Affine LIBOR models with multiple curves: theory and calibration

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Outline

Evolution of interest rates

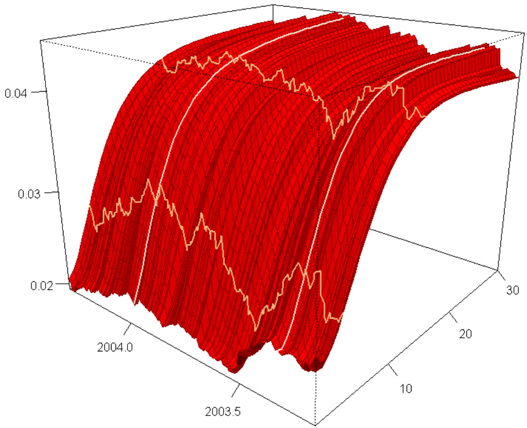


Figure: Evolution of interest rate term structure, 2003–2004

LIBOR rates

- Tenor structure: $\mathcal{T} = \{0 < T_1 < T_2 < \cdots < T_N\}$
- $B(t,T)$: zero coupon bond
- $L(t,T)$: discretely compounded forward LIBOR

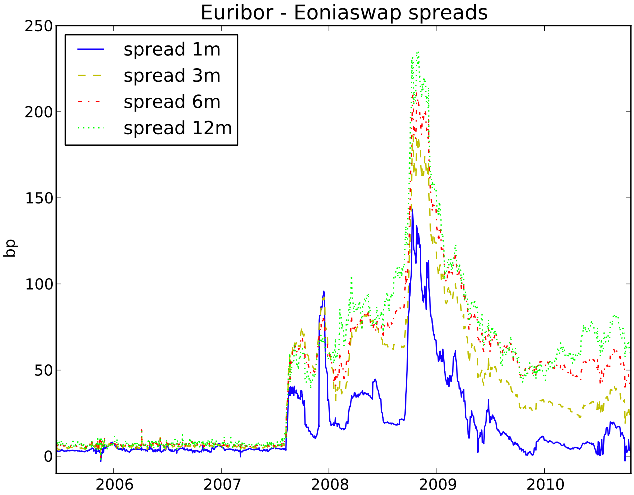
	t	T	$T+\delta$
LIBOR		-1	$1+\delta L(t,T)$
FRA		-1	$\frac{B(t,T)}{B(t,T+\delta)}$

- FRA: buy 1 T-bond, sell $\frac{B(t,T)}{B(t,T+\delta)}$ $[T+\delta]$-bonds

"Master" equation

$$L(t,T) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right) \tag{1}$$

What happened during the ‘credit crunch’?



How is LIBOR *really* computed?

- LIBOR panels: 8-16 banks
  - scale, reputation, expertise
- Question: "*At what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size*?"
- Funds: unsecured interbank cash

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~~*Barclays Bank plc*~~	~~*2.15*~~
~~*Bank of Tokyo-Mitsubishi UFJ Ltd*~~	~~*2.15*~~
~~*HSBC*~~	~~*2.12*~~
~~*Royal Bank of Scotland Group*~~	~~*2.11*~~
UBS AG	*2.105*
Abbey National	*2.1*
Bank of America	*2.1*
Citibank NA	*2.1*
Mizuho Corporate Bank	*2.1*
Rabobank	*2.1*
Royal Bank of Canada	*2.1*
WestLB AG	*2.1*
~~*BNP Paribas*~~	~~*2.05*~~
~~*LLoyds Banking Group*~~	~~*2*~~
~~*Deutsche Bank AG*~~	~~*1.95*~~
~~*JP Morgan Chase*~~	~~*1.95*~~

bbalibor Rate =

2.10063

The new landscape

- Tenor structures:
  - $\mathcal{T}^x = \{0 < T_0^x < T_1^x < \cdots < T_N^x = T_N\}$, $x \in \{1, 3, 6, 12\}M$
- Discount curve: OIS zero coupon bonds

