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The Multivariate Ornstein-Uhlenbeck Type Stochastic Volatility Model

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Based on joint works with O. Barndorff-Nielsen (Aarhus), E. Mayerhofer (Dublin), J. Muhle-Karbe (Zürich), O. Pfaffel (TU München), V. Pérez-Abreu (Guanajuato), Ch. Pigorsch (Bonn)

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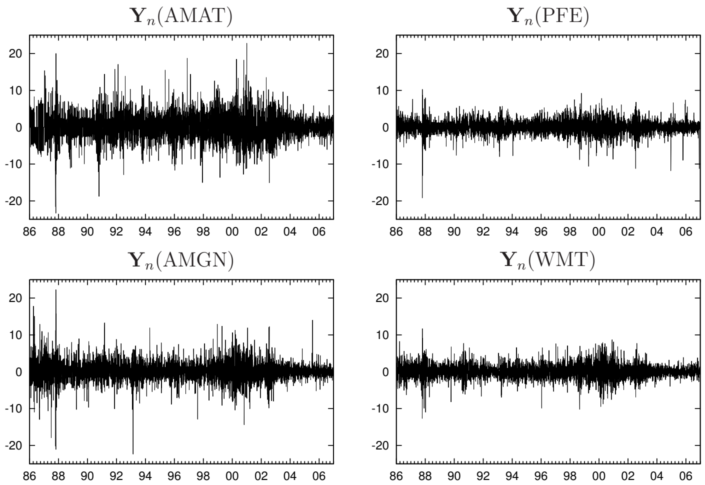
Outline of the talk

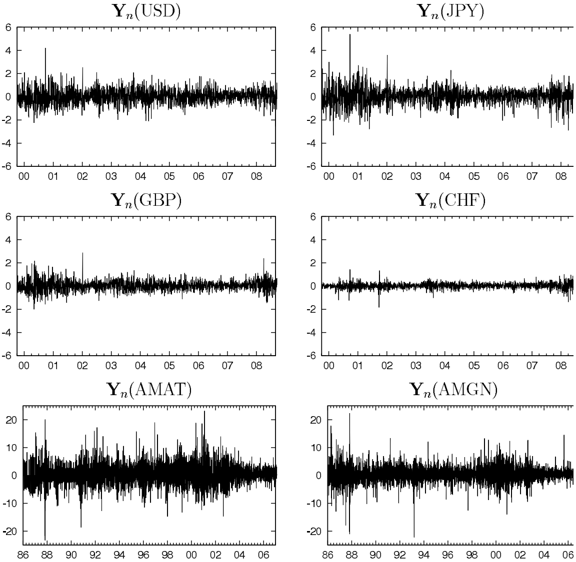
- Motivation
- Positive semi-definite OU type and Wishart processes
- The multivariate OU type stochastic volatility model
- The Wishart affine stochastic correlation model
- Gamma matrix subordinators

Motivation

Log returns of stocks



Log returns of exchange rates



Stylized Facts of Financial Time Series

- non-constant, stochastic volatility (variance)
- volatility exhibits jumps
- asymmetric and heavily tailed marginal distributions
- clusters of extremes
- log returns exhibit marked dependence, but have vanishing autocorrelations (squared returns, for instance, have non-zero autocorrelation)
- long memory (maybe?)
- leverage effect

Stochastic Volatility Models are used to cover these stylized facts.

Univariate OU type Stochastic Volatility Model

- Logarithmic stock price process $(Y_t)_{t\in\mathbb{R}^+}$:

$$dY_t = (\mu + \beta\sigma_{t-})\,dt + \sigma_{t-}^{1/2} dW_t + \rho dL_t$$

  with parameters $\mu, \beta, \rho \in \mathbb{R}$ and $(W_t)_{t\in\mathbb{R}^+}$ being standard Brownian motion.

- Ornstein-Uhlenbeck-type volatility process $(\sigma_t)_{t\in\mathbb{R}^+}$:

$$d\sigma_t = -\lambda\sigma_{t-}dt + dL_t, \;\; \sigma_0 > 0$$

  with parameter $\lambda > 0$ and $(L_t)_{t\in\mathbb{R}^+}$ being a Lévy subordinator.

- One has

$$\sigma_t = e^{-\lambda(t-s)}\sigma_s + \int_s^t e^{-\lambda(t-s)} dL_s.$$

  for all $s, t \in \mathbb{R}^+$ and $t > s$.

Heston Model

Introduced in Heston (1993) ("volatility" process often referred to as CIR process, cf. Cox, Ingersoll, and Ross (1985)).

- The asset price S is given by

$$dS_t = \mu S_t dt + \sqrt{\sigma_t^2} S_t dW_t^{(1)}$$

  with the volatility process solving

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \xi\sqrt{\sigma_t^2} dW_t^{(2)}.$$

- $(W_t^{(1)}, W_t^{(2)})$ are a two-dimensional Brownian motion with covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.
- $\mu \in \mathbb{R}$ is the mean return, $\theta > 0$ is the long run variance, $\kappa > 0$ is the mean reversion speed and $\xi > 0$ is the volatility of volatility.
- If the Feller condition $2\kappa\theta \geq \xi^2$ is satisfied, then σ_t^2 is strictly positive.

The Need for Multivariate Models

Multivariate models are needed

- to study comovements and spill over effects between several assets.
- for optimal portfolio selection and risk management at a portfolio level.
- to price and hedge derivatives on multiple assets.

Desire:
Multivariate models that are flexible, realistic, analytically tractable and amenable to estimation/calibration.

Some matrix notation

- $M_d(\mathbb{R})$: the real $d \times d$ matrices.
- $\mathbb{S}_d$: the real symmetric $d \times d$ matrices.
- $\mathbb{S}_d^+$: the positive semi-definite $d \times d$ matrices (covariance matrices) (a closed cone).
- $\mathbb{S}_d^{++}$: the positive definite $d \times d$ matrices (an open cone).
- $A^{1/2}$: for $A \in \mathbb{S}_d^+$ the unique positive semi-definite square root (functional calculus).
- $\mathrm{tr}(A)$: The trace of a matrix A.

Positive semi-definite processes

Matrix subordinators

- Definition:
  An $\mathbb{S}_d$-valued Lévy process L is said to be a *matrix subordinator*, if $L_t - L_s \in \mathbb{S}_d^+$ for all $s, t \in \mathbb{R}^+$ with $t > s$.
  (Barndorff-Nielsen and Pérez-Abreu (2008)).
- The paths are $\mathbb{S}_d^+$-increasing and of finite variation.
- The characteristic function φ_{L_t} of L_t for $t \in \mathbb{R}^+$ is given by

$$\varphi_{L_t}(z) = e^{t\psi_L(z)} = \exp\left(t \left(i\mathrm{tr}(\gamma_L z) + \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(e^{i\mathrm{tr}(xz)} - 1 \right) \nu_L(dx) \right) \right),$$

for $z \in \mathbb{S}_d$, where γ_L is the drift and ν_L the Lévy measure.

