Applications of the Likelihood Theory in Finance: Modelling and Pricing

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Summary

This paper discusses the connection between mathematical finance and statistical modelling which turns out to be more than a formal mathematical correspondence. We like to figure out how common results and notions in statistics and their meaning can be translated to the world of mathematical finance and vice versa. A lot of similarities can be expressed in terms of LeCam’s theory for statistical experiments which is the theory of the behaviour of likelihood processes. For positive prices the
Outline

• 1. Motivation
• 2. Representation of financial models as statistical experiments
• 3. Option prices as functions of power functions of tests
• 4. Convergence of option prices
1. Motivation

Goal:
1. Application of statistical results in finance.
2. Review parallel working in finance and statistical modelling (Le Cam theory).

Similarities:
- Filtered likelihood processes
- Regression models
- Contiguity
-Completeness
-...
1. Motivation

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References for Le Cam theory:

L. LeCam.
*Asymptotic Methods in Statistical Decision Theory.*
Springer-Verlag, 1986.

L. LeCam and G.L.Yang.
*Asymptotics in statistics.*

A. N. Shiryaev and V. G. Spokoiny.
*Statistical Experiments and Decisions: Asymptotic Theory.*

H. Strasser.
*Mathematical Theory of Statistics.*
Examples of statistical concepts in finance

- geometric Brownian motion
- Contiguity: asymptotic arbitrage
  Kabanov / Kramkov, Hubalek / Schachermeyer
- Neyman Pearson tests
  Föllmer / Leukert, Schied, Rudloff / Karatzas
- binary experiments
  Gushin / Mordecki
- Le Cam’s third Lemma in finance
  Shiryaev
2. Representation of financial models as statistical experiments

Filtered statistical experiment:

- \( E = (\Omega, \mathcal{F}, \{ P_\vartheta : \vartheta \in \Theta \}) \)
- Filtration \((\mathcal{F}_t)_{t \geq 0}\).

Financial model:

- Time interval: \( I \subset [0, T] \) with \( T < \infty \), \( \{0, T\} \subset I \),
- Filtered probability space \((\Omega, \mathcal{F}, P)\) with filtration \((\mathcal{F}_t)_{t \in I}\), \( \mathcal{F} = \mathcal{F}_T \), \( \mathcal{F}_0 = \{ N : P(N) = 0 \text{ or } P(N) = 1 \} \)
- Adapted, positive, discounted price processes \((X^i_t)_{t \in I}, 1 \leq i \leq d\).
2. Representation of financial models as statistical experiments

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- Adapted, positive, discounted price processes \((X^i_t)_{t\in I}, 1 \leq i \leq d\).
Martingale measure: 
$Q$ Martingale measure, if $(X^i_t)_{t \in I}$ $Q$-martingale $\forall 1 \leq i \leq d$.

**Theorem 1**

Let $Q$ be a probability measure equivalent to $P$. The following assertions are equivalent:

1. There are probability measures $Q_1, \ldots, Q_d$ on $(\Omega, \mathcal{F})$ satisfying

   \[
   \frac{dQ_i|_{\mathcal{F}_t}}{dQ|_{\mathcal{F}_t}} = \frac{X^i_t}{X^i_0}, \quad t \in I. \tag{1}
   \]

   and $Q_i \ll Q$ for all $1 \leq i \leq d$.

2. $Q$ is a martingale measure.
From now on: financial models which allow a martingale measure

Notation:

• \( \left( \Omega, \mathcal{F}, \{Q_1, \ldots, Q_d, Q, P\} \right) \) together with \( (\mathcal{F}_t)_{t \in I} \) is called a financial experiment.

• The processes \( \left( \frac{dQ_i|\mathcal{F}_t}{dQ|\mathcal{F}_t} \right)_{t \in I} \) are called filtered likelihood processes.
Example (Cox-Ross-Rubinstein model)

- \( Q, Q_1 \) products of Bernoulli distributions (parameters \( \tau, \kappa \))
- \( Q := ((1 - \tau)\varepsilon_0 + \tau\varepsilon_1)^N \) and \( Q_1 := ((1 - \kappa)\varepsilon_0 + \kappa\varepsilon_1)^N \)
- Result:

\[
\frac{dQ_1|\mathcal{F}_n}{dQ|\mathcal{F}_n}(k_n) = \left(\frac{\kappa}{\tau}\right)^{k_n} \left(\frac{1 - \kappa}{1 - \tau}\right)^{n-k_n} = \tilde{u}^{k_n} \tilde{d}^{n-k_n} = \frac{X_n(k_n)}{X_0(k_n)}.
\]
Example (Cox-Ross-Rubinstein model)

