Ambit fields via Fourier methods in the context of power markets

Heidar Eyjolfsson

University of Oslo

10 June 2013

6th AMaMeF and Banach Center Conference

based on joint work with F. E. Benth (University of Oslo) and A. Veraart (Imperial College)

Introduction

- An ambit field is a stochastic tempo-spatial random field.
  - Very general
  - Initially in turbulence context
  - Has been employed to model tumor growth.
- Power markets display various idiosyncratic features that ambit fields can be used to catch.
  - Dramatic spikes.
  - No buy-and-hold hedging.
  - Complex noise structure
  - Semimartingale / Non-semimartingale setting.
- We develop an incremental approximation scheme for general ambit fields.
  - Integrand depends on tempo-spatial position.
  - Useful for pricing in power markets.

Lévy bases

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $S \in \mathcal{B}(\mathbb{R}^n)$ for $n \geq 1$ and let $\mathcal{S} = \mathcal{B}(S)$.

Definition

A *Lévy basis* on $(S, \mathcal{S})$ is a family $\{L(A)\}_{A \in \mathcal{B}_b(S)}$ of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- $L(\cup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} L(A_n)$ a.s. for disjoint $\{A_n\} \subset \mathcal{B}_b(S)$.
- $L(A_1), L(A_2), \ldots$ are independent for disjoint $\{A_n\} \subset \mathcal{B}_b(S)$.
- For any $A \in \mathcal{B}_b(S)$ if μ is the law of $L(A)$, then there exists a law μ_n that satisfies $\mu = \mu_n^{*n}$ for any $n \geq 1$.

A Lévy basis has Lévy-Kinchin representation

$$\log(\mathbb{E}[\exp(\mathrm{i}\zeta L(A))]) = \mathrm{i}\zeta a^*(A) - \frac{1}{2}\zeta^2 b^*(A) + \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}\zeta x} - 1 - \mathrm{i}\zeta x 1_{[-1,1]}(x)\right) n(dx, A),$$

where a^* is a signed measure on $\mathcal{B}_b(S)$, b^* is a measure on $\mathcal{B}_b(S)$ and $n(dx, A)$ is a Lévy measure on $\mathbb{R}$ for fixed $A \in \mathcal{B}_b(S)$ and a measure on $\mathcal{B}_b(S)$ for fixed dx.

Ambit fields

Definition

An *ambit field* is a tempo-spatial stochastic model on the form

$$Y(\boldsymbol{x}, t) = \int_{A(x,t)} g(\boldsymbol{x}, t; \boldsymbol{\xi}, s)\sigma(\boldsymbol{\xi}, s)L(d\boldsymbol{\xi}, ds),$$

where

- $(\boldsymbol{x}, t) \in \mathbb{R}^d \times \mathbb{R}$,
- $A(\boldsymbol{x}, t) \subset \mathbb{R}^d \times \mathbb{R}$,
- $g : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$,
- σ is a non-negative stochastic space-time volatility field,
- L is a square integrable Lévy basis on $(S, \mathcal{S})$, where $S \subset \mathbb{R}^d \times \mathbb{R}$.

- A general class of models, including null-spatial ($d = 0$) temporal models.
- Initially suggested in the context of turbulence.
- Suggested as a general framework in the energy setting.

A few special cases

Time stationarity and nonanticipative and homogeneous in space

$$Y(\mathbf{x}, t) = \int_{A(x,t)} g(\mathbf{x} - \xi, t - s)\sigma(\xi, s)L(d\xi, ds),$$

where $A(\mathbf{x}, t) = A + (\mathbf{x}, t)$ and A only involves negative time coordinates.

VMV processes

A volatility modulated (VMV) process is a process

$$X(t) = \int_{-\infty}^{t} g(t, s)\sigma(s-)dL(s),$$

for $t \in \mathbb{R}$, where $\{L(t)\}_{t\in\mathbb{R}}$ is a (two-sided) square integrable Lévy process.

LSS processes

If $g(t, s) = h(t - s)$, a VMV process is called a Lévy semistationary (LSS) process.