Positive semi-definite OU type processes

Theorem (Barndorff-Nielsen and St. (2007); Pigorsch and St. (2009b))

Let $(L_t)_{t\in\mathbb{R}}$ be a matrix subordinator and $A \in M_d(\mathbb{R})$. Then the stochastic differential equation of Ornstein-Uhlenbeck-type

$$d\Sigma_t = (A\Sigma_{t-} + \Sigma_{t-}A^T)dt + dL_t$$

with initial value $\Sigma_0 \in \mathbb{S}_d^+$ has a unique solution

$$\Sigma_t = e^{At}\Sigma_0 e^{A^T t} + \int_0^t e^{A(t-s)} dL_s e^{A^T(t-s)}.$$

Provided $E(\max(\log\|L_1\|, 0)) < \infty$ and $\sigma(A) \subset (-\infty, 0) + i\mathbb{R}$ it has a unique stationary solution $\Sigma_t = \int_{-\infty}^t e^{A(t-s)} dL_s e^{A^T(t-s)}$.
Moreover, $\Sigma_t \in \mathbb{S}_d^+$ for all $t \in \mathbb{R}$.

Wishart processes

$$dX_t = (X_{t-}\beta + \beta^T X_{t-} + \delta Q^T Q)dt + \sqrt{X_{t-}} dB_t Q + Q^T dB_t^T \sqrt{X_{t-}},$$
$$X_0 = x \in \mathbb{S}_d^{++},$$

where

- $Q \in M_d$, $\beta \in M_d$, and
- B is an M_d-valued standard Brownian motion.

First introduced and studied in Bru (1991) motivated by problems in principal component analysis and therefore especially focusing on eigenvalues.

It is known that a weak solution of the SDE exists (cf. Bru (1991)) if

$$\delta \geq (d-1)$$

and a strong solution exists (cf. Mayerhofer, Pfaffel, and St. (2011)) if

$$\delta \geq (d+1).$$

Affine processes on S_d^+

Cuchiero, Filipović, Mayerhofer, and Teichmann (2011) developed a concise theory of affine Markov processes on $\mathbb{S}_d^+$:

Corollary

The SDE

$$dX_t = (X_t\beta + \beta^\top X_t + \Gamma(X_t) + b)dt + \sqrt{X_t}dB_tQ + Q^\top dB_t^T\sqrt{X_t} + dL_t,$$
$$X_0 = x \in \mathbb{S}_d^+,$$

with

- *B a $d \times d$ standard Brownian motion B, L a matrix subordinator,*
- *$Q, \beta \in M_d$, $b \in \mathbb{S}_d$ and a positive linear drift $\Gamma : \mathbb{S}_d^+ \to \mathbb{S}_d^+$*

has a unique weak solution with infinite life-time if $b \geq (d-1)Q^TQ$, i.e. $b - (d-1)Q^\top Q \in \mathbb{S}_d^+$.

Strong solutions have been considered in Mayerhofer, Pfaffel, and St. (2011).

Multivariate OU type stochastic volatility model

Multivariate OU type SV Model

- L be a matrix subordinator with triplet $(\gamma_L, 0, \nu_L)$, i.e. with characteristic function

$$E(e^{i\langle Z,L_1\rangle}) = \exp\left(i\mathrm{tr}(\gamma_L Z) + \int_{S_d^+} (e^{i\mathrm{tr}(XZ)} - 1)\,\nu_L(dX)\right),$$

- and W an independent $\mathbb{R}^d$-valued Wiener process.

The d-dimensional log price process Y is then given by

$$dY_t = (\mu - \rho(\gamma_L) + \beta(\Sigma_t))\,dt + \Sigma_t^{\frac{1}{2}}\,dW_t + \rho(dL_t), \quad Y_0 = 0,$$

with volatility process

$$d\Sigma_t = (A\Sigma_t + \Sigma_t A^T)\,dt + dL_t, \quad \Sigma_0 \in \mathbb{S}_d^+$$

where $\beta, \rho : M_d(\mathbb{R}) \to \mathbb{R}^d$ are linear operators, $\mu \in \mathbb{R}^d$ and $A \in M_d(\mathbb{R})$ such that $0 \notin \sigma(A) + \sigma(A)$.

Martingale Conditions and Option Pricing

Martingale Conditions

Denote by $S_t = S_0 e^{Y_t} := (S_{1,0}e^{Y_{1,t}}, S_{2,0}e^{Y_{2,t}}, \dots, S_{d,0}e^{Y_{d,t}})^T$ the asset price process.

Theorem (Muhle-Karbe, Pfaffel, and St. (2011))

Suppose:

- $\int_{||X||>1}(e^{\rho(X)_i} - 1)\,\nu_L(dX), \quad i = 1, \dots, d$ *exists*
- $\beta(X)_i = -\frac{1}{2}X_{ii} \qquad \forall\, i = 1, \dots, d,\ X \in \mathbb{S}_d^+,$
- $r \in \mathbb{R}^+$ *denotes the riskless interest rate and*
  $\mu_i = r - \int_{\mathbb{S}_d}(e^{\rho(X)_i} - 1)\,\nu_L(dX) \qquad \forall\, i = 1, \dots, d$

Then $(e^{-rt}S_t)_{t\in\mathbb{R}^+}$ *is a martingale.*

One can characterize all equivalent martingale measures for a general OU type SV model and also all structure preserving equivalent martingale measures (cf. Muhle-Karbe, Pfaffel, and St. (2011)).

Derivative Pricing using Fourier Inversion

- Let $S = (S_1, \ldots, S_d) = (S_{1,0}e^{Y_1}, \ldots, S_{d,0}e^{Y_d})$ be a d-dimensional semimartingale modeling d financial assets and be r the risk-free interest rate. Suppose that $(e^{-rt}S_t)$ is a martingale.
- $V_f(Y_T; s) := E(f(Y_T - s))$ be the arbitrage-free price (at time zero) of a derivative paying the holder $f(Y_T - s)$ at time T, where $f : \mathbb{R}^d \to \mathbb{R}^+$ is a measurable function and $s := (-ln(S_{1,0}), \ldots, -ln(S_{d,0}))$.
- For some fixed $R \in \mathbb{R}^d$ define the dampened payoff function $g(x) := e^{-\langle R,x\rangle} f(x)$, $x \in \mathbb{R}^d$, and $\widehat{\varrho}(u) := \Phi_{Y_T}(R + iu) := E(e^{\langle R+iu, Y_T\rangle})$, $u \in \mathbb{R}^d$. Assume that $g \in L^1 \cap L^\infty$, $\Phi_{Y_T}(R) < \infty$ and $\widehat{\varrho} \in L^1$.