$$T \mapsto B(0,T) = B^{OIS}(0,T)$$

- Forward measures $\mathbb{P}_k^x$ – numeraire $B(\cdot, T_k^x)$

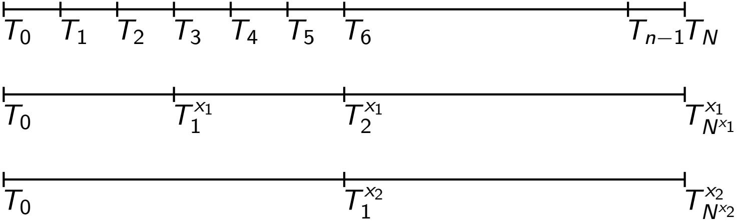

Figure: Illustration of different tenor structures.

The new landscape – II

Definition

The time-t OIS forward rate for $[T^x_{k-1}, T^x_k]$ is

$$F^x_k(t) = \frac{1}{\delta_x}\left(\frac{B(t,T^x_{k-1})}{B(t,T^x_k)} - 1\right) \tag{2}$$

Definition (Mercurio)

The time-t FRA rate L^x_k is the fixed rate to be exchanged at T_k for the LIBOR rate $L(T^x_{k-1}, T^x_k)$ such that the swap has value zero:

$$L^x_k(t) = \mathbb{E}^x_k[L(T^x_{k-1}, T^x_k)|\mathcal{F}_t] \tag{3}$$

- FRA – OIS spread: $S^x_k(t) := L^x_k(t) - F^x_k(t)$

The new landscape – III

Requirements:

- $F_k^x(t) \geq 0$ and F_k^x is a $\mathbb{P}_k^x$-martingale
- $L_k^x(t) \geq 0$ and L_k^x is a $\mathbb{P}_k^x$-martingale
- $S_k^x(t) \geq 0 \Longleftrightarrow L_k^x(t) \geq F_k^x(t)$ (market observations)

Options:

1. Model F_k^x and L_k^x
2. Model L_k^x and S_k^x
3. Model F_k^x and S_k^x

Mercurio [2010]: "The first choice ... is the most convenient in terms of model tractability and calibration... The problem is that there is no guarantee that the implied spreads will have a realistic behavior in the future, in particular preserving the positive sign."

Constructing ordered martingales $\geqslant 1$

1. Take a random variable $Y_T^u \geqslant 1$, $\mathcal{F}_T$-measurable and integrable, and set:

$$M_t^u = \mathbb{E}[Y_T^u | \mathcal{F}_t], \quad (4)$$

then $(M_t)_{0 \leq t \leq T}$ is a martingale and $M_t \geqslant 1$ for all $t \in [0, T]$.

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3. Affine LIBOR model:
   - $Y_T^u = e^{\langle u, X_T \rangle}$
   - X positive affine process

Affine processes I

Model setup:

1. $X = (X_t)_{0 \leq t \leq T_N}$ a time-homogeneous Markov process
2. X takes values in $D = \mathbb{R}^d_{\geqslant 0}$
3. X is affine, if the moment generating function satisfies:

$$\mathbb{E}_x\Big[\exp\langle u, X_t\rangle\Big] = \exp\big(\phi_t(u) + \langle\psi_t(u), x\rangle\big) \tag{6}$$

4. where $\phi_t(u)$ and $\psi_t(u)$ are functions taking values in $\mathbb{R}$ and $\mathbb{R}^d$ and $x \in D$.
5. Also known as CBI processes.

Affine processes II

1. $\phi_t(u)$ and $\psi_t(u)$ are defined on $[0,T] \times \mathcal{I}_T$, where

$$\mathcal{I}_T := \Big\{ u \in \mathbb{R}^d : \mathbb{E}_x\big[\mathrm{e}^{\langle u, X_T\rangle}\big] < \infty, \text{ for all } x \in D \Big\}, \qquad (7)$$

   the 'domain of exponential moments'.
2. Technical assumption: $0 \in \mathcal{I}_T^\circ$.
3. The process X is a regular affine process in the spirit of DFS, and a semimartingale.
4. Old and new examples $\rightsquigarrow$ CIR process, Lévy subordinators, non-Gaussian OU processes, and multivariate extensions.