Lévy semistationary processes

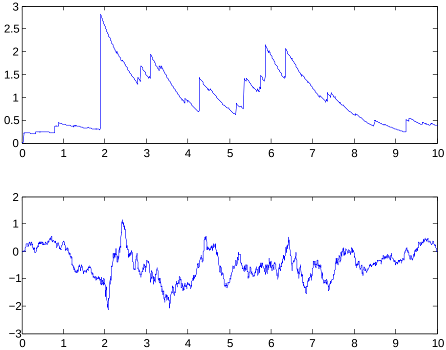

Figure: Above: The process $\sigma^2(t)$, where $\sigma^2(t) = \int_0^t \mathrm{e}^{-(t-s)} dU(s)$ on the interval $[0, 10]$ and U is a inverse Gaussian Lévy process. Below: The LSS process $X(t) = \int_0^t \mathrm{e}^{-(t-s)} \sigma(s-) dB(s)$ on $[0, 10]$, where B is a standard Wiener process.

Deseasonalised spot by means of VMV processes

Barndorff-Nielsen et al. [2] propose modelling the spot by means of both arithmetic and geometric models of the types

$$S(t) = \Lambda(t) + X(t) \quad \text{and} \quad S(t) = \Lambda(t)\exp(X(t)),$$

where $\Lambda : [0, \infty) \to [0, \infty)$ is a deterministic seasonality and trend function.

- Forward price dynamics, $f(t, T)$, may be derived as an expression involving the VMV process

$$\int_{-\infty}^{t} g(T, s)\sigma(s-)dL(s) = \int_{-\infty}^{t} g(t + x, s)\sigma(s-)dL(s),$$

  where $x = T - t$ denotes time-to-maturity.
- A contract delivering electricity over $[T_1, T_2]$, is priced by

$$F(t, T_1, T_2) = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} f(t, u)du.$$

Ambit field application in energy II

Forward dynamics modelled directly by ambit fields

Barndorff-Nielsen et al. [1] suggest using ambit fields of the type

$$f(t,x) = \int_{A(x,t)} k(x, t-s, \xi)\sigma(\xi, s)L(d\xi, ds),$$

where ($d = 1$ and)

$$A(x,t) = \{(\xi, s) \in \mathbb{R}^2 : \xi \geq 0, s \leq t\},$$

as a general modelling framework for electricity forwards.

Here the forward is modelled directly under the risk neutral measure.

Parameter dependece of model integrands

Tempo-spatial dependency of the integral kernel

For general ambit fields

$$Y(\boldsymbol{x},t) = \int_{A(x,t)} g(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds),$$

the tempo-spatial dependency of the kernel means that for given $(\Delta\boldsymbol{x},\Delta t) \geq \mathbf{0}$ one can not use the value of $Y(\boldsymbol{x},t)$ together with an increment to obtain the value of

$$Y(\boldsymbol{x}+\Delta\boldsymbol{x}, t+\Delta t).$$

Ambit fields in the "Fourier domain"

To get around this problem we suggest approximating g by means of a linear comibination of complex exponential functions, which accommodate incremental approximation.

Approximating general kernel functions

For a general kernel $g : \mathbb{R}^n \to \mathbb{R}$, where $n \geq 1$, we proceed as follows.

- Approximate g in L^2 by a continuous and compactly supported function h, such that
$$h(u_1, \dots, u_k, \dots, u_n) = h(u_1, \dots, -u_k, \dots, u_n),$$
for all $1 \leq k \leq n$.
- For $\boldsymbol{\lambda} > \mathbf{0}$, introduce $h_{\boldsymbol{\lambda}}(\boldsymbol{u}) := h(\boldsymbol{u})\mathrm{e}^{\boldsymbol{\lambda}\cdot|\boldsymbol{u}|}$, where $|\boldsymbol{u}| = (|u_1|, \dots, |u_n|)$ in the case of multivariate h.
- Represent $h_{\boldsymbol{\lambda}}$ by its inverse Fourier transform
$$h_{\boldsymbol{\lambda}}(\boldsymbol{u}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{h}_{\boldsymbol{\lambda}}(v)\mathrm{e}^{\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{v}} d\boldsymbol{v},$$
where $\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}) = \int_{\mathbb{R}^n} h_{\boldsymbol{\lambda}}(\boldsymbol{u})\mathrm{e}^{-\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{v}} d\boldsymbol{u}$.
- Multiply both sides with $\mathrm{e}^{-\boldsymbol{\lambda}\cdot|\boldsymbol{u}|}$ to obtain
$$h(\boldsymbol{u}) \approx \sum_{\alpha\in\mathcal{I}} c_\alpha \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v}_\alpha)\cdot\boldsymbol{u}},$$
for $\boldsymbol{u} \geq 0$, where $\mathcal{I} \subset \mathbb{Z}^n$ is a finite set, and $\{c_\alpha\}_{\alpha\in\mathcal{I}}$ and $\{v_\alpha\}_{\alpha\in\mathcal{I}}$ are appropriately selected coefficients.