Then

$$V_f(Y_T; s) = \frac{e^{-\langle R,s\rangle - rT}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle u,s\rangle} \Phi_{Y_T}(R + iu) \widehat{f}(iR - u)\, du.$$

Joint Characteristic Function

Theorem (Muhle-Karbe, Pfaffel, and St. (2011))

L be a matrix subordinator with characteristic exponent Ψ_L, i.e. $E(e^{i\mathrm{tr}(L_t Z)}) = e^{t\Psi_L(Z)}$ *for all $Z \in M_d(\mathbb{R}) + i\mathbb{S}_d^+$, $t \in \mathbb{R}^+$.*
Then for every $(y, z) \in \mathbb{R}^d \times M_d(\mathbb{R})$ and $t \in \mathbb{R}^+$:

$$E[\exp(i\langle(y,z),(Y_t,\Sigma_t)\rangle)|\Sigma_0, Y_0] = \exp\Bigg\{ iy^T(Y_0 + (\mu - \rho(\gamma_L))t) + i\mathrm{tr}(\Sigma_0 e^{A^T t} z e^{At})$$
$$+ i\mathrm{tr}\left(\Sigma_0\left(e^{A^T t}\mathbf{A}^{-*}\left(\beta^*(y) + \frac{i}{2}yy^T\right)e^{At} - \mathbf{A}^{-*}\left(\beta^*(y) + \frac{i}{2}yy^T\right)\right)\right)$$
$$+ \int_0^t \Psi_L\left(e^{A^T s} z e^{As} + \rho^*(y) + e^{A^T s}\mathbf{A}^{-*}\left(\beta^*(y) + \frac{i}{2}yy^T\right)e^{As} - \mathbf{A}^{-*}\left(\beta^*(y) + \frac{i}{2}yy^T\right)\right)ds\Bigg\}$$

where $\mathbf{A}^{-*} := (\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1}$ denotes the inverse of the adjoint of $\mathbf{A} : X \mapsto AX + XA^T$.

$z = 0$ gives the characteristic function of Y_t given Y_0, Σ_0.

Analyticity of the Characteristic Function

Theorem (Muhle-Karbe, Pfaffel, and St. (2011))

Suppose

$$\int_{\{||X||\geq 1\}} e^{\mathrm{tr}(RX)}\,\nu_L(dX) < \infty \quad \text{for all } R \in M_d(\mathbb{R}) \text{ with } ||R|| < \epsilon$$

for some $\epsilon > 0$ *and* $\|\cdot\|$ *denotes appropriate (matrix, operator) norms. Then* $\Phi_{Y_{t|0}}(y) := E(e^{\langle Y_T, y\rangle}|(Y_0, \Sigma_0))$ *is analytic on the open strip*

$$S := \{y \in \mathbb{C}^d : ||\Re(y)|| < \theta\},$$

where $\theta := -\frac{||\rho||}{(e^{2||A||t}+1)||\mathbf{A}^{-1}||} - ||\beta|| + \sqrt{\Delta} > 0$,

$$\Delta := \left(\frac{||\rho||}{(e^{2||A||t}+1)||\mathbf{A}^{-1}||} + ||\beta||\right)^2 + \frac{2\epsilon}{(e^{2||A||t}+1)||\mathbf{A}^{-1}||}.$$

and $\widehat{\rho}_R = \Phi_{Y_{t|0}}(R + i\cdot) \in L^1$ *for all* $\|R\| < \theta$.

Options Amenable to "Fourier Pricing"

- Single-Asset: Call/Put, Digital Call/Put, Asset-or-Nothing Call/Put, Self-quanto Call/Put, Power 2 Call/Put, ...
- Multi-Asset:
  - Eberlein, Glau, and Papapantoleon (2010): Call/Put on the minimum/maximum of d assets
  - Hubalek and Nicolato (2009): Spread Call/Put Option with non-zero strike, Basket Put/Call
  - Spread Call/Put Option with zero strike:

Suppose $\Phi_{(Y_T^1, Y_T^2)}(R, 1-R) < \infty$ *for some* $R > 1$. Then the price of a spread option with payoff $(S_{1,0}e^{Y_{1,T}} - S_{2,0}e^{Y_{2,T}})^+$ at $t = 0$ is given by

$$E(e^{-rT}(S_{1,T} - S_{2,T})^+) = \frac{e^{R(s_2-s_1)-s_2-rT}}{2\pi} \int_{\mathbb{R}} e^{iu(s_2-s_1)} \frac{\Phi_{(Y_{1,T},Y_{2,T})}(R+iu, 1-R-iu)}{(R+iu)(R+iu-1)}\, du,$$

where $s_1 = -\ln(S_0^1)$, $s_2 = -\ln(S_0^2)$.

A Calibration Example

The OU-Wishart Model

- L: Compound Poisson process with rate λ and Wishart $\mathcal{W}_2(2,\Theta)$-distributed jumps
- The dynamics are given by:

$$\begin{pmatrix} dY_t^1 \\ dY_t^2 \end{pmatrix} = \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \Sigma_t^{11} \\ \Sigma_t^{22} \end{pmatrix} \right) dt + \begin{pmatrix} \Sigma_t^{11} & \Sigma_t^{12} \\ \Sigma_t^{12} & \Sigma_t^{22} \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix} + \begin{pmatrix} \rho_1 \, dL_t^{11} + \rho_{12} \, dL_t^{12} \\ \rho_2 \, dL_t^{22} + \rho_{21} \, dL_t^{12} \end{pmatrix}$$

$$d \begin{pmatrix} \Sigma_{11,t} & \Sigma_{12,t} \\ \Sigma_{12,t} & \Sigma_{22,t} \end{pmatrix} = \left(\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} + \begin{pmatrix} 2a_1\Sigma_{11,t} & (a_1+a_2)\Sigma_{12,t} \\ (a_1+a_2)\Sigma_{12,t} & 2a_2\Sigma_{22,t} \end{pmatrix} \right) dt + d \begin{pmatrix} L_{11,t} & L_{12,t} \\ L_{12,t} & L_{22,t} \end{pmatrix}$$

- Initial values: $Y_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \Sigma_0 = \begin{pmatrix} \Sigma_{11,0} & \Sigma_{12,0} \\ \Sigma_{12,0} & \Sigma_{22,0} \end{pmatrix}$
- Parameters: $\gamma_1, \gamma_2 \geq 0,\ a_1, a_2 < 0,\ \rho_1, \rho_2, \rho_{12}, \rho_{21} \in \mathbb{R}, \Theta \in \mathbb{S}_2^+$

Reasons for the Choice

- Y_1 and Y_2 are univariate Γ-BNS models (in distribution) if $\rho_{12} = \rho_{21} = 0$.
- Popular (and often used) model for the univariate margins and marginal characteristic function known in closed form.
- Single-Asset Option pricing computationally not slower than in the univariate model:
  Speeds up calibration significantly
- If one assumes $a_1 = a_2$, characteristic function known in closed form.