Ansatz

- Modeling OIS forward rates: $(u_k^x)_{k\geq 0}$

$$1+\delta_x F_k^x(t)=\frac{B(t,T_{k-1}^x)}{B(t,T_k^x)}=\frac{M_t^{u_{k-1}^x}}{M_t^{u_k^x}} \tag{8}$$

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- Modeling FRA rates: $(v_k^x)_{k\geq 0}$

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- The $\mathbb{P}_k^x$-martingale property is obvious
- Orderings:

$$\left.\begin{array}{lll} u_k^x \geq u_{k+1}^x & \Rightarrow & F_k^x \geq 0 \\ v_k^x \geq u_k^x & \Rightarrow & L_k^x \geq F_k^x \end{array}\right\} \text{automatic: initial values} \tag{$\star$}$$

Analytical tractability

Proposition

The process X is a time-inhomogeneous affine process under the measure $\mathbb{P}_k^x$, for every x, k, with

$$\mathbb{E}_{\mathrm{x},k}^{x}\left[e^{\langle w, X_t\rangle}\right] = \exp\left(\phi_t^{k,x}(w) + \langle \psi_t^{k,x}(w), \mathrm{x}\rangle\right), \tag{10}$$

where

$$\phi_t^{k,x}(w) := \phi_t\big(\psi_{T_N-t}(u_k^x) + w\big) - \phi_t\big(\psi_{T_N-t}(u_k^x)\big) \tag{11}$$

$$\psi_t^{k,x}(w) := \psi_t\big(\psi_{T_N-t}(u_k^x) + w\big) - \psi_t\big(\psi_{T_N-t}(u_k^x)\big). \tag{12}$$

Caplet pricing

Easy: express the payoff of a caplet as:

$$\begin{aligned} \delta_x(L(T^x_{k-1}, T^x_k) - K)^+ &= (1 + \delta_x L(T^x_{k-1}, T^x_k) - 1 + \delta_x K)^+ \\ &= \left(e^{A^x_k + B^x_k \cdot X_{T_{k-1}}} - \mathcal{K}\right)^+ \end{aligned} \tag{13}$$

where $\mathcal{K} := 1 + \delta_x K$. Apply Fourier methods

$$\begin{aligned} \mathbb{C}_0(K, T^x_k) &= \delta_x B(0, T^x_k)\ \mathbb{E}^x_k\left[(L(T^x_{k-1}, T^x_k) - K)^+\right] \\ &= \frac{\mathcal{K} B(0, T^x_k)}{2\pi} \int_{\mathbb{R}} \mathcal{K}^{iv-R} \frac{\Lambda_{A^x_k + B^x_k \cdot X_{T_{k-1}}}(R - iv)}{(R - iv)(R - 1 - iv)} \mathrm{d}v \end{aligned} \tag{14}$$

where $\Lambda_{A^x_k + B^x_k \cdot X_{T_k}}$ is the $\mathbb{P}^x_k$-mgf (known).

Swaption pricing

In-crisis: no cancelations and ...

$$\mathbb{S}_0^+(K, \mathcal{T}_{pq}^x) = B(0, T_k^x)\,\mathbb{E}_k^x\Big[\Big(\sum_{i=k+1}^{m} e^{\widetilde{A}_i + \widetilde{B}_i \cdot X_{T_k}} - \sum_{i=k+1}^{m} e^{\widetilde{C}_i + \widetilde{D}_i \cdot X_{T_k}}\Big)^+\Big]$$

$$= B(0, T_k^x)\,\mathbb{E}_k^x\Big[\Big(-\sum_{i=1}^{2l} \mathbf{c}_i e^{\mathbf{A}_i + \mathbf{B}_i \cdot X_{T_k}}\Big)^+\Big]$$

can be treated as a basket option on $\mathbf{Y}_i := \mathbf{A}_i + \mathbf{B}_i \cdot X_{T_k}$

$$= \frac{B(0, T_k)}{(2\pi)^{2l}} \int_{\mathbb{R}^{2l}} \widehat{g}(\mathrm{i}R - v) M_{\mathbf{Y}}(R + \mathrm{i}v)\mathrm{d}v. \tag{15}$$

