup_{\mathbb{Q} \in \mathcal{Q}} E_{\mathbb{Q}}(-Y)$.

Proposition 2

$\bar{\rho}$ is a $\sigma(L^1, L^\infty)$-l.s.c, coherent measure of risk.

Consider the static problem:

$$\inf_{\phi \in \mathcal{R}_0} \bar{\rho}((\phi - 1)H),$$

Proposition 3

Let H be and arbitrary contingent claim for which $\bar{\rho}((\widehat{\phi} - 1)H) < \infty$ with some $\widehat{\phi} \in \mathcal{R}_0$.
1. The function $\bar{\rho} : L^1 \to \mathbb{R} \cup \{\infty\}$ is not continuous at $(\phi_0 - 1)H$ for any $\phi_0 \in \mathcal{R}_0$ satisfying $\bar{\rho}((\phi_0 - 1)H) < \infty$.
2. $\bar{\rho}|_{L^2} < \infty$

Overview

1. For $n = 1, 2, 3, \ldots$ let $H_n := H \wedge n$.
2. Consider problems
$$\inf_{\phi \in \mathcal{R}_0} \rho(H_n(\phi - 1)). \qquad (4)$$
3. We apply results of (Rudloff (2007)) to obtain the existence and the structure of the solution ϕ_n of (5) for $n \in \mathbb{N}$.
4. Komlos' type argument:
there exists a sequence $\tilde{\phi}_n \in conv(\phi_n, \phi_{n+1}, \phi_{n+2}, \ldots)$ and $\tilde{\phi} \in \mathcal{R}$ such that $\lim_{m\to\infty} \tilde{\phi}_n = \tilde{\phi}\ \mathbb{P} -$ a.s. and in L^1.
5. $\tilde{\phi} \in \mathcal{R}_0$
6. $\tilde{\phi}$ solves the static problem.

Overview

1. For $n = 1, 2, 3, \ldots$ let $H_n := H \wedge n$.
2. Consider problems

$$\inf_{\phi \in \mathcal{R}_0} \rho(H_n(\phi - 1)). \tag{4}$$

3. We apply results of (Rudloff (2007)) to obtain the existence and the structure of the solution ϕ_n of (5) for $n \in \mathbb{N}$.
4. Komlos' type argument:
there exists a sequence $\tilde{\phi}_n \in conv(\phi_n, \phi_{n+1}, \phi_{n+2}, \ldots)$ and $\tilde{\phi} \in \mathcal{R}$ such that $\lim_{m\to\infty} \tilde{\phi}_n = \tilde{\phi}\ \mathbb{P} -$ a.s. and in L^1.
5. $\tilde{\phi} \in \mathcal{R}_0$
6. $\tilde{\phi}$ solves the static problem.

Overview

1. For $n = 1, 2, 3, \ldots$ let $H_n := H \wedge n$.
2. Consider problems
$$\inf_{\phi \in \mathcal{R}_0} \rho(H_n(\phi - 1)). \quad (4)$$
3. We apply results of (Rudloff (2007)) to obtain the existence and the structure of the solution ϕ_n of (5) for $n \in \mathbb{N}$.
4. Komlos' type argument:
there exists a sequence $\tilde{\phi}_n \in conv(\phi_n, \phi_{n+1}, \phi_{n+2}, \ldots)$ and $\tilde{\phi} \in \mathcal{R}$ such that $\lim_{m\to\infty} \tilde{\phi}_n = \tilde{\phi}$ $\mathbb{P}$ − a.s. and in L^1.
5. $\tilde{\phi} \in \mathcal{R}_0$
6. $\tilde{\phi}$ solves the static problem.

Overview

1. For $n = 1, 2, 3, \ldots$ let $H_n := H \wedge n$.
2. Consider problems
$$\inf_{\phi \in \mathcal{R}_0} \rho(H_n(\phi - 1)). \tag{4}$$
3. We apply results of (Rudloff (2007)) to obtain the existence and the structure of the solution ϕ_n of (5) for $n \in \mathbb{N}$.
4. Komlos' type argument:
there exists a sequence $\tilde{\phi}_n \in conv(\phi_n, \phi_{n+1}, \phi_{n+2}, \ldots)$ and $\tilde{\phi} \in \mathcal{R}$ such that $\lim_{m \to \infty} \tilde{\phi}_n = \tilde{\phi}$ $\mathbb{P} -$ a.s. and in L^1.
5. $\tilde{\phi} \in \mathcal{R}_0$
6. $\tilde{\phi}$ solves the static problem.

Overview

1. For $n = 1, 2, 3, \ldots$ let $H_n := H \wedge n$.
2. Consider problems
$$\inf_{\phi \in \mathcal{R}_0} \rho(H_n(\phi - 1)). \tag{4}$$
3. We apply results of (Rudloff (2007)) to obtain the existence and the structure of the solution ϕ_n of (5) for $n \in \mathbb{N}$.
4. Komlos' type argument:
there exists a sequence $\tilde{\phi}_n \in conv(\phi_n, \phi_{n+1}, \phi_{n+2}, \ldots)$ and $\tilde{\phi} \in \mathcal{R}$ such that $\lim_{m\to\infty} \tilde{\phi}_n = \tilde{\phi}\ \mathbb{P} -$ a.s. and in L^1.
5. $\tilde{\phi} \in \mathcal{R}_0$
6. $\tilde{\phi}$ solves the static problem.