Problem for calibration: Multi-asset option prices are not readily available (OTC Market)

Calibration: Used Data

- Y_1: EUR/USD exchange rate and Y_2 GBP/USD
- European Call-Options on EUR/USD (148 different strikes/maturities), GBP/USD (67) and EUR/GBP (105) (per April 28th, 2010)
- Riskless interest rates: 3 month LIBOR in the different currencies
- Options on EUR/GBP are zero strike spread options on EUR/USD and GBP/USD
- martingale conditions:

$$\mu_1 = r_{\$} - r_{€} - \lambda \frac{2\rho_1\Theta_{11} + 2\rho_{12}\Theta_{12}}{1 - 2\rho_1\Theta_{11} - 2\rho_{12}\Theta_{12}}.$$
$$\mu_2 = r_{\$} - r_{£} - \lambda \frac{2\rho_2\Theta_{22} + 2\rho_{21}\Theta_{12}}{1 - 2\rho_2\Theta_{22} - 2\rho_{21}\Theta_{12}}.$$

Calibration

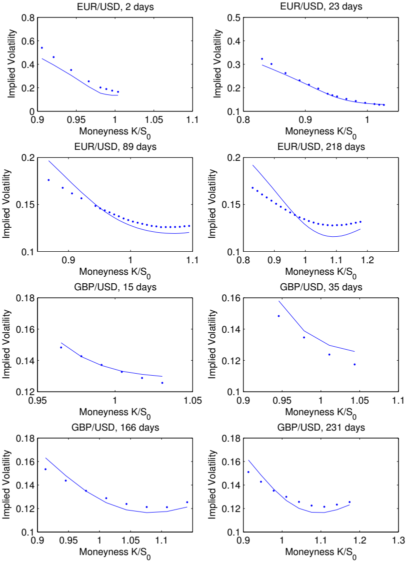
- Calibration by minimizing squared distance between model and market implied volatilities
- Step A: Assuming $a_1 = a_2$, $\rho_{12} = \rho_{21} = 0$
- Step B: Assuming $a_1 = a_2$
- Step C: Assuming $\rho_{12} = \rho_{21} = 0$
- Step D: no restrictions

Calibration: Results

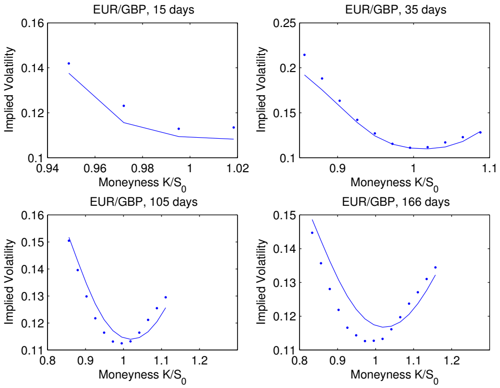
Step	λ	a_1	ρ_1	ρ_{12}	Θ^{11}	Σ_0^{11}	γ_1
A	0.774	-2.392	-3.741	/	0.011	0.019	0.027
B	0.901	-3.008	-5.364	0.679	0.011	0.019	0.034
C	0.774	-2.392	-3.741	/	0.011	0.019	0.027
D	1.231	-7.562	-6.806	0.948	0.010	0.024	0.097
univ. 1	0.781	-32.177	-5.995	/	0.007	0.034	/
univ. 2	0.864	/	/	/	/	/	/
initial	0.800	-2.500	-3.000	/	0.010	0.020	0.020

Step	a_2	ρ_2	ρ_{21}	Θ^{22}	Σ_0^{22}	γ_2	Θ^{12}	Σ_0^{12}
A	/	-0.494	/	0.063	0.017	0.000	0.022	0.013
B	/	-0.661	0.896	0.067	0.018	0.000	0.023	0.013
C	-2.392	-0.494	/	0.063	0.017	0.000	0.022	0.013
D	-6.553	-0.535	1.188	0.102	0.021	0.000	0.030	0.016
univ. 1	/	/	/	/	/	/	/	/
univ. 2	-2.482	-0.471	/	0.050	0.017	0.012	/	/
initial	/	-0.500	/	0.030	0.015	0.011	0.010	0.010

Calibration: Results



Calibration: Results



Calibration: Results

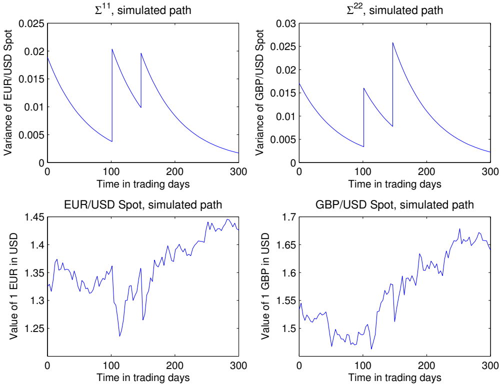


Figure : Simulated sample paths of the EUR/USD and the GBP/USD spot rates and their variances.

Statistics

The Logarithmic Returns

Assume from now on $\rho = 0$.
Let $\Delta > 0$ (grid size). Define for $n \in \mathbb{N}$:

- log-returns over periods $[(n-1)\Delta, n\Delta]$ of length Δ:

$$\mathbf{Y}_n = Y_{n\Delta} - Y_{(n-1)\Delta} = \int_{(n-1)\Delta}^{n\Delta} (\mu + \Sigma_t \beta) dt + \int_{(n-1)\Delta}^{n\Delta} \Sigma_t^{1/2} dW_t.$$

- Integrated volatility over $[(n-1)\Delta, n\Delta]$:

$$\mathbf{\Sigma}_n := \int_{(n-1)\Delta}^{n\Delta} \Sigma_t dt.$$

It holds that

$$\mathbf{Y}_n | \mathbf{\Sigma}_n \sim N_d \left(\mu\Delta + \mathbf{\Sigma}_n \beta, \mathbf{\Sigma}_n \right)$$

with N_d denoting the d-dimensional normal distribution.