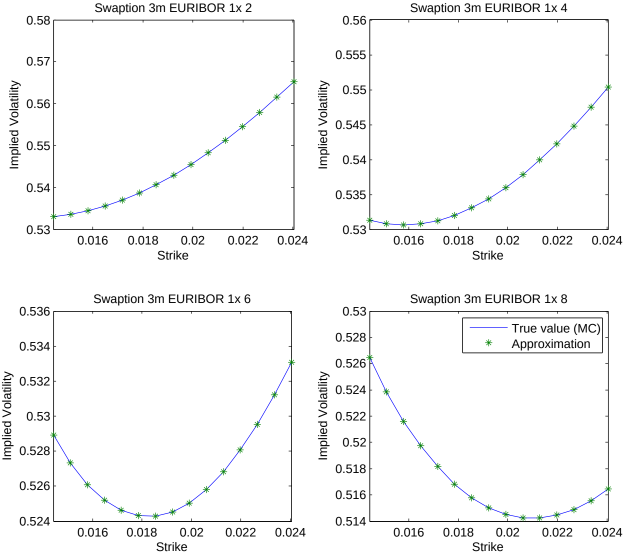
- Efficient numerical approximation?

Swaption pricing II

$$
\begin{aligned}
\mathbb{S}_0^+(K, \mathcal{T}_{pq}^x) &= B(0, T_N)\, \mathbb{E}_N \left[\left(\sum_{j=p+1}^{q} M_{T_p^x}^{v_{j-1}^x} - \sum_{j=p+1}^{q} K_x M_{T_p^x}^{u_j^x} \right)^+ \right] \\
&= B(0, T_N)\, \mathbb{E}_N \left[\left(\sum_{j=p+1}^{q} M_{T_p^x}^{v_{j-1}^x} - \sum_{j=p+1}^{q} K_x M_{T_p^x}^{u_j^x} \right) 1_{\{f(X_{T_p^x}) \geq 0\}} \right] \\
&= B(0, T_N) \sum_{j=p+1}^{q} M_0^{v_{j-1}^x} \overline{\mathbb{E}}_{j-1}^x \left[1_{\{f(X_{T_p^x}) \geq 0\}} \right] \\
&\quad - K_x \sum_{j=p+1}^{q} B(0, T_j^x) \mathbb{E}_j^x \left[1_{\{f(X_{T_p^x}) \geq 0\}} \right]
\end{aligned}
$$

- Linear approximation of the exercise boundary (Singleton & Umantsev)

Swaption pricing III



Connection to LMMs

Assume that X is an affine diffusion: OIS dynamics

$$\frac{dF_k^x}{F_k^x} = \frac{1+\delta_x F_k^x}{\delta_x F_k^x} \sum_{l=1}^{d} \left(\psi_{T-t}^l\left(u_{k-1}^x\right) - \psi_{T-t}^l\left(u_k^x\right)\right)\left(\sigma^l\right)^{\mathsf{T}} \sqrt{X^{x,k;l}} dW^{x,k}$$

FRA dynamics

$$\frac{dL_k^x}{L_k^x} = \frac{1+\delta_x L_k^x}{\delta_x L_k^x} \sum_{l=1}^{d} \left(\psi_{T-t}^l\left(v_k^x\right) - \psi_{T-t}^l\left(u_k^x\right)\right)\left(\sigma^l\right)^{\mathsf{T}} \sqrt{X^{x,k;l}} dW^{x,k}$$

- Built-in displacement
- Structure of volatility: completely determined by X

Affine processes III

Recall:

$$\mathbb{E}_x\left[\exp\langle u, X_t\rangle\right] = \exp\left(\phi_t(u) + \langle\psi_t(u), x\rangle\right) \tag{16}$$

Lemma (Riccati equations)

The functions ϕ and ψ satisfy the (generalized) **Riccati ODEs**

$$\frac{\partial}{\partial t}\phi_t(u) = F(\psi_t(u)), \qquad \phi_0(u) = 0, \tag{17a}$$

$$\frac{\partial}{\partial t}\psi_t(u) = R(\psi_t(u)), \qquad \psi_0(u) = u, \tag{17b}$$

for all suitable $0 \leq t + s \leq T$ and $u \in \mathcal{I}_T$.