Ambit fields approximated by a linear combination of complex ambit fields driven by exponential kernels

Linear combination of complex ambit fields

For an ambit field driven by h we find that

$$\int_{A(\boldsymbol{x},t)} h(\boldsymbol{x},t;\boldsymbol{\xi},s)\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds) \approx \sum_{\alpha\in\mathcal{I}} c_\alpha \widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x},t,\boldsymbol{v}_\alpha), \tag{1}$$

where

$$\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x},t,\boldsymbol{v}_\alpha) = \int_{A(\boldsymbol{x},t)} \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v}_\alpha)\cdot p(\boldsymbol{x},t;\boldsymbol{\xi},s)}\sigma(\boldsymbol{\xi},s)L(d\boldsymbol{\xi},ds)$$

and $p : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^n$ with $1 \leq n \leq 2d+2$ is a linear map that represents a possible dimension reduction.

One can quantify the approximation (1) in L^2 by a constant, depending on the model parameters, times

$$\sum_{\alpha\in\mathbb{Z}^n\setminus\mathcal{I}} |\widehat{h}_{\boldsymbol{\lambda}}(\boldsymbol{v}_\alpha)|.$$

Incremental property of ambit fields driven by complex exponential kernels

Incremental property

If $\Delta \boldsymbol{x} \geq \boldsymbol{0}$ and $\Delta t \geq 0$ denote increments in space and time respectively, then

$$\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}+\Delta\boldsymbol{x}, t+\Delta t, \boldsymbol{v}) = C_{\boldsymbol{\lambda}}(\Delta\boldsymbol{x}, \Delta t, \boldsymbol{v}) \left(\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}, t, \boldsymbol{v}) + \epsilon_{\boldsymbol{\lambda}}(\boldsymbol{x}, \Delta\boldsymbol{x}, t, \Delta t, \boldsymbol{v}) \right)$$

holds, where

$$C_{\boldsymbol{\lambda}}(\Delta\boldsymbol{x}, \Delta t, \boldsymbol{v}) = \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\Delta\boldsymbol{x}, \Delta t; \boldsymbol{0}, 0)}$$

and

$$\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{x}, \Delta\boldsymbol{x}, t, \Delta t, \boldsymbol{v}) = \int_{A(\boldsymbol{x}+\Delta\boldsymbol{x}, t+\Delta t)\setminus A(\boldsymbol{x}, t)} \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}, t; \boldsymbol{\xi}, s)} \sigma(\boldsymbol{\xi}, s) L(d\boldsymbol{\xi}, ds).$$

One can further approximate $\epsilon_{\boldsymbol{\lambda}}$ by

$$\epsilon_{\boldsymbol{\lambda}}(\boldsymbol{x}, \Delta\boldsymbol{x}, t, \Delta t, \boldsymbol{v}) \approx \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}, t; \boldsymbol{x}^*, t^*)} \sigma(\boldsymbol{x}^*, t^*) \Delta L(\boldsymbol{x}, t),$$

where $(\boldsymbol{x}^*, t^*) \in \overline{A(\boldsymbol{x}+\Delta\boldsymbol{x}, t+\Delta t) \setminus A(\boldsymbol{x}, t)}$.