Second Order Structure of $\boldsymbol{\Sigma}_n$

(see Pigorsch and St. (2009a))

Assume henceforth $E(\|L_1\|^2) < \infty$.

$$\begin{aligned}
E(\boldsymbol{\Sigma}_n) &= \Delta E(\Sigma_0) = -\Delta \mathbf{A}^{-1} E(L_1) \\
\mathbb{V}\mathrm{ar}(\mathrm{vec}(\boldsymbol{\Sigma}_n)) &= r^{++}(\Delta) + \left(r^{++}(\Delta)\right)^T \\
r^{++}(t) &= \left(\mathscr{A}^{-2}\left(e^{\mathscr{A}t} - I_{d^2}\right) - \mathscr{A}^{-1}t\right)\mathbb{V}\mathrm{ar}(\mathrm{vec}(\Sigma_0)) \\
&= -\left(\mathscr{A}^{-2}\left(e^{\mathscr{A}t} - I_{d^2}\right) - \mathscr{A}^{-1}t\right)\mathcal{A}^{-1}\mathbb{V}\mathrm{ar}(\mathrm{vec}(L_1)) \\
\mathrm{acov}_{\boldsymbol{\Sigma}}(h) &= e^{\mathscr{A}\Delta(h-1)}\mathscr{A}^{-2}\left(I_{d^2} - e^{\mathscr{A}\Delta}\right)^2 \mathbb{V}\mathrm{ar}(\mathrm{vec}(\Sigma_0)) \\
&= -e^{\mathscr{A}\Delta(h-1)}\mathscr{A}^{-2}\left(I_{d^2} - e^{\mathscr{A}\Delta}\right)^2 \mathcal{A}^{-1}\mathbb{V}\mathrm{ar}(\mathrm{vec}(L_1)), \; h \in \mathbb{N}.
\end{aligned}$$

where $\mathscr{A} = A \otimes I_d + I_d \otimes A$ and
$\mathcal{A} : M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R}), X \mapsto \mathscr{A}X + X\mathscr{A}^T$.
$\Longrightarrow$ $\mathrm{vec}(\boldsymbol{\Sigma}_n)$ is a causal ARMA(1,1) process with AR parameter $e^{\mathscr{A}\Delta}$.

Second Order Structure of $\mathbf{Y}_n$ and $\mathbf{Y}_n\mathbf{Y}_n^T$

$$
\begin{aligned}
E(\mathbf{Y}_n) &= (\mu + E(\Sigma_0)\beta)\Delta \\
\mathbb{V}\mathrm{ar}(\mathbf{Y}_n) &= E(\Sigma_0)\Delta + (\beta^T \otimes I_d)\mathbb{V}\mathrm{ar}(\mathrm{vec}(\boldsymbol{\Sigma}_n))(\beta \otimes I_d) \\
\mathrm{acov}_{\mathbf{Y}}(h) &= (\beta^T \otimes I_d)\mathrm{acov}_{\boldsymbol{\Sigma}}(h)(\beta \otimes I_d),\ h \in \mathbb{N}
\end{aligned}
$$

Assume $\mu = \beta = 0$. Then:

$$
\begin{aligned}
E(\mathbf{Y}_n\mathbf{Y}_n^T) &= -\Delta\mathbf{A}^{-1}E(L_1) \\
\mathbb{V}\mathrm{ar}(\mathrm{vec}(\mathbf{Y}_n\mathbf{Y}_n^T)) &= (I_{d^2} + \mathbf{Q} + \mathbf{PQ})\,\mathbb{V}\mathrm{ar}(\mathrm{vec}(\boldsymbol{\Sigma}_n)) \\
&\quad + (I_{d^2} + \mathbf{P})\,(E(\Sigma_0) \otimes E(\Sigma_0))\,\Delta^2 \\
\mathrm{acov}_{\mathbf{YY}^T}(h) &= \mathrm{acov}_{\boldsymbol{\Sigma}}(h) \text{ for } h \in \mathbb{N}
\end{aligned}
$$

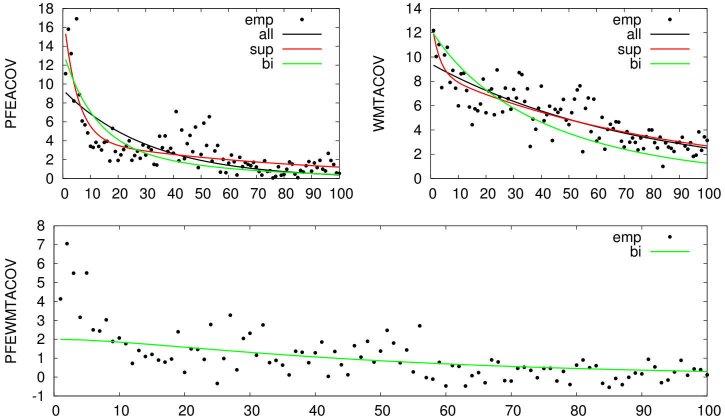
where $\mathbf{P}$ and $\mathbf{Q}$ are linear operators on $M_{d^2}(\mathbb{R})$ rearranging the entries.
$\Longrightarrow$ $\mathrm{vec}(\mathbf{Y}_n\mathbf{Y}_n^T)$ is a causal ARMA(1,1) process with AR parameter $e^{\mathscr{A}\Delta}$.

Moment Estimators

Assume $\mu = \beta = 0$.