Affine processes IV

Lemma (Riccati equations II)

[DFS] showed that

$$F(u) := \frac{\partial}{\partial t}\Big|_{t=0+}\phi_t(u) \qquad \text{and} \qquad R(u) := \frac{\partial}{\partial t}\Big|_{t=0+}\psi_t(u) \tag{18}$$

exist for all $u \in \mathcal{I}_T$ and are continuous in u. Moreover, F and R satisfy Lévy–Khintchine-type equations:

$$F(u) = \langle b, u\rangle + \int_D \big(e^{\langle \xi, u\rangle} - 1\rangle\big) m(d\xi) \tag{19}$$

and

$$R_i(u) = \langle \beta_i, u\rangle + \Big\langle \frac{\alpha_i}{2} u, u \Big\rangle + \int_D \big(e^{\langle \xi, u\rangle} - 1 - \langle u, h^i(\xi)\rangle\big)\mu_i(d\xi), \tag{20}$$

where $(b, m, \alpha_i, \beta_i, \mu_i)_{1\le i\le d}$ are admissible parameters.

Affine processes V

Admissible parameters for $\mathbb{R}^2_{\geqslant 0}$-valued affine processes:

$$b = \begin{pmatrix} + \\ + \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} * \\ + \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} + \\ * \end{pmatrix}$$

$$a = 0, \quad \alpha_1 = \begin{pmatrix} + & * \\ * & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & * \\ * & + \end{pmatrix}$$

m, μ_1, μ_2 are Lévy measures on $\mathbb{R}^2_{\geqslant 0}$

μ_1, μ_2 can have infinite variation

- 'Classical' examples: CIR, CIR with jumps, OU processes, ...

New examples I

2D CIR with dependent jumps

$$\mathrm{d}X_t^1 = -\lambda_1(X_t^1 - \theta_1)\mathrm{d}t + 2\eta_1\sqrt{X_t^1}\mathrm{d}W_t^1 + \alpha_1\mathrm{d}Z_t,$$
$$\mathrm{d}X_t^2 = -\lambda_2(X_t^2 - \theta_2)\mathrm{d}t + 2\eta_2\sqrt{X_t^2}\mathrm{d}W_t^2 + \alpha_2\mathrm{d}Z_t.$$

Functional characteristics

$$F(u_1, u_2) = \lambda_1\theta_1 u_1 + \lambda_2\theta_2 u_2 + \ell\frac{u_1\alpha_1 + u_2\alpha_2}{\frac{1}{\mu} - u_1\alpha_1 - u_2\alpha_2},$$
$$R_i(u_1, u_2) = -\lambda_i u_i + 2\eta_i^2 u_i^2, \quad i = 1, 2.$$

- Explicit solution of the Riccati equations when $\lambda_1 = \lambda_2$

New examples II

Stochastic volatility/intensity on $\mathbb{R}^2_{\geqslant 0}$

$$\mathrm{d}X_t = -\lambda_1(X_t - \theta_1)\mathrm{d}t + 2\eta_1\sqrt{X_t}\mathrm{d}W_t^1 + \mathrm{d}Z_t,$$
$$\mathrm{d}V_t = -\lambda_2(V_t - \theta_2)\mathrm{d}t + 2\eta_2\sqrt{V_t}\mathrm{d}W_t^2,$$

where the intensity of Z is an affine function of V: $\Lambda_t = \ell_0 + \ell_1 \cdot V_t$.
Functional characteristics

$$F(u_1, u_2) = \lambda_1\theta_1 u_1 + \lambda_2\theta_2 u_2 + \ell_0 \frac{u_1}{\frac{1}{\mu} - u_1},$$
$$R_1(u_1, u_2) = -\lambda_1 u_1 + 2\eta_1^2 u_1^2,$$
$$R_2(u_1, u_2) = -\lambda_2 u_2 + 2\eta_2^2 u_2^2 + \ell_1 \frac{u_1}{\frac{1}{\mu} - u_1},$$