Incremental property bound

Proposition

Let $\{(\boldsymbol{x}_j, t_j)\}_{j=0}^J \subset \mathbb{R}^d \times \mathbb{R}$ *be a space time grid and denote by* $(\Delta \boldsymbol{x}_j, \Delta t_j) := (\boldsymbol{x}_j - \boldsymbol{x}_{j-1}, t_j - t_{j-1})$ *where* $j = 1, \ldots, J$ *the increments in the space time domain, where* $(\Delta \boldsymbol{x}_j, \Delta t_j) \geq (\mathbf{0}, 0)$ *holds for all* $j = 1, \ldots, J$*. For a given* $\mathcal{I} \subset \mathbb{Z}^n$*, it holds that*

$$E\left[\left|\sum_{\alpha \in \mathcal{I}} c_\alpha \left(\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}_J, t_J, \boldsymbol{v}_\alpha) - \eta_J(\boldsymbol{v}_\alpha)\right)\right|^2\right]$$
$$\leq C_1 \max_{1 \leq j \leq J} ||(\Delta \boldsymbol{x}_j, \Delta t_j)||^2 + C_2 \max_{1 \leq j \leq J} \mathbb{E}\left[|\sigma(\boldsymbol{x}_{j-1}, t_{j-1}) - \sigma(\boldsymbol{x}_j, t_j)|^2\right],$$

where $C_1, C_2 \geq 0$ *are model dependent constants and* $||\cdot||$ *is the Euclidian norm on* $\mathbb{R}^n$*, where* $A_j := A(\boldsymbol{x}_{J-j+1}, t_{J-j+1}) \setminus A(\boldsymbol{x}_{J-j}, t_{J-j})$ *and*

$$\eta_J(\boldsymbol{v}) := \sum_{j=1}^J \mathrm{e}^{(-\boldsymbol{\lambda}+\mathrm{i}\boldsymbol{v})\cdot p(\boldsymbol{x}_J, t_J; \boldsymbol{x}^*_{J-j}, t^*_{J-j})} \sigma(\boldsymbol{x}^*_{J-j}, t^*_{J-j}) \Delta L(A_j).$$

Fourier method - Summary

Fourier method

To simulate a discrete field $Y(\boldsymbol{x}_0, t_0), Y(\boldsymbol{x}_1, t_1), \ldots, Y(\boldsymbol{x}_J, t_J)$ given all information available at $(\boldsymbol{x}_0, t_0)$, we do the following: For each $(\boldsymbol{x}_j, t_j)$ where $j = 1, \ldots, J$.

1. Simulate the increment in the Lévy basis.
2. For each $\alpha \in \mathcal{I}$, simulate $\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}_j, t_j, \boldsymbol{v}_\alpha)$ from $\widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}_{j-1} t_{j-1}, \boldsymbol{v}_\alpha)$ and the Lévy basis increment.
3. Compute numerically the inverse Fourier transform

$$\int_{A(x,t)} h(\boldsymbol{x}, t; \xi, s)\sigma(\xi, s)L(d\xi, ds) \approx \sum_{\alpha \in \mathcal{I}} c_\alpha \widehat{Y}_{\boldsymbol{\lambda}}(\boldsymbol{x}, t, \boldsymbol{v}_\alpha).$$

Example - Application to forward pricing

Given $\{x_j\}_{j=0}^J$ and $\{t_k\}_{k=0}^K$, consider simulating

$$Y(x,t) = \int_{-\infty}^{t} \int_0^{\infty} g(t-s+x)\varphi(\xi)\sigma_s(\xi)L(d\xi, ds).$$

If $\mathcal{I} = \{n \in \mathbb{Z} : |n| \leq N\}$, $0 < \tau_0 < \tau$ are constants such that $t_K + x_J \leq \tau_0$ and $h|_{(0,\tau_0)} = g|_{(0,\tau_0)}$, a.e., we employ that $\widehat{h}_\lambda$ is symmetric around 0 to write

$$Y(x,t) \approx \frac{c_0}{2}\widehat{Y}_\lambda(x,t,0) + \mathrm{Re}\sum_{n=1}^{N} c_n \widehat{Y}_\lambda(x,t,n\pi/\tau),$$

where $c_n = \widehat{h}_\lambda(n\pi/\tau)/\tau$ for $n = 0, \ldots, N$. Furthermore