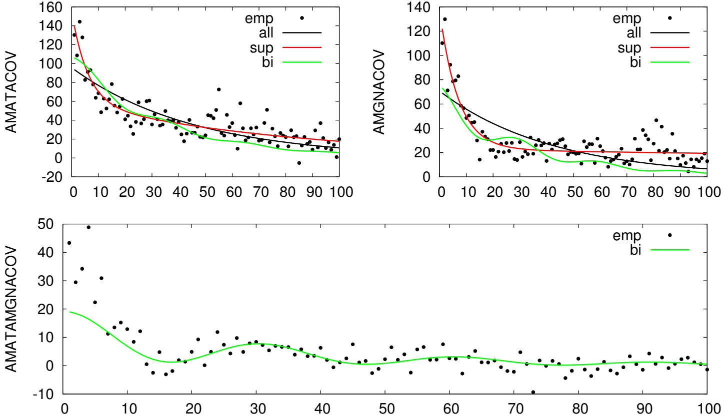
- $E(L_1), \mathbb{V}\mathrm{ar}(\mathrm{vec}(L_1))$ and A can be estimated from the empirically observed $E(\mathbf{Y}_n \mathbf{Y}_n^T), \mathrm{acov}_{\mathbf{Y}\mathbf{Y}^T}(1)$ and $\mathrm{acov}_{\mathbf{Y}\mathbf{Y}^T}(2)$.
- They are identified provided one assumes that $e^{\mathbf{A}_{\mathrm{vech}}\Delta}$ has a unique real logarithm (no issue in one dimension!) and $\mathbb{V}\mathrm{ar}(\mathrm{vech}(\Sigma_0))$ is invertible.
- In practice one uses more lags of the autocovariance function and GMM estimation.
- The log-returns $\mathbf{Y}$ are strongly mixing. Thus the estimators are under appropriate technical conditions consistent and asymptotically normal.

Empirical Illustration I



Empirical and estimated autocorrelation functions: PFE and WMT

Empirical Illustration II



Empirical and estimated autocorrelation functions: AMAT and AMGN

Statistics in the Presence of a Leverage Effect

Second Order Structure

Proposition

Provided Y is given by $dY_t = (\mu + \beta\Sigma_t)dt + \Sigma_t^{1/2}dW_t + \rho dL_t$ and L has finite second moments. Then **Y** *is stationary and has finite second moments.*
Setting $\beta_{\mathrm{vec}} = \beta \circ \mathrm{vec}^{-1}$ and $\rho_{\mathrm{vec}} = \rho \circ \mathrm{vec}^{-1}$ (which are $d \times d^2$ matrices) it holds that:

$$\begin{aligned}
E(\mathbf{Y}_1) =& (\mu + \beta E(\Sigma_0) + \rho E(L_1))\Delta \\
\mathbb{V}\mathrm{ar}(\mathbf{Y}_1) =& \beta_{\mathrm{vec}} \mathbb{V}\mathrm{ar}(\mathrm{vec}(\mathbf{\Sigma}_1))\beta_{\mathrm{vec}}^T + E(\mathbf{\Sigma}_1) + \rho_{\mathrm{vec}} \mathbb{V}\mathrm{ar}(\mathrm{vec}(L_1))\rho_{\mathrm{vec}}^T \\
&+ \mathscr{D}\beta_{\mathrm{vec}} \left(e^{(A\otimes I_d + I_d \otimes A)\Delta} - I_{d^2} - (A\otimes I_d + I_d\otimes A)^{-1}\Delta\right)\rho_{\mathrm{vec}}^T \\
\mathrm{acov}_{\mathbf{Y}}(h) =& \mathbb{C}\mathrm{ov}(\mathbf{Y}_{h+1}, \mathbf{Y}_1) = \beta_{\mathrm{vec}}\mathrm{acov}_{\mathbf{\Sigma}}(h)\beta_{\mathrm{vec}}^T \\
&+ \beta_{\mathrm{vec}} e^{(A\otimes I_d + I_d\otimes A)(h-1)\Delta} \\
&\cdot \left(e^{2(A\otimes I_d + I_d\otimes A)\Delta} - 2e^{(A\otimes I_d + I_d\otimes A)\Delta} + I_{d^2}\right)\rho_{\mathrm{vec}}^T,\ h \in \mathbb{N}
\end{aligned}$$

where $\mathscr{D}X = X + X^T$.

Moment based estimation?

- Estimating the model based on moments requires knowledge about higher order moments like the second order structure of the squared returns
- Problem: These moments are basically not computable
- Solution: Use estimation based on the empirical characteristic function (in the univariate case similar to Taufer, Leonenko, and Bee (2011)) and the moment generating function (Laplace transform).
- Reason: These functions are readily computable for our model

Characteristic Function

Proposition

Assume the driving matrix subordinator L has characteristic exponent ψ_L, i.e. $E(e^{i\operatorname{tr}(L_t z)}) = e^{t\psi_L(z)}$ for all $z \in M_d(\mathbb{R}) + i\mathbb{S}_d^+$.
(i) Then the characteristic function of $\mathbf{Y}_1$ is for all $u \in \mathbb{R}^d$

$$E\left(e^{i\langle u,\mathbf{Y}_1\rangle}\right) = e^{i\mu^T u\Delta} \exp\left\{\int_{-\infty}^{\Delta} \psi_L\left(e^{A^T(\Delta-s)}\left[\mathbf{A}^{-*}\left(\beta^* u + \frac{i}{2}uu^T\right)\right]e^{A(\Delta-s)}\right.\right.$$
$$- 1_{(-\infty,0]}(s)e^{-A^T s}\left[\mathbf{A}^{-*}\left(\beta^* u + \frac{i}{2}uu^T\right)\right]e^{-As}$$
$$\left.\left. -1_{(0,\Delta]}(s)\left(\mathbf{A}^{-*}\left[\beta^* u + \frac{i}{2}uu^T\right] - \rho^* u\right)\right) ds\right\}$$

with $\mathbf{A}^{-*}$ denoting the inverse of the adjoint of $\mathbf{A}$, i.e. $\mathbf{A}^{-*}$ is the inverse of the linear operator $\mathbf{A}^*$ given by $X \mapsto A^T X + XA$.
(ii) A similar result holds for the joint characteristic function $E\left(e^{i(\langle u,\mathbf{Y}_1\rangle + \langle z,\mathbf{Y}_2\rangle)}\right)$ of two log returns.

Analyticity of the Characteristic Function I

Proposition

Assume the driving matrix subordinator L satisfies $\int_{\|X\|\geq 1} e^{\mathrm{tr}(RX)}\nu_L(dx) < \infty$ *for all* $R \in M_d(\mathbb{R}), \|R\|_F < \epsilon$ *with some* $\epsilon > 0$. *Let* Ψ *be the cumulant function of* L, *i.e.* $E(e^{\mathrm{tr}(L_t z)}) = e^{t\Psi_L(z)}$ *for all* $z \in U_L := \{v \in M_d(\mathbb{C}) : \Re(v) = x + y$ *with* $x \in \mathbb{S}_d^-, \|y\|_F < \epsilon\}$.