- Explicit solution of the Riccati equations when $2\eta_1^2 = \lambda_1\mu$ and $\lambda_2 = \lambda_1/2$

Calibration to caplet data

M maturities, $2M$-drivers (JCIR + CIR)

$$X = ((X^{1,1}, X^{2,1}), \dots, (X^{1,M}, X^{2,M}))$$

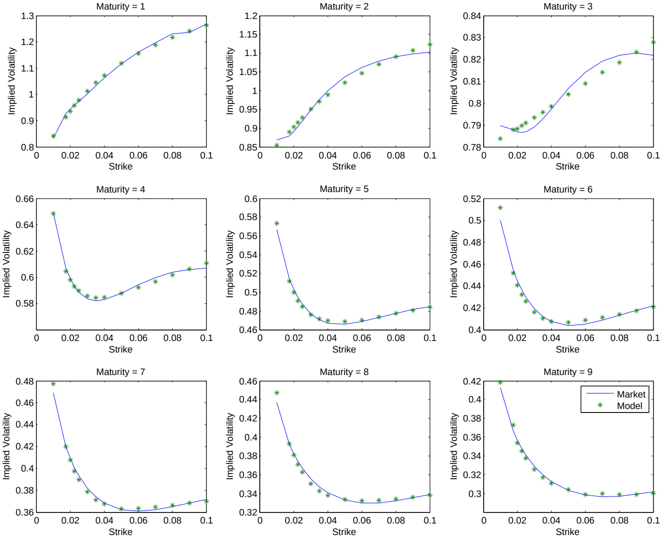
structured via

$$\begin{array}{rl} u^x_{1/\delta_x} & = \begin{pmatrix} \bar{u}_1 & \dots & \bar{v}_{M-2} & \bar{v}_{M-1} & \bar{v}_M \end{pmatrix} \\ \vdots & \quad \vdots \qquad\qquad \ddots \qquad \vdots \qquad \vdots \\ u^x_{(M-2)/\delta_x} & = \begin{pmatrix} (0,0) & \dots & \bar{u}_{M-2} & \bar{v}_{M-1} & \bar{v}_M \end{pmatrix} \\ u^x_{(M-1)/\delta_x} & = \begin{pmatrix} (0,0) & \dots & (0,0) & \bar{u}_{M-1} & \bar{v}_M \end{pmatrix} \\ u^x_{M/\delta_x} & = \begin{pmatrix} (0,0) & \dots & (0,0) & (0,0) & \bar{u}_M \end{pmatrix} \end{array} \tag{21}$$

and

$$\begin{array}{rl} v^x_{1/\delta_x-1} & = \begin{pmatrix} \bar{v}_1 & \dots & \bar{v}_{M-2} & \bar{v}_{M-1} & \bar{v}_M \end{pmatrix} \\ \vdots & \quad \vdots \qquad\qquad \ddots \qquad \vdots \qquad \vdots \\ v^x_{(M-2)/\delta_x-1} & = \begin{pmatrix} (0,0) & \dots & \bar{v}_{M-2} & \bar{v}_{M-1} & \bar{v}_M \end{pmatrix} \\ v^x_{(M-1)/\delta_x-1} & = \begin{pmatrix} (0,0) & \dots & (0,0) & \bar{v}_{M-1} & \bar{v}_M \end{pmatrix} \\ v^x_{M/\delta_x-1} & = \begin{pmatrix} (0,0) & \dots & (0,0) & (0,0) & \bar{v}_M \end{pmatrix} \end{array} \tag{22}$$

Calibration to caplet data II



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Thank you for your attention!