$$\widehat{Y}_\lambda(x_j, t_k, v) = \mathrm{e}^{(-\lambda+\mathrm{i}v)\Delta t}\left(\widehat{Y}_\lambda(x_j, t_{k-1}, v) + \int_{t_{k-1}}^{t_k}\int_0^{\infty} \mathrm{e}^{(-\lambda+\mathrm{i}v)(t_{k-1}-s+x_j)}\sigma_s(\xi)L_\varphi(d\xi, ds)\right)$$

for all $j = 0, \ldots, J$ and $k = 1, \ldots, K$, and

$$\widehat{Y}_\lambda(x_j, t_k, v) = \mathrm{e}^{(-\lambda+\mathrm{i}v)\Delta x}\widehat{Y}_\lambda(x_{j-1}, t_k, v).$$

Example - Asian options

As an application consider

$$P(T) = f\left(\int_0^T X(t)dt\right),$$

where f is Lipschitz and X is an LSS process. E.g. $f(x) = \max(x/T - K, 0)$ or $f(x) = \max(K - x/T, 0)$ (Asian call/put). For a constant $C > 0$ it holds that

$$\mathbb{E}\left[\left|f\left(\int_0^T X(t)dt\right) - f\left(\int_0^T \widetilde{X}(t)dt\right)\right|\right] \leq C\left(\int_0^T \mathbb{E}\left[|X(t) - \widetilde{X}(t)|^2\right] dt\right)^{1/2}.$$

Can utilise our approximation on X to simulate option price

$$P(T) = \mathrm{e}^{-r(T-t)}\mathbb{E}\left[f\left(\int_0^T X(t)dt\right)|\mathcal{F}_t\right].$$

Gamma driven Asian option

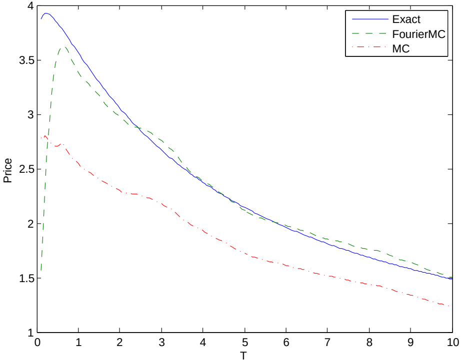

Figure: The price curve $T \mapsto \mathbb{E}[\max(A(T) - K, 0)|\mathcal{F}_t]$, where $A(T) = \frac{1}{T}\int_0^T X(t)dt$, $K = 1$, $t = 0$, $L = W$ on $[0, 10]$ and $L = 0$ on $(-\infty, 0)$, $\sigma = 1$ and $g(x) = Cx^{\nu-1}\mathrm{e}^{-\alpha x}$, where $C = 10$, $\alpha = 1$ and $\nu = 0.55$. As obtained by the explicit Gaussian calculations, numerical integration to evaluate $X(t)$ and the Fourier approximation method, with $\lambda = 1.9$, $\Delta t = 0.05$ and $N = 30$, where the expectations are obtained by averaging over 1000 Brownian motion paths.

References

- Barndorff-Nielsen, O. E., Benth, F. E., and Veraart, A. E. D. Modelling electricity forward markets by ambit fields. Available at: http://ssrn.com/abstract=1938704, (2011).
- Barndorff-Nielsen, O.E., Benth, F. E., and Veraart, A. E. D. Modeling energy spot prices by volatility modulated Lévy-driven Volterra processes. Accepted for publication in *Bernoulli*, (2012).
- Benth, F.E. and Eyjolfsson, H. Stochastic modelling of power markets using stationary processes. To appear in the proceedings of the Ascona Conference 2011, R. Dalang, M. Dozzi, F. Russo (eds.). Springer (2012)
- Benth, F.E., Eyjolfsson, H. and Veraart, A., Approximating Lévy semistationary processes via Fourier methods. Available at *E-print*, No. 4, April, Department of Mathematics, University of Oslo, Norway, (2013).
- Eyjolfsson, H. Ambit fields via Fourier methods in the context of power markets. Available at *E-print*, No. 6, May, Department of Mathematics, University of Oslo, Norway, (2013).