(i) Then there exists an open convex neighbourhood $U_{\mathbf{Y}}$ *of zero in* $\mathbb{C}^d$ *where the Laplace transform* $E\left(e^{\langle u, \mathbf{Y}_1\rangle}\right)$ *exists and is analytic:*

$$E\left(e^{\langle u,\mathbf{Y}_1\rangle}\right) = e^{\mu^T u\Delta} \exp\left\{\int_{-\infty}^{\Delta} \Psi_L\left(e^{A^T(\Delta-s)}\left[\mathbf{A}^{-*}\left(\beta^* u + \frac{1}{2}uu^T\right)\right]e^{A(\Delta-s)}\right.\right.$$
$$- 1_{(-\infty,0]}(s)e^{-A^Ts}\left[\mathbf{A}^{-*}\left(\beta^* u + \frac{1}{2}uu^T\right)\right]e^{-As}$$
$$\left.\left.-1_{(0,\Delta]}(s)\left(\mathbf{A}^{-*}\left[\beta^* u + \frac{1}{2}uu^T\right] - \rho^* u\right)\right) ds\right\}$$

and $U_{\mathbf{Y}} \supseteq \{u \in \mathbb{C}^d : \|\Re(u)\| < U_{max}\} =: U_{\mathbf{Y},\epsilon}$.

Analyticity of the Characteristic Function II

Proposition (continued)

Here

$$U_{max} = \min\left(-\|\beta^*\|_{2,F} + \sqrt{\|\beta^*\|_{2,F}^2 + \frac{2\epsilon}{\kappa_A^2 \|(A \otimes I_d + I_d \otimes A)^{-1}\left(e^{A\Delta} \otimes e^{A\Delta} - I_d^2\right)\|}}, \right.$$
$$- \|\beta^*\|_{2,F} - \frac{\|\rho^*\|_{2,F}}{(\kappa_A^2+1)\|(A \otimes I_d + I_d \otimes A)^{-1}\|}$$
$$\left. + \sqrt{\left(\|\beta^*\|_{2,F} + \frac{\|\rho^*\|_{2,F}}{(\kappa_A^2+1)\|(A \otimes I_d + I_d \otimes A)^{-1}\|}\right)^2 + \frac{2\epsilon}{(\kappa_A^2+1)\|(A \otimes I_d + I_d \otimes A)^{-1}\|}} \right) > 0$$

with $\kappa_A \geq 1$ *and* $\rho_A < 0$ *satisfying* $\|e^{As}\| \leq \kappa_A e^{\rho_A s} \, \forall \, s \geq 0$.
(ii) A similar result holds for the joint Laplace transform $E\left(e^{(\langle u, \mathbf{Y}_1\rangle + \langle z, \mathbf{Y}_2\rangle)}\right)$ *of two log returns.*

Estimation approach for the general OU type SV model

- GMM based estimation which minimises the distance between the theoretical and empirical Laplace transform (at a chosen set of points) using both $E\left(e^{\langle u, \mathbf{Y}_1 \rangle}\right)$ and $E\left(e^{(\langle u, \mathbf{Y}_1 \rangle + \langle z, \mathbf{Y}_2 \rangle)}\right)$ to capture the dependence structure.
- In many respects this is similar to the calibration procedure of Muhle-Karbe, Pfaffel, and St. (2011).
- Unfortunately this has yet to be implemented on a computer and applied to data ...
- Characteristic function based estimation works well in the univariate case (see Taufer, Leonenko, and Bee (2011)).

Some additional remarks on the OU type SV model

- Adding long memory: supOU stochastic volatility model (see Barndorff-Nielsen and St. (2011, 2013); St., Tosstorff, and Wittlinger (2013)).
- Tail behaviour under regularly varying matrix subordinators (see Moser and St. (2011)).

The Wishart affine stochastic correlation model

The Model

The d-dimensional price process is given by

$$dS_t = \mathrm{diag}(S_t)(\mu dt + \Sigma_t^{1/2} dZ_t)$$

with $\mu \in \mathbb{R}$ and the volatility given by a Wishart process

$$d\Sigma_t = (\Sigma_t \beta + \beta^\top X_t + bQ^T Q)dt + \sqrt{\Sigma_t} dB_t Q + Q^\top dB_t^T \sqrt{\Sigma_t}$$

with $b > d - 1$, $Q \in GL_d(\mathbb{R})$ and $\beta \in M_d(\mathbb{R})$.
W is a standard $d \times d$-dimensional Brownian motion. Furthermore, there exists a d-dimensional standard Brownian motion B independent of W and $\rho \in \mathbb{R}^d$ with $\|\rho\|_2 \leq 1$ such that

$$Z_t = B_t \rho + \sqrt{1 - \rho^T \rho} W_t.$$

Wishart Affine Stochastic Correlation Model – What has been done?

- Estimation via characteristic function (see da Fonseca, Grasseli, and Ielpo (2012))
- Some option pricing (see da Fonseca, Grasselli, and Tebaldi (2007); Gourieroux and Sufana (2010))
- Calibration to indices (using plain vanilla options and "Fourier pricing"; see da Fonseca and Grasselli (2011)):
  - Restrictions on ρ seem to be rather tricky.
  - Claim: It is sufficient to use only plain vanilla options on a single underlying.

Gamma Matrix Subordinators

Defining matrix subordinators via the Lévy measure

- "Gamma distributions" on $\mathbb{S}_d^+$ to be found in the literature typically not infinitely divisible (Wishart distributions are a special case, Lévy (1948))
- Analogues of univariate subordinators can be defined via the characteristic functions: e.g. (tempered) stable, Gamma or IG matrix subordinators
- Recall that the univariate $\Gamma(\alpha, \beta)$ distribution has characteristic function

$$\hat{\mu}(z) = \left(\frac{\beta}{\beta - iz}\right)^{\alpha} = \exp\left(\ln\left(\frac{\beta}{\beta - iz}\right)\alpha\right).$$

and Lévy measure

$$\nu_L(dx) = \alpha \frac{e^{-\beta x}}{x} 1_{[0,\infty)}(x) dx$$

Gamma matrix subordinators

- Barndorff-Nielsen and Pérez-Abreu (2008):

$$\nu_L(dx) = \frac{e^{-\mathrm{tr}(x)}}{\mathrm{tr}(x)^{d(d+1)/2}} 1_{\mathbb{S}_d^+}(x)dx$$

- Pérez-Abreu and Sakuma (2008):

$$\nu_L(E) = \int_{v\in\mathbb{S}_d^+ :\, \|v\|_2=1} \int_{\mathbb{R}^+} 1_E(rv)\frac{e^{-r}}{r} dr\omega_d(dv), \quad E \in \mathcal{B}(\mathbb{S}_d)$$

where $\omega_d(dv)$ is a certain measure concentrated on the positive semi-definite rank one matrices of norm one.