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\]
Example (Itô type price processes)

Discounted price processes:

\[ X^i_t = X^i_0 \exp \left( \int_0^t \sigma^i_t(s) dW(s) + \int_0^t \left( \mu^i(s) - \rho(s) - \frac{\|\sigma^i(s)\|^2}{2} \right) ds \right) \]

where \( W \) is a \( d \)-dimensional Brownian motion

Usual assumptions:

- **Parameter space** \( \Theta \): space of volatility matrices
  \[ \sigma = (\sigma_{ij})_{i,j=1,\ldots,d} \] which are progressively measurable, uniformly positive definite processes such that an integrability condition holds.
  \[ \sigma_i := (\sigma_{i1}, \ldots, \sigma_{id})' \]
- **Interest rate and drift**: \( \rho \) and \( \mu = (\mu_1, \ldots, \mu_d)' \) progressively measurable processes
- **Bond price**:
  \[ V^0_t = \exp \left( \int_0^t \rho(s) ds \right), \quad 0 \leq t \leq T \]
\[
\frac{X_T^i}{X_0^i} = \frac{dQ_i}{dQ} := \exp \left( \int_0^T \sigma'_i(s)d\bar{W}(s) - \frac{1}{2} \int_0^T \|\sigma_i(s)\|^2 ds \right)
\]

Changing martingale measure:
\( \mathbf{1} = (1, \ldots, 1)' \in \mathbb{R}^d \) and \( \theta(s) := \sigma^{-1}(s)[\rho(s)\mathbf{1} - \mu(s)] \)

- Set \( \frac{dQ}{dP} := \exp \left( \int_0^T \theta'(s)dW(s) - \frac{1}{2} \int_0^T \|\theta(s)\|^2 ds \right) \)

By Girsanov’s Theorem \( \bar{W}(t) = W(t) - \int_0^t \theta(s)ds \) is a \( d \)-dimensional Brownian motion with respect to \( Q \)

Statistical meaning: regression model in survival analysis

- \( \xi(t) = W(t) + \int_0^t (\tau(s) - \theta(s))ds = \bar{W}(t) + \int_0^t \tau(s)ds, \quad 0 \leq t \leq T \)
- \( Q = \mathcal{L}((\xi(t))_{t \leq T}|\tau = 0) \)
- \( Q_i = \mathcal{L}((\xi(t))_{t \leq T}|\tau = \sigma_i), \quad \sigma_i := (\sigma_{i1}, \ldots, \sigma_{id})' \)
\[
\frac{X_i^T}{X_i^0} = \frac{dQ_i}{dQ} := \exp \left( \int_0^T \sigma_i'(s)d\bar{W}(s) - \frac{1}{2} \int_0^T \|\sigma_i(s)\|^2 ds \right)
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### Finance | Statistics
---|---
Itô process models of Black-Scholes type | regression models
volatility | hazard parameters

**Definition**

\[ \{P_\vartheta : \vartheta \in \Theta\} \text{ is } G\text{-complete w.r.t. some class of function } G, \text{ if for } g \in G \text{ we have } \int g \, dP_\vartheta = \text{constant for all } P_\vartheta \text{ implies } g = \text{constant a.e.} \]
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3. Option prices as functions of power functions of tests

Example (European Call option)

- **Payoff European Call:**
  \[ H_C = (X_T^1 - K)^+ = (X_T^1 - K)1_{\{X_T^1 > K\}} = (X_T^1 - K)\phi_C \left( \frac{dQ_1}{dQ} \right) \]

  where \( \phi_C \left( \frac{dQ_1}{dQ} \right) = 1 \left\{ \frac{dQ_1}{dQ} > \frac{K}{X_0^1} \right\} \) is a Neyman Pearson test for the null hypothesis \( \{ Q \} \) versus \( \{ Q_1 \} \)

- **Fair price:**
  \[ p(H_C) = X_0^1 E_{Q_1} \left( \phi_C \left( \frac{dQ_1}{dQ} \right) \right) - KE_{Q_1} \left( \phi_C \left( \frac{dQ_1}{dQ} \right) \right) \]

- \( p(H_C) + K \) Bayes risk of the test \( \phi_C \) with respect to the prior \( \Lambda_0 = X_0^1 \) and \( \Lambda_1 = K \)
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Example (European Call option)

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Option:
Random payoff $H$ at time $T$.
Assumption (A): $H$ is of the form

$$H = \sum_{j=1}^{m} \sum_{i=1}^{d} [a_{ij} X^i_T - K_{ij}] \phi_{ij} \left( \left( \frac{X^i_t}{X^i_0} \right)_{t \in I} \right),$$

where $\phi_{ij} : \mathbb{R}^l \to [0, 1]$ tests and $a_{ij}, K_{ij}$ real coefficients $1 \leq i \leq d, 1 \leq j \leq m$.