Overview

1. For $n = 1, 2, 3, \ldots$ let $H_n := H \wedge n$.
2. Consider problems
$$\inf_{\phi \in \mathcal{R}_0} \rho(H_n(\phi - 1)). \tag{4}$$
3. We apply results of (Rudloff (2007)) to obtain the existence and the structure of the solution ϕ_n of (5) for $n \in \mathbb{N}$.
4. Komlos' type argument:
there exists a sequence $\tilde{\phi}_n \in conv(\phi_n, \phi_{n+1}, \phi_{n+2}, \ldots)$ and $\tilde{\phi} \in \mathcal{R}$ such that $\lim_{m\to\infty} \tilde{\phi}_n = \tilde{\phi}\ \mathbb{P} -$ a.s. and in L^1.
5. $\tilde{\phi} \in \mathcal{R}_0$
6. $\tilde{\phi}$ solves the static problem.

Overview

1. For $n = 1, 2, 3, \ldots$ let $H_n := H \wedge n$.
2. Consider problems

$$\inf_{\phi \in \mathcal{R}_0} \rho(H_n(\phi - 1)). \tag{4}$$

3. We apply results of (Rudloff (2007)) to obtain the existence and the structure of the solution ϕ_n of (5) for $n \in \mathbb{N}$.
4. Komlos' type argument:
there exists a sequence $\tilde{\phi}_n \in conv(\phi_n, \phi_{n+1}, \phi_{n+2}, \ldots)$ and $\tilde{\phi} \in \mathcal{R}$ such that $\lim_{m\to\infty} \tilde{\phi}_n = \tilde{\phi}\ \mathbb{P} -$ a.s. and in L^1.
5. $\tilde{\phi} \in \mathcal{R}_0$
6. $\tilde{\phi}$ solves the static problem.

The main result I

Let Λ_+ denote the set of measures of finite variation on $\mathcal{P}_\sigma$

Fix $\tilde{V}_0 > 0$ and let $\rho : L^1 \to \mathbb{R} \cup \{\infty\}$ be a convex measure of risk with a determining set $\mathcal{Q}$.

Let H be an integrable contingent claim and denote $H_n := H \wedge n$.

Theorem 1

Assume: ρ *is* L^p*-continuous for some* $p \geq 1$ *at* $\phi_0(H-1)$ *with some* $\phi_0 \in \mathcal{R}_0$*.*
1. *For every* $n \in \mathbb{N}$ *there exists a solution* $(\lambda_n, \mathbb{Q}_n)$ *to:*

$$\inf_{(\lambda,\mathbb{Q})\in(\Lambda_+,\mathcal{Q})} \left\{ E\left[H_n Z_{\mathbb{Q}} \wedge H_n \int_{\mathcal{P}_\sigma} Z_{\mathbb{P}^*} d\lambda \right] - \tilde{V}_0 \lambda(\mathcal{P}_\sigma) - \rho^*(-Z_{\mathbb{Q}}) \right\}.$$

The main result

Theorem 1 (continued)

2. For every n there exists a solution ϕ_n to the static problem

$$\inf_{\phi \in \mathcal{R}_0} \rho(H_n(\phi - 1)) \tag{5}$$

satisfying

$$\phi_n(\omega) = \begin{cases} 1 & : \; H_n(Z_{\mathbb{Q}_n} - \int_{\mathcal{P}_\sigma} Z_{\mathbb{P}^*} \lambda_n(d\mathbb{P}^*))(\omega) > 0, \\ 0 & : \; H_n(Z_{\mathbb{Q}_n} - \int_{\mathcal{P}_\sigma} Z_{\mathbb{P}^*} \lambda_n(d\mathbb{P}^*))(\omega) < 0 \end{cases}$$

and $E_{\mathbb{P}^*} H_n \phi_n = \tilde{V}_0 \quad$ for $\lambda_n - a.e. \mathbb{P}^* \in \mathcal{P}_\sigma$

3. Solution to the static problem

$$\inf_{\phi \in \mathcal{R}_0} \bar{\rho}((\phi - 1)H),$$

is given by $\tilde{\phi} = \lim_{n \to \infty} \tilde{\phi}_n$, for some $\tilde{\phi}_n \in conv(\phi_n, \phi_{n+1}, \ldots)$.

The Bibliography I

H.Föllmer, P. Leukert
Efficient hedging: Cost versus shortfall risk.
Finance and Stochastics, 4 (2000), 117-146

Y. Nakano
Efficient hedging with coherent risk measure.
J. Math. Anal. Appl., 293 (2004), 345-354

B. Rudloff
Convex hedging in incomplete Markets.
Applied Mathematical Finance 14 (2007), 437-452.

A. A. Gushchin, E. Mordecki
Bounds on Option Prices for Semimartingale Market Models.
Proc. Steklov Math. Inst 237 (2002), 73-113.

Thank you for your attention!