Gamma matrix subordinators

- $\Gamma_{\mathbb{S}_d^+}(\alpha, \beta)$-distribution, Pérez-Abreu and St. (2012):

$$\nu_L(E) = \int_{\mathbf{S}_{\mathbb{S}_d^+, \|\cdot\|}} \int_{\mathbb{R}^+} 1_E(rv) \frac{e^{-\beta(v)r}}{r} dr\alpha(dv)$$

  where α is a finite measure on the unit sphere $\mathbf{S}_{\mathbb{S}_d^+, \|\cdot\|}$ with respect to the norm $\|\cdot\|$ and β a Borel-measurable function $\beta : \mathbf{S}_{\mathbb{S}_d^+, \|\cdot\|} \to \mathbb{R}^+$ such that

$$\int_{\mathbf{S}_{\mathbb{S}_d^+, \|\cdot\|}} \ln\left(1 + \frac{1}{\beta(v)}\right) \alpha(dv) < \infty.$$

- The property of being a Gamma distribution does not depend on the norm used.
- Scaling and convolution properties; closed under invertible linear transformations
- Self-decomposable and, whenever non-degenerate, absolutely continuous

The $A\Gamma$-distribution I

Proposition (Pérez-Abreu and St. (2012))

Let $\eta > (d-1)/2$. There exists a homogeneous Gamma matrix distribution $\Gamma_{\mathbb{S}_d^+}(\alpha_\eta, \beta)$ with respect to the trace norm where $\beta(U) = 1$ for each $U \in \mathbf{S}^+_{\|\cdot\|}$ and α_η is the measure on $\mathbf{S}^+_{\|\cdot\|}$ given by

$$\alpha_\eta(\mathrm{d}U) = c_{d,\eta} \, |U|^\eta \frac{\mathrm{d}U}{|U|^{(d+1)/2}}, c_{d,\eta} = \omega_{d,\eta} \frac{\Gamma(\eta d)}{\Gamma_d(\eta)} \tag{1}$$

with $\alpha_\eta(\mathbf{S}^+_{\|\cdot\|}) = \omega_{d,\eta}$. Moreover, the Lévy measure of $\Gamma_{\mathbb{S}_d^+}(\alpha_\eta, \beta)$ is ρ_η and has a polar decomposition

$$\rho_\eta(E) = \int_E g_\eta(X)\mathrm{d}X = c_{d,\eta} \int_{\mathbf{S}^+_{\|\cdot\|}} \int_0^\infty 1_E(rU) \frac{e^{-r}}{r} \mathrm{d}r \frac{|U|^\eta \, \mathrm{d}U}{|U|^{(d+1)/2}}.$$

The $A\Gamma$-distribution II

Definition (Pérez-Abreu and St. (2012))

Let $\eta > (d-1)/2$ and $\Sigma \in \mathbb{S}_d^{++}$. An infinitely divisible $p \times p$ positive definite random matrix M is said to follow the distribution $A\Gamma_d(\eta, \Sigma)$ if it has Gamma distribution $\Gamma_{\mathbb{S}_d^+}(\alpha_{\eta,\Sigma}, \beta_\Sigma)$ with respect to the trace norm where $\beta_\Sigma(U) = \mathrm{tr}(\Sigma^{-1}U)$ and

$$\alpha_{\eta,\Sigma}(\mathrm{d}U) = \frac{1}{|\Sigma|^\eta \, \mathrm{tr}(\Sigma^{-1}U)^{\eta d}} \alpha_\eta(\mathrm{d}U) \qquad (2)$$

and α_η is given by (1).

Note that if $M \sim A\Gamma(\eta, \mathrm{I}_d)$, then $\Sigma^{1/2} M \Sigma^{1/2} \sim A\Gamma_d(\eta, \Sigma)$.

Properties of the $A\Gamma$-distribution I

- The case $\eta = (d+1)/2$ was considered in Barndorff-Nielsen and Pérez-Abreu (2008).
- The distribution $A\Gamma_d(\eta, \sigma \mathrm{I}_d)$ with $\sigma \in \mathbb{R}^+$ is invariant under orthogonal conjugations.
- If $M \sim A\Gamma_d(\eta, \sigma \mathrm{I}_d)$, then $\mathrm{tr}(M)$ follows a one-dimensional Gamma distribution $\Gamma(\omega_{d,\eta}, \sigma)$.
- For $\eta \in ((d-1)/2, (d+1)/2)$ the Lévy density becomes infinity at the non-invertible elements of $\mathbb{S}_d^+$ (i.e. the matrices which are positive semi-definite, but not strictly).
  For $\eta > (d+1)/2$ the Lévy density becomes zero at the non-invertible elements of $\mathbb{S}_d^+$.
  For $\eta = (d+1)/2$ we have that α_{η, I_d} is the uniform measure on the unit sphere.

Properties of the $A\Gamma$-distribution II

The last observation is very interesting in relation to applications using e.g. $A\Gamma_d(\eta, \Sigma)$ matrix subordinators.

- If all jumps should really be strictly positive definite , then a model with $\eta > (d+1)/2$ should be appropriate.
- Whereas one should use $\eta \in ((d-1)/2, (d+1)/2)$, if it seems desirable to have most of the strictly positive definite jumps rather close to non-invertible matrices.
- The latter should be especially useful in stochastic volatility models in finance where very often news (resulting in jumps of the covariance) should only really affect a single asset or a special group of assets (like stocks of companies from the same country or the same branch of industry) or make all assets perfectly correlated.

Moments of the $A\Gamma$-distribution

The m-th moments and cumulants of a $d \times d$ random matrix are d^{2m}-dimensional. We now define the moments and cumulants using the tensor product, e.g. the m-th moment of a random matrix X is understood to be $E(X^{\otimes m})$.

Corollary (Pérez-Abreu and St. (2012))

The cumulants of the random matrix $M \sim A\Gamma_d(\eta, \Sigma)$ are proportional to the tensor moments of the Wishart distribution $W_d(2\eta, \Sigma)$. In particular

$$\mathbb{E}(M) = \frac{\omega_{d,\eta}}{d}\Sigma$$

and the matrix of covariances between elements of M is given by

$$\begin{aligned} \mathbb{V}\mathrm{ar}(M) &= \mathbb{E}(M \otimes M) - \mathbb{E}(M) \otimes \mathbb{E}(M) \\ &= \omega_{d,\eta}\frac{\eta}{d(nd+1)}\left(\left(1+\frac{1}{2\eta}\right)\mathrm{I}_{d^2} + K\right)(\Sigma \otimes \Sigma). \end{aligned}$$

K is the $d^2 \times d^2$ commutation matrix.

Thank you very much for your attention!

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