Examples:
- European Call, European Put
- Strangle Option, Straddle Option, Bull-Spread Option
- Digital Option
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Theorem 2
Under assumption (A) and for a fixed martingale measure \( Q \) the option price \( p(H) \) of \( H \) is given by

\[
p(H) = \sum_{j=1}^{m} \sum_{i=1}^{d} a_{ij} X_0^i E_{Q_i} \left( \phi_{ij} \left( \left( \frac{dQ_i|\mathcal{F}_t}{dQ|\mathcal{F}_t} \right)_{t \leq T} \right) \right) - K_{ij} E_Q \left( \phi_{ij} \left( \left( \frac{dQ_i|\mathcal{F}_t}{dQ|\mathcal{F}_t} \right)_{t \leq T} \right) \right).
\]

- \( E_Q \left( \phi_{ij} \left( \left( \frac{dQ_i|\mathcal{F}_t}{dQ|\mathcal{F}_t} \right)_{t \leq T} \right) \right) \) level of the test \( \phi_{ij} \)
- \( E_{Q_i} \left( \phi_{ij} \left( \left( \frac{dQ_i|\mathcal{F}_t}{dQ|\mathcal{F}_t} \right)_{t \leq T} \right) \right) \) power of the test \( \phi_{ij} \)
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\]

\[
- K_{ij} E_Q \left( \phi_{ij} \left( \left( \frac{dQ_i|\mathcal{F}_t}{dQ|\mathcal{F}_t} \right)_{t \leq T} \right) \right) .
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## Dictionary

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4. Convergence of option prices

Le Cam: Convergence of experiments (likelihood processes)
Consequences: Convergence of Neyman Pearson Tests
Convergence of Bayes risks

- \( E_n = \{ P_{n,\vartheta} : \vartheta \in \Theta \} \) sequence of experiments
- weak convergence of experiments \( E_n \to E = \{ P_{\vartheta} : \vartheta \in \Theta \} \)
- weak convergence of all finite dimensional likelihood ratio processes

\[
\mathcal{L} \left( \left( \frac{dP_{n,\vartheta}}{dP_{n,\vartheta_0}} \right)_{\vartheta} \mid P_{n,\tau} \right) \to \mathcal{L} \left( \left( \frac{dP_{\vartheta}}{dP_{\vartheta_0}} \right)_{\vartheta} \mid P_{\tau} \right) \quad \text{for all } \tau
\]

- compact topology (on the set of classes of experiments)
Example

Central limit Theorem for statistical experiments (LAN) “local asymptotic normality” under some condition:

- $E_n \to E = \{ Q_{\vartheta g} : \vartheta \in \mathbb{R} \}$ weakly
- $\frac{dQ_{\vartheta g}}{dQ_0} = \exp(\vartheta L(g) - \vartheta^2 \| g \|^2 / 2)$ Gaussian shift, $g \in L_2$
- $g \mapsto L(g)$ Gaussian process, mean zero and $\text{Cov}(L(g_1), L(g_2)) = \langle g_1, g_2 \rangle$ under $Q_0$
- Covers the geometric Brownian motion
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- $g \mapsto L(g)$ Gaussian process, mean zero and $\text{Cov}(L(g_1), L(g_2)) = \langle g_1, g_2 \rangle$ under $Q_0$
- Covers the geometric Brownian motion
Example (Brownian motion regression model)

- \( g : [0, 1] \rightarrow \mathbb{R}, \ P_0 = \lambda_{|[0,1]} \)
- \( X_t = B(t) + \int_0^t g(u) du, \ 0 \leq t \leq 1 \)  
  noise + signal
- \( Q_g = \mathcal{L}((X_t)_{t \leq 1} | g) \)
- \( L(g) = \int_0^1 g B(dt) \)
- \( \frac{dQ_g}{dQ_0} = \exp \left( \int_0^1 g B(dt) - \frac{||g||^2}{2} \right) \)
For simplicity, one asset $d = 1$

$$X_{n,t} = \frac{dQ_{1,n|F_{t,n}}}{dQ_{n|F_{t,n}}}, \quad n \in \mathbb{N}_0$$

**Theorem 3**

Suppose that $X_{n,t}$ “converges weakly to” $X_{0,t}$ (in terms of financial experiments). $X_{0,t}$ is a price process iff $Q_{1,n} \triangleleft Q_n$ and $Q_n \triangleleft Q_{1,n}$ (Contiguity).

Contiguity $P_n \triangleleft Q_n$:
When $Q_n(A_n) \to 0$ then $P_n(A_n) \to 0$ holds.

“Asymptotic arbitrage freeness”
For simplicity, one asset $d = 1$

$$X_{n,t} = \frac{dQ_1,n|\mathcal{F}_{t,n}}{dQ_n|\mathcal{F}_{t,n}}, \quad n \in \mathbb{N}_0$$

**Theorem 3**

*Suppose that $X_{n,t}$ “converges weakly to” $X_{0,t}$ (in terms of financial experiments). $X_{0,t}$ is a price process iff $Q_{1,n} \prec Q_n$ and $Q_n \prec Q_{1,n}$ (Contiguity).*

**Contiguity $P_n \prec Q_n$:**

When $Q_n(A_n) \to 0$ then $P_n(A_n) \to 0$ holds.

“Asymptotic arbitrage freeness”
Example

\[ t_k = \frac{k}{n} T \ \text{discrete times} \]

\[ X_{n,t_k} = \prod_{j=1}^{k} Z_{n,j}, \quad Z_{n,j} - 1 = \frac{X_{n,t_k} - X_{n,t_{k-1}}}{X_{n,t_{k-1}}} \quad \text{returns} \]

Under regularity assumptions:
Convergence to the Gaussian shift (= geometric Brownian motion)

Concrete Example: Convergence of Cox-Ross-Rubinstein models
Theorem (Le Cam) (Main Theorem of Testing)

Suppose that $\{P_{n,\vartheta} : \vartheta \in \Theta\} \rightarrow \{P_\vartheta : \vartheta \in \Theta\}$ weakly. $P_\vartheta \ll P_{\vartheta_0}$

Let $\varphi_n : \Omega_n \to [0, 1]$ be a sequence of tests with

$$\lim_{n \to \infty} E_{P_{n,\vartheta}}(\varphi_n) = a_\vartheta \text{ exists.}$$

Then there exists a test $\varphi$ for $(P_\vartheta)_{\vartheta}$ with

$$E_{P_\vartheta}(\varphi) = a_\vartheta \text{ for all } \vartheta.$$ 

$\varphi$ is related to the option for the “limit model”.
Theorem (Le Cam) (Main Theorem of Testing)

Suppose that \( \{ P_{n, \vartheta} : \vartheta \in \Theta \} \rightarrow \{ P_{\vartheta} : \vartheta \in \Theta \} \) weakly. \( P_{\vartheta} \ll P_{\vartheta_0} \)

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\( \varphi \) is related to the option for the “limit model”.
All kind of convergence results are known in statistics.

Application in finance:

- Convergence of financial experiments (price processes)
- Convergence of power functions
  Neyman Pearson tests, Bayes risks
- Convergence of option prices
- Discrete approximation of complicated option prices
| Dictionary |
|-----------------|-----------------|
| **Finance** | **Statistics** |
| price process | filtered likelihood process |
| Itô process models of Black-Scholes type | regression models |
| volatility | hazard parameters |
| complete markets | completeness of experiments |
| options | tests |
| option price | by power functions of test |
| European call option | Neyman Pearson test |
| Black-Scholes price | Bayes risk |
| martingale measure | null hypothesis |
| approximation of continuous time price models | convergence of experiments |
| asymptotic arbitrage free models | contiguity |
References

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Bounds on option prices for semimartingale market models.

F. Hubalek and W. Schachermayer
When does convergence of asset prices imply convergence of option prices?

A. Janssen and M. Tietje.
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Thank you for your attention!
Appendix

\[ X_{n,t_k} = \prod_{j=1}^{k} Z_{n,j}, \quad Z_{n,j} - 1 = \frac{X_{n,t_k} - X_{n,t_{k-1}}}{X_{n,t_{k-1}}} \text{ returns} \]

\[ \frac{dQ_{1,n|\mathcal{F}_{t_k,n}}}{dQ_{n|\mathcal{F}_{t_k,n}}} = X_{n,t_k} = \prod_{j=1}^{k} Z_{n,j} = \frac{d \bigotimes_{j=1}^{k} Q_{1,n(j)}}{d \bigotimes_{j=1}^{k} Q_{n(j)}} \]

\[ \frac{dQ_{1,n(j)}}{dQ_{n(j)}} - 1 = Z_{n,j} - 1 \quad \text{“returns at stage } t_{\frac{i}{n}} \text{ for the } n\text{-th price process”} \]